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SOME CHARACTERIZATIONS OF ARTINIAN RINGS

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Abstract. In this paper, we investigate rings in which the prime radical is an annihilator and present a characterization of Artinian rings satisfying this property. We also study rings in which the singular ideal and the prime radical coincide. Finally we show that Artinian rings are the direct product of a semiprime ring and a semiprime-free ring (ring in which every nonzero ideal contains a nonzero nilpotent ideal) and present a result on quasi-Baer Artinian rings.

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1. Preliminaries

We adopt the following notations and definitions for convenience. In this paper the rings involved might not have identity unless explicitly pointed out.

Let R be a ring.

(1) R is said to be *semiprime-free* if every nonzero ideal contains a nonzero nilpotent ideal of R.
(2) $A \subseteq R$ is said to be a *right annihilator* if $\mathrm{ann_r}(\mathrm{ann_l}(A)) = A$.
(3) $A \subseteq R$ is said to be *left self-faithful* if $A \cap \mathrm{ann_l}(A) = 0$.
(4) For $A \subseteq R$, the sum of ideals N with $A \cap N = 0$ is shown by $\mathrm{C_I}(A)$.
(5) A *right annihilator ideal* means an ideal which is a right annihilator.
(6) For $A, B \subseteq R$, $A \sqsubseteq^{\mathrm{I}}_{\mathrm{e}} B$ indicates that for every ideal N, $A \cap N = 0$ implies $B \cap N = 0$. If also $A \subseteq B$, then we write $A \subseteq^{\mathrm{I}}_{\mathrm{e}} B$. $A \subseteq^{\mathrm{rAI}}_{\mathrm{e}} B$ and $A \subseteq^{\mathrm{rI}}_{\mathrm{e}} B$ are defined similarly but the adjective "ideal" is replaced respectively, by right annihilator ideal and right ideal.
(7) R is said to be *rAI-semiprime* if zero is the only nilpotent right annihilator ideal.

(8) $A \subseteq R$ is said to be lI-I-closed, if for every left ideal N, $A \subseteq_{\mathrm{e}}^{\mathrm{I}} N$ implies $N = A$.
(9) The prime radical of R is shown by $\mathrm{P}(R)$.
(10) Clearly for every ideal N with $\mathrm{P}(R) \cap N = 0$ we have $\mathrm{ann_l}(N) = \mathrm{ann_r}(N)$. This ideal is shown by $\mathrm{ann}(N)$.
(11) Clearly $\mathrm{ann_l}(\mathrm{C_I}(\mathrm{P}(R))) = \mathrm{ann_r}(\mathrm{C_I}(\mathrm{P}(R)))$. This ideal is shown by $\mathrm{EP}(R)$.

Lemma 1.1. *Let R be a ring and N be an ideal. The following conditions are equivalent.*

(1) *N is left self-faithful and $\mathrm{ann_l}(N)$ is a semiprime ideal.*
(2) *N contains no nonzero nilpotent ideal of R.*
(3) *$N \cap \mathrm{P}(R) = 0$.*

Proof. ($1 \Rightarrow 3$) Since $\mathrm{P}(R) \subseteq \mathrm{ann_l}(N)$ and $N \cap \mathrm{ann_l}(N) = 0$.
($3 \Rightarrow 2$) Let J be an nilpotent ideal of R contained in N. Then $J \subseteq N \cap \mathrm{P}(R) = 0$.
($2 \Rightarrow 1$) Since $(N \cap \mathrm{ann_l}(N))^2 = 0$, $N \cap \mathrm{ann_l}(N) = 0$. Now let J be an ideal such that $J^2 \subseteq \mathrm{ann_l}(N)$. Then $(JN)^2 \subseteq N \cap \mathrm{ann_l}(N) = 0$, implying $JN = 0$. Consequently $J \subseteq \mathrm{ann_l}(N)$. □

Lemma 1.2. *Let R be a ring and J be an ideal containing $\mathrm{P}(R)$. Then the following hold.*

(1) $J \subseteq \mathrm{ann}(\mathrm{C_I}(J))$.
(2) $\mathrm{ann}(\mathrm{C_I}(J))$ *is a semiprime ideal.*
(3) $\mathrm{ann}(\mathrm{C_I}(J))$ *is a left annihilator and a right annihilator.*

Proof. For every ideal N, $N \cap J = 0$ implies that $N \cap \mathrm{P}(R) = 0$, hence $\mathrm{ann}(N)$ is a semiprime ideal by Lemma 1.1 and $J \subseteq \mathrm{ann}(N)$ since $N \cap J = 0$. Thus, $\mathrm{ann}(\mathrm{C_I}(J))$ is a semiprime ideal and $J \subseteq \mathrm{ann}(\mathrm{C_I}(J))$. □

Lemma 1.3. *Let R be a ring.*

(1) $\mathrm{EP}(R)$ *is a semiprime ideal and contains* $\mathrm{P}(R)$.
(2) $\mathrm{P}(R) \subseteq_{\mathrm{e}}^{\mathrm{I}} \mathrm{EP}(R)$.
(3) *For every ideal N,* $\mathrm{EP}(R) \sqsubseteq_{\mathrm{e}}^{\mathrm{I}} N$ *implies* $N \subseteq \mathrm{EP}(R)$.
(4) *Every nonzero ideal of R contained in* $\mathrm{EP}(R)$ *contains a nonzero nilpotent ideal of R.*
(5) *If N is an ideal such that every nonzero ideal of R contained in N contains a nonzero nilpotent ideal of R, then $N \subseteq \mathrm{EP}(R)$.*
(6) $\mathrm{EP}(R)$ *is a left annihilator and a right annihilator.*

Proof. (1) and (6) are by Lemma 1.2.
(2) Let N be an ideal with $N \cap \mathrm{P}(R) = 0$. Set $K = N \cap \mathrm{EP}(R)$. Then $KN = 0$ because $N \subseteq \mathrm{C_I}(\mathrm{P}(R))$ and $K \subseteq \mathrm{EP}(R) = \mathrm{ann}(\mathrm{C_I}(\mathrm{P}(R)))$, implying $K^2 = 0$, consequently $K = 0$ by Lemma 1.1.
(3) Let N be an ideal such that $\mathrm{EP}(R) \sqsubseteq^{\mathrm{I}}_{\mathrm{e}} N$. For every ideal J, that $J \cap \mathrm{P}(R) = 0$ we have $J \cap \mathrm{EP}(R) = 0$, implying $J \cap N = 0$, consequently $N \subseteq \mathrm{ann}(J)$. Thus $N \subseteq \mathrm{EP}(R)$.
(4) Let J be an ideal contained in $\mathrm{EP}(R)$. Then, $J \cap \mathrm{P}(R)$ is a nonzero ideal, thus it contains a nonzero nilpotent ideal of R by Lemma 1.1.
(5) By absurd suppose that $N \not\subseteq \mathrm{EP}(R)$. There exists a nonzero ideal J contained in N with $J \cap \mathrm{EP}(R) = 0$. Then $J \cap \mathrm{P}(R) = 0$, hence J contains a nonzero nilpotent ideal of R which is a contradiction. □

Note that conditions (2) and (3) are strong enough to characterize $\mathrm{EP}(R)$. In other words $\mathrm{EP}(R)$ is the unique ideal of R satisfying conditions (2) and (3), because if N is an ideal satisfying (2) and (3), then $N \sqsubseteq^{\mathrm{I}}_{\mathrm{e}} \mathrm{P}(R)$ and $\mathrm{P}(R) \sqsubseteq^{\mathrm{I}}_{\mathrm{e}} \mathrm{EP}(R)$, implying $N \sqsubseteq^{\mathrm{I}}_{\mathrm{e}} \mathrm{EP}(R)$, consequently $\mathrm{EP}(R) \subseteq N$, also similarly $N \subseteq \mathrm{EP}(R)$.

Lemma 1.4. *Let R be a ring. The following conditions are equivalent.*

(1) *R is semiprime-free.*
(2) $\mathrm{P}(R) \subseteq^{\mathrm{I}}_{\mathrm{e}} R$.
(3) $\mathrm{EP}(R) = R$.

Proof. $(2 \Rightarrow 1)$ Let J be a nonzero ideal. $J \cap \mathrm{P}(R)$ contains a nonzero nilpotent ideal of R by Lemma 1.1, because it is a nonzero ideal.
$(1 \Rightarrow 2)$ Let N be a nonzero ideal. N contains a nonzero nilpotent ideal J of R, then $J \subseteq N \cap \mathrm{P}(R)$, implying $N \cap \mathrm{P}(R) \neq 0$.
$(2 \Leftrightarrow 3)$ By Lemma 1.3. □

Lemma 1.5. *Let R be a ring. If $\mathrm{P}(R) \cap \mathrm{C_I}(\mathrm{P}(R)) = 0$, then $R/\mathrm{C_I}(\mathrm{P}(R))$ is a semiprime-free ring.*

Proof. Let $\overline{J}$ be an ideal of $\overline{R} = R/\mathrm{C_I}(\mathrm{P}(R))$. Then $J \cap \mathrm{P}(R) \neq 0$, thus $J \cap \mathrm{P}(R)$ contains a nonzero nilpotent ideal of R by Lemma 1.1. On the other hand $(J \cap \mathrm{P}(R))/\mathrm{C_I}(\mathrm{P}(R)) \neq 0$. Consequently, $\overline{J}$ contains a nonzero nilpotent ideal of $\overline{R}$. □

Lemma 1.6. *Let R be a ring and N be an ideal such that every ideal of R contained in N is idempotent. Then, for every ideals J and P, $N \cap P = 0$ and $J \cap P = 0$ implies $(N + J) \cap P = 0$.*

Proof. Set $N_0 = \{a \in N \mid \exists b \in J, a + b \in P\}$. Let $k \in (N + J) \cap P$. There exist $a \in N$ and $b \in J$ with $k = a + b$. Since $a \in N_0$, there exists $a_i, c_i \in N_0$ such that $a = \sum_{i=1}^{n} c_i a_i$ because N_0 is idempotent. Now there is $b_i \in J$ with $a_i + b_i \in P$, then $c_i(a_i + b_i) = 0$, implying

$$a + \sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i a_i + \sum_{i=1}^{n} c_i b_i = \sum_{i=1}^{n} c_i (a_i + b_i) = 0$$

Thus, $a \in J$ consequently $k = 0$. Therefore $(N + J) \cap P = 0$. $\square$

Lemma 1.7. *Let R be a ring. If P is an ideal such that every ideal having zero intersection with P is idempotent, then $P \cap \mathrm{C_I}(P) = 0$.*

Proof. There is an ideal N maximal with respect to the property $P \cap N = 0$. Clearly $N \subseteq \mathrm{C_I}(P)$. Now let J be an ideal such that $P \cap J = 0$. Then, $P \cap (N+J) = 0$ by Lemma 1.6, implying $N+J = N$ by the maximality of N, consequently $J \subseteq N$. Thus $\mathrm{C_I}(P) = N$, implying $P \cap \mathrm{C_I}(P) = 0$. $\square$

Clearly a ring R with essential prime radical ($\mathrm{P}(R) \subseteq_{\mathrm{e}}^{\mathrm{rI}} R$) is semiprime-free. The following lemma consider the other way.

Lemma 1.8. *Let R be a semiprime-free ring. If for every right ideal J with $J \cap \mathrm{P}(R) = 0$, we have $J \subseteq_{\mathrm{e}}^{\mathrm{rAI}} J + RJ$, then R is with essential prime radical.*

Proof. Let J be a right ideal such that $J \cap \mathrm{P}(R) = 0$. Then, $\mathrm{ann_r}(J) \cap J = 0$ and $\mathrm{ann_r}(J)$ is a right annihilator ideal, thus $(J + RJ) \cap \mathrm{ann_r}(J) = 0$, hence $(J + RJ) \cap \mathrm{P}(R) = 0$, because $\mathrm{P}(R) \subseteq \mathrm{ann_r}(J)$, implying $J \subseteq J + RJ = 0$. $\square$

Lemma 1.9. *Let R be a ring. For every right ideal N and $n \geq 1$, $\mathrm{ann_r}(N) \subseteq_{\mathrm{e}}^{\mathrm{lI}} \mathrm{ann_r}(N^n)$.*

Proof. It is enough to prove the claim for $n = 2$. Let J be a left ideal contained in $\mathrm{ann_r}(N^2)$ such that $J \cap \mathrm{ann_r}(N) = 0$. Then, $NJ \subseteq J \cap \mathrm{ann_r}(N) = 0$, hence $J \subseteq \mathrm{ann_r}(N)$, consequently $J = 0$. $\square$

2. Results

Theorem 2.1. *Let R be a ring. If $R/\mathrm{P}(R)$ is left Artinian, then $R = \mathrm{EP}(R) \oplus \mathrm{C_I}(\mathrm{P}(R))$.*

Proof. First we claim that every ideal N with $N \cap \mathrm{P}(R) = 0$ is idempotent, because $R/\mathrm{P}(R)$ is a semisimple left Artinian ring, thus $N/\mathrm{P}(R)$ is idempotent, hence $N \subseteq N^2 \oplus \mathrm{P}(R)$ implying $N^2 = N$. Thus, $\mathrm{P}(R) \cap \mathrm{C_I}(\mathrm{P}(R)) = 0$ by Lemma 1.7, implying $\mathrm{EP}(R) \cap \mathrm{C_I}(\mathrm{P}(R)) = 0$ by Lemma 1.3. Set $\overline{R} = R/\mathrm{EP}(R)$. Showing $\overline{\mathrm{C_I}(\mathrm{P}(R))} \subseteq_{\mathrm{e}}^{\mathrm{I}} \overline{R}$

completes the proof because $\overline{R}$ is a semisimple left Artinian ring by Lemma 1.3, implying $\overline{\mathrm{C_I}(\mathrm{P}(R))} = \overline{R}$. Let $\overline{J}$ be an ideal of $\overline{R}$ such that $\overline{J} \cap \overline{\mathrm{C_I}(\mathrm{P}(R))} = 0$. Then, $J \cap \mathrm{C_I}(\mathrm{P}(R)) \subseteq \mathrm{EP}(R)$, thus $J \cap \mathrm{C_I}(\mathrm{P}(R)) = 0$, hence $J \subseteq \mathrm{EP}(R)$, consequently $\overline{J} = 0$. □

Proposition 2.2. *Every ring R in which $R/\mathrm{P}(R)$ is left Artinian, is the direct product of a semiprime ring and a semiprime-free ring.*

Proof. We have $R = \mathrm{EP}(R) \oplus \mathrm{C_I}(\mathrm{P}(R))$. On the other hand $R/\mathrm{C_I}(\mathrm{P}(R))$ is a semiprime-free ring by Lemma 1.5, and $R/\mathrm{EP}(R)$ is a semiprime ring by Lemma 1.3. □

Recall that a ring R is *quasi-Baer* if every right annihilator ideal is a right direct summand [1].

Proposition 2.3. *Let R be a nonzero left faithful quasi-Baer ring with ACC on right annihilator ideals. Then, R is not right essential extension of* $\mathrm{P}(R)$.

Proof. There exists a maximal right annihilator ideal Q. Then, there exists a right ideal N such that $R = Q \oplus N$. On the other hand Q is a prime ideal, hence $\mathrm{P}(R) \subseteq Q$, consequently $N \cap \mathrm{P}(R) = 0$. □

Theorem 2.4. *Let R be a quasi-Baer left Artinian ring with identity. If for every right ideal J with $J \cap \mathrm{P}(R) = 0$, we have $J \subseteq_{\mathrm{e}}^{\mathrm{rAI}} RJ$, then R is semisimple.*

Proof. By Proposition 2.2, assuming R is semiprime-free, it is enough to show that $R = 0$ and that can be obtained by applying Lemma 1.8 and Proposition 2.3. □

Theorem 2.5. *A quasi-Baer left Artinian ring with identity in which every right annihilator ideal is left self-faithful, is semisimple.*

Proof. By Theorem 2.4, it s enough to show that for every right ideal J, $J \subseteq_{\mathrm{e}}^{\mathrm{rAI}} RJ$. Let K be a right annihilator ideal such that $J \cap K = 0$. Then $K \subseteq \mathrm{ann_r}(J)$, thus $K \cap \mathrm{ann_l}(\mathrm{ann_r}(J)) = 0$ because $\mathrm{ann_r}(J)$ is a right annihilator ideal, hence $K \cap RJ = 0$ because $RJ \subseteq \mathrm{ann_l}(\mathrm{ann_r}(J))$. □

Proposition 2.6. *Let R be a nonzero left faithful quasi-Baer and idempotent ring. If* $\mathrm{P}(R)$ *is nilpotent, then every maximal right annihilator ideal is a minimal prime ideal.*

Proof. Let Q be maximal right annihilator ideal. There exists a right ideal N such that $R = Q \oplus N$. If Q is not a minimal prime ideal, then Q is contained in no minimal prime ideal because Q is a prime ideal, consequently $N \subseteq \mathrm{P}(R)$. On the other hand there exists $n \geq 1$ such that $\mathrm{P}(R)^n = 0$, implying $R = R^n = Q + N^n = Q$ which is a contradiction. $\square$

Proposition 2.7. *Let R be a ring. The following conditions are equivalent.*

(1) *R is rAI-semiprime.*
(2) *Zero is the only right annihilator ideal with zero square.*
(3) *Every right annihilator ideal is right self-faithful.*

Proof. ($2 \Rightarrow 3$) Let N be a right annihilator ideal. $N \cap \mathrm{ann_r}(N)$ is a right annihilator ideal with zero square, implying $N \cap \mathrm{ann_r}(N) = 0$.
($3 \Rightarrow 1$) Temporary suppose that there exists a nonzero nilpotent right annihilator ideal N. There exists $n \geq 2$ such that $N^n = 0$ and $N^{n-1} \neq 0$. Then $N^{n-1} \subseteq N \cap \mathrm{ann_r}(N) = 0$ which is a contradiction.
($1 \Rightarrow 2$) It is obvious. $\square$

Theorem 2.8. *A left Artinian ring is left nonsingular if and only if is rAI-semiprime.*

Proof. ($\Rightarrow$) Let N be a right annihilator ideal with zero square. We show that $\mathrm{ann_l}(N) \subseteq_{\mathrm{e}}^{\mathrm{ll}} R$. Let J be a left ideal with $\mathrm{ann_l}(N) \cap J = 0$, then $\mathrm{ann_l}(N)J = 0$, thus $J \subseteq \mathrm{ann_r}(\mathrm{ann_l}(N)) = N \subseteq \mathrm{ann_l}(N)$, implying $J = 0$. Thus, $N \subseteq \mathrm{ann_r}(N) \subseteq \mathrm{Z}({}_RR) = 0$.
($\Leftarrow$) It is easy to see that $\mathrm{ann_l}(\mathrm{Z}({}_RR))$ is the intersection of essential left annihilators. Thus, $\mathrm{ann_l}(\mathrm{Z}({}_RR))$ is an essential left ideal, then $\mathrm{ann_r}(\mathrm{ann_l}(\mathrm{Z}({}_RR))) \subseteq \mathrm{Z}({}_RR)$, implying $\mathrm{Z}({}_RR) = \mathrm{ann_r}(\mathrm{ann_l}(\mathrm{Z}({}_RR)))$. Thus $\mathrm{Z}({}_RR)$ is a right annihilator ideal. On the other hand $\mathrm{Z}({}_RR)$ is nilpotent by [4, p. 252, Theorem 7.15]. Therefore $\mathrm{Z}({}_RR) = 0$. $\square$

Recall that a ring R is called *semiprimary* if $R/\mathrm{P}(R)$ is a left Artinian ring and $\mathrm{P}(R)$ is nilpotent [4].

Theorem 2.9. *Let R be a semiprimary and idempotent ring. If $\mathrm{P}(R)$ is a right annihilator, then R is the only left faithful ideal.*

Proof. Let N be a left faithful ideal. We claim that $N/\mathrm{P}(R)$ is left faithful, because if K is an ideal with $KN \subseteq \mathrm{P}(R)$, then $\mathrm{ann_l}(\mathrm{P}(R))KN = 0$, hence $\mathrm{ann_l}(\mathrm{P}(R))K = 0$, implying $K \subseteq \mathrm{ann_r}(\mathrm{ann_l}(\mathrm{P}(R))) = \mathrm{P}(R)$. Thus $R = N + \mathrm{P}(R)$ because $R/\mathrm{P}(R)$ is a semisimple left Artinian ring. On the other hand, there exists $n \geq 1$ such that $\mathrm{P}(R)^n = 0$, then $R = R^n = (N + \mathrm{P}(R))^n \subseteq N + \mathrm{P}(R)^n = N$. $\square$

Corollary 2.10. *Let R be a semiprimary and idempotent ring. If* $\mathrm{P}(R)$ *is a right annihilator, then every left and right self-faithful left annihilator ideal, is a right direct summand.*

Proof. Let N be a left and right self-faithful left annihilator ideal. $N \oplus \mathrm{ann_r}(N)$ is a left faithful ideal. Thus $R = N \oplus \mathrm{ann_r}(N)$ by Theorem 2.9. $\square$

Corollary 2.11. *In a QF-ring, every left and right self-faithful left annihilator ideal is a right direct summand.*

For a semiprimary R, $\mathrm{P}(R)$ is a prime ideal if and only $R/\mathrm{P}(R)$ is simple, also if and only if every ideal is either left faithful or nilpotent. These rings are investigated in [4] and a nice theorem, [4, p. 349, Theorem (23.10)] is obtained. Also this idea is generalized as rings in which for every ideal N, a power of N is left self-faithful and [3, Theorem 8] is obtained and shown that the prime radical is a right annihilator [3, Lemma 5].

A nonzero right ideal N of a ring R is said to be *right self-prime*, if N_R is a prime module, in other words if for every nonzero right ideal J contained in N, $\mathrm{ann_r}(J) = \mathrm{ann_r}(N)$. Below, the sum of right self-prime right ideals is shown by E. It is easy to see that $\mathrm{ann_r}(N)$ is a prime ideal for every right self-prime right ideal N. Thus, $\mathrm{P}(R) \subseteq \mathrm{ann_r}(E)$.

Proposition 2.12. *Let R be a ring with ACC on right annihilator ideals. Then,* $E \subseteq^{\mathrm{rI}}_{\mathrm{e}} R$.

Proof. Let J be a nonzero right ideal. J contains a nonzero right ideal N such that $\mathrm{ann_r}(N)$ is maximal respect to this condition. It can be see easily that N is a right self-prime, implying $E \cap J \neq 0$. $\square$

Theorem 2.13. *Let R be a ring with DCC on left annihilators. Every right essential left annihilator ideal is left essential if and only if* $\mathrm{Z}({}_RR) = \mathrm{ann_r}(E)$. *In this case* $\mathrm{Z}({}_RR) = \mathrm{P}(R)$ *and* $\mathrm{P}(R)$ *is a right annihilator.*

Proof. ($\Rightarrow$) Clearly R is with ACC on right annihilators so R is with ACC on right annihilator ideals. Thus, $E \subseteq^{\mathrm{rI}}_{\mathrm{e}} R$ by Proposition 2.12, hence $\mathrm{ann_l}(\mathrm{ann_r}(E)) \subseteq^{\mathrm{rI}}_{\mathrm{e}} R$, implying $\mathrm{ann_l}(\mathrm{ann_r}(E)) \subseteq^{\mathrm{lI}}_{\mathrm{e}} R$, consequently $\mathrm{P}(R) \subseteq \mathrm{ann_r}(E) \subseteq \mathrm{Z}({}_RR)$. On the other hand $\mathrm{Z}({}_RR)$ is nilpotent by [4, p. 252, Theorem 7.15], implying $\mathrm{Z}({}_RR) \subseteq \mathrm{P}(R)$.

($\Leftarrow$) Let N be a right essential left annihilator ideal. For every right self-prime right ideal $N \cap J \neq 0$, thus $\mathrm{ann_r}(N) \subseteq \mathrm{ann_r}(N \cap J) = \mathrm{ann_r}(J)$, hence $\mathrm{ann_r}(N) \subseteq \mathrm{ann_r}(E) = \mathrm{Z}({}_RR)$, implying $\mathrm{ann_l}(\mathrm{Z}({}_RR)) \subseteq N$. Thus N is left essential. $\square$

A ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for all $a, b \in R$ [2]. Below, we consider rings in which $AB = 0$ implies $BA = 0$ for all ideals A and B. It is easy to see that in this case, for every ideal N, $\mathrm{ann_l}(N) = \mathrm{ann_r}(N)$.

Theorem 2.14. *Let R be a ring with DCC on left annihilators. If $AB = 0$ implies $BA = 0$ for all ideals A and B, then, $\mathrm{Z}({}_RR) = \mathrm{P}(R)$ and $\mathrm{P}(R)$ is a right annihilator.*

Proof. By [4, p. 252, Theorem 7.15], it is enough to show that $\mathrm{Z}({}_RR)$ is a semiprime ideal. Let N be an ideal such that $N^2 \subseteq \mathrm{Z}({}_RR)$. Then $\mathrm{ann_l}(\mathrm{Z}({}_RR)) \subseteq \mathrm{ann_l}(N^2)$, thus $\mathrm{ann_l}(N^2) \subseteq_{\mathrm{e}}^{\mathrm{II}} R$, hence $\mathrm{ann_r}(N^2) \subseteq_{\mathrm{e}}^{\mathrm{II}} R$, implying $\mathrm{ann_r}(N) \subseteq_{\mathrm{e}}^{\mathrm{II}} R$ by Lemma 1.9, consequently $\mathrm{ann_l}(N) \subseteq_{\mathrm{e}}^{\mathrm{II}} R$. Therefore $N \subseteq \mathrm{ann_r}(\mathrm{ann_l}(N)) \subseteq \mathrm{Z}({}_RR)$. $\square$

The following is an application for Theorem 2.1.

Theorem 2.15. *Let R be a left faithful (right faithful) semiprimary ring. If $\mathrm{P}(R)$ is a nonzero prime ideal, then R is semiprime-free.*

Proof. There exist a semiprime ring S and a semiprime-free ring Q such that $R \cong S \times Q$ by Proposition 2.12. It is enough to show that $S = 0$. By absurd suppose that it is not so. Since $\mathrm{P}(S \times Q) = \mathrm{P}(S) \times \mathrm{P}(Q) = 0 \times \mathrm{P}(Q)$, $S \times 0 \nsubseteq \mathrm{P}(S \times Q)$ implying $0 \times Q \subseteq \mathrm{P}(S \times Q)$ because $\mathrm{P}(S \times Q)$ is a prime ideal, consequently $\mathrm{P}(Q) = Q$. Thus Q is nilpotent contradicting the fact that R is left faithful (right faithful). $\square$

As an example, let F be a field of characteristic 3 and D_6 be the dihedral group of degree 6. We show that $F[D_6]$ is semiprime-free. By Theorem 2.15, it is enough to show that $\mathrm{P}(F[D_6])$ is a prime ideal. We know that $D_6 = \{e, a, a^2, b, ab, a^2b\}$ with $a^3 = b^2 = e$ and $ba = a^{-1}b$. Set $K = \{e, a, a^2\}$, $L = \{e, b\}$ and

$$P = \{xe + ya + za^2 + ub + vab + wa^2b \mid x + y + x = 0 \,\&\, u + v + w = 0\}$$

The map $D_6 \longrightarrow L$ given by $a^ib^k \longrightarrow b^k$ is a group epimorphism and the kernel is K. Thus the map $\theta : F[D_6] \longrightarrow F[L]$ given by

$$\theta(xe+ya+za^2+ub+vab+wa^2b) = x\theta(e)+y\theta(a)+z\theta(a^2)+u\theta(b)+v\theta(ab)+w\theta(a^2b) =$$

$$(x + y + x)e + (u + v + w)b$$

is a ring epimorphism with $\mathrm{Ker}(\theta) = P$. On the other hand $F[L]$ is a prime ring by [5 p. 6, Theorem 2.5]. Consequently, P is a prime ideal and $\mathrm{P}(F[D_6]) \subseteq P$. On the other hand in $F[K]$ we have,

$$(e - a)^3 = (e - a)(e + a + a^2) = 0$$

implying $e - a \in \mathrm{P}(F[K])$, also the calculation

$$(e-a)(xe+ya+za^2+ub+vab+wa^2b) =$$

$$(x-z)e+(y-x)a+(z-y)a^2+(u-w)b+(v-u)ab+(w-u)a^2b$$

implies $(e-a)F[D_6] = P$, consequently $P = (e-a)F[D_6] \subseteq \mathrm{P}(F[D_6])$ by [5, p. 84, Theorem 20.2]. Therefore $\mathrm{P}(F[D_6]) = P$. Now in fact $F[D_6]$ is a local ring by [4, p. 349, Theorem 23.10] because $\dim_F(F[D_6]) = 6$.

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