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SPECTRA OF BOOLEAN GRAPHS AND CERTAIN MATRICES OF BINOMIAL COEFFICIENTS

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Abstract. Given an integer $k > 1$, let A be the adjacency matrix of the zero-divisor graph of the finite Boolean ring of order 2^k. In this paper, the spectra of two $(k-1) \times (k-1)$ matrices P and Q of binomial coefficients are shown to be linked to the spectrum of the larger matrix A. Since earlier investigations provide the eigenvalues and eigenvectors of Q, certain eigenvalues and eigenvectors of A are obtained.

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1. Introduction

Given any commutative ring R with $1 \neq 0$, the *zero-divisor graph* of R is the (undirected) graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R such that distinct vertices x and y are adjacent if and only if $xy = 0$. This construction has received a significant amount of attention during the past ten years (cf. [1] and [2]), and provides a means by which tools from graph theory become available to solve problems in algebra, and vice versa. Furthermore, by representing graph-theoretic information with a matrix, techniques from linear algebra become accessible to study graphs, and can therefore be used to investigate rings.

While there are many ways to represent graphs with matrices, one popular construction is achieved by recording the adjacency relations among the vertices of a graph. If Γ is a graph with vertex set $V(\Gamma) = \{v_1, \ldots, v_K\}$, then an *adjacency matrix* of Γ is a $K \times K$ matrix $A = [A_{i,j}]$ such that

$$A_{i,j} = \begin{cases} 0, & \text{if } v_i \notin N(v_j) \\ 1, & \text{if } v_i \in N(v_j) \end{cases},$$

where $N(v_j)$ is the set of all vertices of Γ that are adjacent to v_j. An *eigenvalue* of Γ is then defined to be any eigenvalue of A. It is straightforward to check that

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any two adjacency matrices of Γ are unitarily equivalent ([9, Lemma 8.1.1]). In particular, the eigenvalues of Γ are independent of the sequence $(v_1, \ldots, v_K)$, and there will be no harm in referencing A as *the* adjacency matrix of Γ.

In [10], it was shown that if a ring R is finite and not isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$, then R is a *Boolean ring* (that is, $R \simeq \mathbb{Z}_2^k$ for some $k \in \mathbb{Z}^+$) if and only if the spectrum of the adjacency matrix of $\Gamma(R)$ is such that λ is an eigenvalue of multiplicity m if and only if either $1/\lambda$ or $-1/\lambda$ is an eigenvalue of multiplicity m. More general graphs which satisfy these *reciprocal eigenvalue properties* were studied in [3], [4], and [5]. Note that the zero-divisor graphs of finite Boolean rings can be regarded as generalizations of the well known *Kneser graphs*, whose vertices are the j-element subsets of $\{1, \ldots, k\}$ for some fixed integer $1 \le j \le k/2$, and two vertices are adjacent if and only if their intersection is empty (e.g., the 2-element subsets of $\{1, \ldots, 5\}$ yield the famous *Peterson graph*). The eigenvalues of the Kneser graphs are computed in [9, Theorem 9.4.3]. In the investigation that follows, the numerical values of λ are sought for the zero-divisor graphs of finite Boolean rings, and are linked to the eigenvalues of certain matrices of binomial coefficients. Throughout, all matrices are real and symmetric. In particular, all eigenvalues are real, and can be associated with real eigenvectors.

2. The matrices of binomial coefficients

Let $1 < k \in \mathbb{Z}^+$, and set $\Gamma = \Gamma(\mathbb{Z}_2^k)$. For all integers $i, j \in \{1, \ldots, k-1\}$, define

$$P_{i,j} = \begin{cases} \left(\begin{array}{c} i \\ k-j \end{array} \right), & \text{if } i+j \ge k \\ 0, & \text{if } i+j < k \end{cases}$$

and

$$Q_{i,j} = \begin{cases} \left(\begin{array}{c} i-1 \\ k-j-1 \end{array} \right), & \text{if } i+j \ge k \\ 0, & \text{if } i+j < k \end{cases}.$$

Consider the $(k-1) \times (k-1)$ matrices $P = [P_{i,j}]$ and $Q = [Q_{i,j}]$. Of course, producing P and Q amounts to the construction of the first k rows of Pascal's triangle. For example, if $k = 4$ then

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Let φ denote the *golden ratio* $1/2 + 1/2\sqrt{5}$ and let $\xi = -\varphi^{-1}$. While it appears that the problem of finding all eigenvalues of P has not been solved, it was shown in [6] that the eigenvalues of Q are precisely the real numbers φ^{k-2}, $\varphi^{k-3}\xi$, $\varphi^{k-4}\xi^2, \ldots, \varphi\xi^{k-3}$, and ξ^{k-2}. More recently, this result was generalized in [7], and the eigenvectors of Q were later computed in [8].

In this paper, it is shown that if λ is any eigenvalue of Q, then $-\lambda$ is an eigenvalue of Γ (Theorem 3.4). Moreover, it is proved that every eigenvalue of P is an eigenvalue of Γ (Theorem 3.1). In [11], it is shown that these eigenvalues make up the complete set of eigenvalues of Γ. Since the eigenvalues of Q are known, the problem of finding all eigenvalues of Γ (that is, the eigenvalues of the $(2^k - 2) \times (2^k - 2)$ adjacency matrix of Γ) is equivalent to the problem of finding all eigenvalues of the (much smaller) matrix P.

3. The spectrum of Γ

Fix an integer $k > 1$, and let A denote the adjacency matrix of the zero-divisor graph $\Gamma := \Gamma(\mathbb{Z}_2^k)$. Let $V(\Gamma) = \{v_1, \ldots, v_K\}$ (so $K = 2^k - 2$). Given any integer $1 \le l \le K$, define $\mathbf{r}_l \in \{0,1\}^K$ to be the l-row of A. Thus $\mathbf{r}_l(j) = 1$ if and only if $v_l \in N(v_j)$.

For each integer $1 \le j \le k-1$, define D_j to be the set of all elements in $\mathbb{Z}_2^k$ having precisely j coordinates that are equal to zero. From a graph-theoretic perspective, D_j is the set of all $v \in V(\Gamma)$ such that $|N(v)| = 2^j - 1$. Clearly the sets $D_1, \ldots, D_{k-1}$ partition $V(\Gamma)$.

The following theorem shows that each eigenvalue of P is an eigenvalue of A. Furthermore, it is shown that the corresponding eigenvectors of A are easily obtained from those of P. For the remainder of this paper, the usual inner product of any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ will be given by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{m=1}^n \mathbf{x}(m)\mathbf{y}(m)$.

Theorem 3.1. *Let $\mathbf{u} \in \mathbb{R}^{k-1}$ be any nonzero vector with $P\mathbf{u} = \lambda\mathbf{u}$. Define $\mathbf{v} \in \mathbb{R}^K$ by $\mathbf{v}(m) = \mathbf{u}(j)$ if and only if $v_m \in D_j$. Then $A\mathbf{v} = \lambda\mathbf{v}$.*

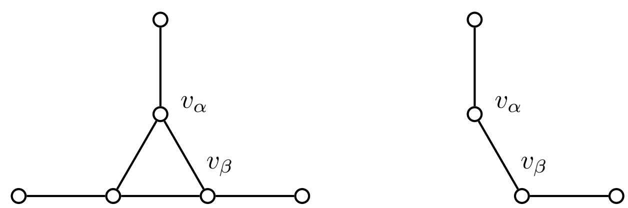
Proof. Fix an integer $1 \le l \le K$. Say $v_l \in D_i$, and let $k - i \le j \le k-1$. By counting the combinations of nonzero coordinates that yield vertices adjacent to v_l, it follows that v_l is adjacent to precisely $\binom{i}{k-j}$ vertices in D_j. If $j < k - i$, then v_l is not adjacent to any elements of D_j. Thus

$$\langle \mathbf{r}_l, \mathbf{v} \rangle = \sum_{j=k-i}^{k-1} \binom{i}{k-j} \mathbf{u}(j) = \sum_{j=k-i}^{k-1} P_{i,j}\mathbf{u}(j) = \lambda\mathbf{u}(i) = \lambda\mathbf{v}(l).$$

Therefore, $A\mathbf{v} = \lambda\mathbf{v}$. $\square$

In the next lemma, it will be shown that there are subgraphs of Γ whose spectra include λ and $-\lambda$ for every eigenvalue λ of Q. To construct such a subgraph, fix two distinct vertices $v_\alpha, v_\beta \in D_{k-1}$. Let $V' = \big(N(v_\alpha) \setminus N(v_\beta)\big) \cup \big(N(v_\beta) \setminus N(v_\alpha)\big)$. Define Γ' to be the subgraph of Γ induced by the vertices in V' (see Figure 1).

Set $N = |V'|$; say $V' = \{v_{\gamma_1}, \ldots, v_{\gamma_N}\}$. Notice that the adjacency matrix A' of Γ' can be obtained by deleting the i-row and i-column of A if and only if either $v_\alpha, v_\beta \in N(v_i)$ or $v_\alpha, v_\beta \notin N(v_i)$. Let $\mathbf{r}'_1, \ldots, \mathbf{r}'_N \in \{0,1\}^N$ be the rows of A'. Thus $\mathbf{r}'_l(m) = 1$ if and only if $v_{\gamma_l} \in N(v_{\gamma_m})$.



(a) Γ (b) Γ'

FIGURE 1. The graphs Γ and Γ' for the two fixed vertices v_α and v_β when $k = 3$.

Lemma 3.2. *If λ is an eigenvalue of Q, then λ and $-\lambda$ are eigenvalues of Γ'.*

Proof. Suppose that $\mathbf{u} \in \mathbb{R}^{k-1}$ is any nonzero vector with $Q\mathbf{u} = \lambda\mathbf{u}$. Define $\mathbf{v} \in \mathbb{R}^N$ by $\mathbf{v}(m) = \mathbf{u}(j)$ if and only if $v_{\gamma_m} \in D_j$. Fix a $v_{\gamma_l} \in D_i$, and l et $k - i \leq j \leq k-1$. By counting the combinations of nonzero coordinates that yield vertices adjacent to v_{γ_l}, it follows that v_{γ_l} is adjacent to precisely $\binom{i-1}{k-j-1}$ vertices $v \in V'$ such that $v \in D_j$. If $j < k - i$, then v_{γ_l} is not adjacent to any vertices $v \in V'$ with $v \in D_j$. Therefore,

$$\langle \mathbf{r}'_l, \mathbf{v} \rangle = \sum_{j=k-i}^{k-1} \binom{i-1}{k-j-1} \mathbf{u}(j) = \sum_{j=k-i}^{k-1} Q_{i,j}\mathbf{u}(j) = \lambda\mathbf{u}(i) = \lambda\mathbf{v}(l).$$

So $A'\mathbf{v} = \lambda\mathbf{v}$.

To show that $-\lambda$ is an eigenvalue of A', let $\mathbf{w} \in \mathbb{R}^N$ be the vector defined by

$$\mathbf{w}(l) = \begin{cases} \mathbf{v}(l), & \text{if } v_{\gamma_l} \in N(v_\alpha) \\ -\mathbf{v}(l), & \text{if } v_{\gamma_l} \in N(v_\beta) \end{cases}.$$

Suppose that $v_{\gamma_l} \in N(v_\alpha)$. So $\mathbf{w}(l) = \mathbf{v}(l)$. Also, in the ring $\mathbb{Z}_2^k$, it follows that $v_{\gamma_l}v_\beta = v_\beta$. Thus, if $v_{\gamma_m} \in N(v_{\gamma_l})$, then $v_{\gamma_m}v_\beta = v_{\gamma_m}v_{\gamma_l}v_\beta = 0$. Hence, if

$v_{\gamma_m} \in N(v_{\gamma_l})$ then $\mathbf{w}(m) = -\mathbf{v}(m)$. Therefore, $\langle \mathbf{r}'_l, \mathbf{w}\rangle = -\langle \mathbf{r}'_l, \mathbf{v}\rangle = -(\lambda \mathbf{v}(l)) = (-\lambda)\mathbf{w}(l)$. Similarly, if $v_{\gamma_l} \in N(v_\beta)$ then $\mathbf{w}(l) = -\mathbf{v}(l)$, and $\mathbf{w}(m) = \mathbf{v}(m)$ whenever $v_{\gamma_m} \in N(v_{\gamma_l})$. In this case, $\langle \mathbf{r}'_l, \mathbf{w}\rangle = \langle \mathbf{r}'_l, \mathbf{v}\rangle = \lambda\mathbf{v}(l) = \lambda(-\mathbf{w}(l)) = (-\lambda)\mathbf{w}(l)$. Hence $A'\mathbf{w} = (-\lambda)\mathbf{w}$. □

Remark 3.3. There are well known theorems that support the ideas in the above results. However, proofs of Theorem 3.1 and Lemma 3.2 that employ these known theorems are less enlightening, and the simplicity that might be gained from any such application is at best marginal. On the other hand, it is intriguing to observe how the above results are connected to existing theory.

In the language of *equitable partitions* (cf. [9, Section 9.3]), it follows from the proof of Theorem 3.1 that P is the *adjacency matrix of the quotient of* Γ over the partition $\{D_j\}_{j=1}^{k-1}$ of $V(\Gamma)$. Furthermore, the proof of Lemma 3.2 shows that Q is the adjacency matrix of the quotient of Γ' over the partition $\{D_j \cap V'\}_{j=1}^{k-1}$ of V'. By [9, Theorem 9.3.3], it follows that every eigenvalue of P is an eigenvalue of Γ, and that each eigenvalue of Q is an eigenvalue of Γ'. Moreover, it is not difficult to show that Γ' is bipartite with bipartition given by $\{N(v_\alpha) \setminus N(v_\beta), N(v_\beta) \setminus N(v_\alpha)\}$, and thus [9, Theorem 8.8.2] implies that λ is an eigenvalue of Γ' if and only if $-\lambda$ is an eigenvalue of Γ'.

Theorem 3.4. *Suppose that* $\mathbf{u} \in \mathbb{R}^{k-1}$ *is any nonzero vector with* $Q\mathbf{u} = \lambda\mathbf{u}$*, and define* $\mathbf{v}$ *and* $\mathbf{w}$ *as in the proof of Lemma 3.2. If* $\mathbf{x} \in \mathbb{R}^K$ *is defined such that*

$$\mathbf{x}(m) = \begin{cases} \mathbf{w}(j), & \text{if } v_m = v_{\gamma_j} \in V' \\ 0, & \text{otherwise} \end{cases},$$

then $A\mathbf{x} = (-\lambda)\mathbf{x}$.

Proof. Let $v_i \in V$.

Case 1. Let $v_i \in V'$; say $v_i = v_{\gamma_l}$. Then it is straightforward to check that $\langle \mathbf{r}_i, \mathbf{x}\rangle = \langle \mathbf{r}'_l, \mathbf{w}\rangle = (-\lambda)\mathbf{w}(l) = (-\lambda)\mathbf{x}(i)$, where the second equality holds as in the proof of Lemma 3.2.

Case 2. Suppose that $v_i \notin N(v_\alpha) \cup N(v_\beta)$, i.e., v_i is not adjacent to either v_α or v_β. Then $\mathbf{x}(i) = 0$. If $v_m \in V'$, then either $v_\alpha v_m$ or $v_\beta v_m$ is a nonzero element in the ring $\mathbb{Z}_2^k$. Without loss of generality, assume that $v_\alpha v_m$ is nonzero. Clearly $v_i v_\alpha = v_\alpha$, and thus $v_i v_\alpha v_m = v_\alpha v_m$ is nonzero. In particular, $v_i v_m$ is nonzero. Hence $\mathbf{r}_i(m) = 0$ whenever $v_m \in V'$. Therefore, $\langle \mathbf{r}_i, \mathbf{x}\rangle = 0 = (-\lambda)\mathbf{x}(i)$.

Case 3. Suppose that $v_i \in N(v_\alpha) \cap N(v_\beta)$, i.e., v_i is adjacent to both v_α and v_β. Assume that $v_\alpha(s) = v_\beta(t) = 1$, and let $\mathcal{V}$ be the collection of all two-element subsets $\{v, w\}$ of $V' \subseteq \mathbb{Z}_2^k$ such that $v(j) = w(j)$ if and only if $j \in \{1, \ldots, k\} \setminus \{s, t\}$.

Clearly the elements of $\mathcal{V}$ partition V'. Also, if $\{v, w\} \in \mathcal{V}$, then $v \in N(v_i)$ if and only if $w \in N(v_i)$ (indeed, $v_i(s) = v_i(t) = 0$ since $v_i \in N(v_\alpha) \cap N(v_\beta)$).

Let $v_{\gamma_l} \in V'$ such that $v_{\gamma_l} \in N(v_i)$; say $\{v_{\gamma_l}, v_{\gamma_m}\} \in \mathcal{V}$ and $v_{\gamma_l} \in N(v_\alpha)$ (so $v_{\gamma_m} \in N(v_\beta)$). Thus $\mathbf{r}_i(\gamma_l) = \mathbf{r}_i(\gamma_m) = 1$. Also, v_{γ_l} and v_{γ_m} have the same number of coordinates that are equal to zero, and thus $\mathbf{v}(l) = \mathbf{v}(m)$. Therefore, the equalities $\mathbf{w}(l) = \mathbf{v}(l)$ and $\mathbf{w}(m) = -\mathbf{v}(m)$ imply that $\mathbf{x}(\gamma_l) = -\mathbf{x}(\gamma_m)$. Thus $\mathbf{r}_i(\gamma_l)\mathbf{x}(\gamma_l) + \mathbf{r}_i(\gamma_m)\mathbf{x}(\gamma_m) = 0$. But $\mathbf{x}(i) = 0$ since $v_i \notin V'$, and it follows that

$$\langle \mathbf{r}_i, \mathbf{x} \rangle = \sum_{\{v_{\gamma_l}, v_{\gamma_m}\} \in \mathcal{V}} \big(\mathbf{r}_i(\gamma_l)\mathbf{x}(\gamma_l) + \mathbf{r}_i(\gamma_m)\mathbf{x}(\gamma_m)\big) = 0 = (-\lambda)\mathbf{x}(i).$$

Therefore, $A\mathbf{x} = (-\lambda)\mathbf{x}$. □

It is proved in [11] that every eigenvalue of Γ is either an eigenvalue of P or the negative of an eigenvalue of Q. By the above results, since the eigenvalues of Q are known (see [6]), the problem of finding all eigenvalues of A is equivalent to the problem of finding the eigenvalues of the (much smaller) matrix P.

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