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ANALOGUES OF THE FRATTINI SUBALGEBRA

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Abstract. For a Lie algebra, L, the Frattini subalgebra $F(L)$ is the intersection of all maximal subalgebras of L. We develop two analogues of the Frattini subalgebra, namely $nFrat(L)$ and $R(L)$. Specifically, we develop properties involving non-generators, containment relations, and nilpotency.

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1. Introduction

Giovanni Frattini introduced his subgroup, now called the Frattini subgroup, in the nineteenth century. A good reference is [14]. The Lie algebra analogue, the Frattini subalgebra, appeared in [13] and early contributions to the theory can be found in [2] - [4] and [18] - [23] and recent work is in [15] and [24]. Infinite dimensional results can be found in [1], [10], and [16]. The generalization to non-associative algebras was begun in [22] with particular non-associative algebras investigated in [6], [7], [8], [12], [17], [20], [25], and [26], among others. Meanwhile, the group concept has been generalized in various ways; for example, see [5] and [11].

The present paper contains Lie algebra analogues to the results in [11]. We find characterizations of the concepts by non-generators, find to what extent our concepts are nilpotent, find containments relations and a characterization of nilpotency. All Lie algebras consider here are finite dimensional over a field, F.

Definition 1.1. For a Lie algebra L we define the following:

(1) $\mathcal{M} = \{M \mid M \text{ is a maximal subalgebra of } L\}$
(2) $\mathcal{N} = \{N \mid N \neq L, N \text{ is a maximal ideal of } L\}$
(3) $\mathcal{R} = \{R \mid R \text{ is in } \mathcal{M} \cap \mathcal{N}\}$ = all maximal subalgebras that are ideals

Lemma 1.2. *Let N be an ideal of L.*

(1) $N \in \mathcal{N}$ *if and only if* L/N *is simple.*
(2) $N \in \mathcal{R}$ *if and only if* $dim(L/N) = 1$.

Definition 1.3. For a Lie algebra L we define the following subalgebras:

(1) $nFrat(L) = \bigcap_{N \in \mathcal{N}} N$
(2) $R(L) = \bigcap_{N \in \mathcal{R}} N$ if $\mathcal{R} \neq \emptyset$ and $R(L) = L$ if $\mathcal{R} = \emptyset$

2. Characterization by non-generators

It is true in group theory that Frat(G) is the set of non-generators. It is also known that this concept carries over for $nFrat(G)$ and $R(G)$ [11]. We are going to study the idea of non-generators for these concepts in Lie algebras. It is widely recognized that the Frattini subalgebra is equal to the set of non-generators. We will provide this proof as it fits with the rest.

Definition 2.1. A subset S of a Lie algebra L is *normal* in L if $ad_x(S) \subseteq S$ for all $x \in L$.

Definition 2.2. An element $x \in L$ is a *normal non-generator* if $L = \langle x, T \rangle$ for a normal subset T in L implies $L = \langle T \rangle$.

Proposition 2.3. *$F(L)$ equals the set of non-generators.*

Proof. Let $x \in F(L)$. Let $L = \langle H, x \rangle$. If $H \neq L$, then $H \subseteq M$, where M is a maximal subalgebra. Also, $x \in F(L) \subseteq M$. Hence $\langle H, x \rangle = M$, a contradiction. So $H = L$ and x is a non-generator.

Now suppose $x \notin F(L)$. Let M be a maximal subalgebra such that $x \notin M$. Then $M \subset \langle x, M \rangle \subseteq L$, so $\langle x, M \rangle = L$. But $M \neq L$, so x is a generator for L. □

Theorem 2.4. *$R(L)$ equals the set of all normal non-gernators of L.*

Proof. Suppose $x \notin R(L)$. Then $x \notin M$, a maximal subalgebra that is an ideal. M is a normal subset of L and $L = \langle x, M \rangle$ but $L \neq M$. Hence x is a normal generator.

Now suppose $x \in R(L)$ and $L = \langle x, S \rangle$ for a normal subset S of L. If $\langle S \rangle \neq L$ then $\dim(L) = \dim(S) + 1$ so ,$\langle S \rangle \in \mathcal{R}$. But $x \in R(L) \subseteq \langle S \rangle$ so $\langle S \rangle = L$, a contradiction. □

Definition 2.5. Let X be a subset of L. $X^L = \langle [x, l_1, \ldots, l_k] \rangle$ where $x \in X$ and $l_i \in L$ and $k = 0, 1, \ldots$.

Definition 2.6. An element $x \in L$ is called an *n-nongenerator* of L if for every subset X of L, $L = X^L$ whenever $L = \langle x, X \rangle^L$.

Lemma 2.7. *For any element $g \in L$ any subset X of L, $\langle g, X \rangle^L = \langle g^L, X^L \rangle = g^L + X^L$.*

Proof. Both g^L and X^L are contained in $\langle g, X\rangle^L$, so $\langle g^L, X^L\rangle \subseteq \langle g, X\rangle^L$ and $g^L + X^L \subseteq \langle g, X\rangle^L$. Since, $\langle g, X\rangle \subseteq \langle g^L, X^L\rangle$ this implies $\langle g, X\rangle^L \subseteq \langle g^L, X^L\rangle$. Also, $g^L \subseteq g^L + X^L$ and $X^L \subseteq g^L + X^L$, thus $\langle g^L, X^L\rangle \subseteq g^L + X^L$.

Alternate Proof: Let g and h be in L and X a subset of L with $x \in X$.

$$x^L = \sum a_i[x, h_{i_1}, h_{i_2}, \ldots, h_{i_k}]$$

$$g^L = \sum b_i[g, h_{i_1}, h_{i_2}, \ldots, h_{i_k}]$$

$$\begin{aligned} \langle g, x\rangle^L &= [\alpha g + \beta x, h_1, \ldots, h_n] \\ &= [\alpha g, h_1, \ldots, h_n] + [\beta x, h_1, \ldots, h_n] \\ &\in g^L + X^L \text{ and } \in \langle g^L, X^L\rangle \end{aligned}$$

So $\langle g, X\rangle^L \subseteq g^L + X^L$ and $\langle g, X\rangle^L \subseteq \langle g^L, X^L\rangle$. Since $g \in \langle h, X\rangle^L$ then $g^L \in \langle g, X\rangle^L$ and similarly for X and X^L. So $g^L + X^L \subseteq \langle g, X\rangle^L$. Since $g^L + X^L$ is a vector space, $\langle g^L, X^L\rangle \subseteq g^L + X^L$. Thus we get

$$\langle g, X\rangle^L \subseteq \langle g^L, X^L\rangle \subseteq g^L + X^L \subseteq \langle g, X\rangle^L$$

and the theorem holds. □

Theorem 2.8. *For a Lie algebra L, $nFrat(L)$ is the set of n-nongenerators of L.*

Proof. Let $T = \{x \,|\, x \text{ is a } n\text{-nongenerator of } L\}$. Since L is finite dimensional, there exists maximal ideals so $nFrat(L) \neq L$. Suppose $x \in T$ and $x \notin nFrat(L)$. There exists $N \in \mathcal{N}$ such that $x \notin N$. Now either $x^L + N = N$ or $x^L + N = L$. If $x^L + N = N$ then $x \in N$. But $x \notin N$ so $x^L + N \neq N$. Thus $x^L + N = L$. This implies $\langle x, N\rangle^L = L$, so $N = N^L = L$ since x is an n-nongenerator. But $N \neq L$ so this contradiction establishes $x \in N$ for all $N \in \mathcal{N}$ and $x \in nFrat(L)$. Thus $T \subseteq nFrat(L)$.

Conversely, let $x \in nFrat(L)$ and suppose x is not an n-nongenerator. Thus there exists $S \subseteq L$ such that $L = \langle x, S\rangle^L$, but $L \neq S^L$. Hence S^L is a proper ideal of L and $x \notin S^L$. By lemma 2.7 $L = \langle x, S\rangle^L = x^L + S^L$. Let M be maximal with respect to the properties for S^L: $x \notin M, M \lhd L, S^L \subseteq M, L = x^L + M$.

We claim $M \in \mathcal{N}$. If not, there exists N such that $M \subsetneq N \subsetneq L$, $N \trianglelefteq L$. Then $L = x^L + M = x^L + N$. If $x \notin N$, then N can replace M in the condition above the claim, which is a contradiction. Thus $x \in N$ and $x^L \subseteq N$. Hence $L = N$, a contradiction. Hence $M \in \mathcal{N}$, but $x \notin M$ so $x \notin nFrat(L)$, a contradiction. Hence whenever $L = \langle x, S\rangle^L$ implies $L = S^L$ and x is an n-nongenerator. Thus $nFrat(L) \subseteq T$. □

3. Nilpotency and containments

In this section, we collect information about the Frattini subalgebra and its generalizations which are inspired by the results in group theory. We begin by reviewing known properties of the Frattini subalgebra or Frattini ideal.

The following theorem is proven in [19].

Theorem 3.1. *Let L be a Lie algebra and N an ideal of L.*

(1) $F(L) + N/N \subseteq F(L/N)$.
(2) *If* $N \subseteq F(L)$*, then* $F(L)/N = F(L/N)$.

We have similar results for $nFrat(L)$.

Theorem 3.2. *Let L be a Lie algebra and N an ideal of L. Then*

(1) $(nFrat(L) + N)/N \subseteq nFrat(L/N)$*;*
(2) *If* $N \subseteq nFrat(L)$*, then* $nFrat(L)/N = nFrat(L/N)$.

Proof. (1) For each M with $M/N \in \mathcal{N}(L/N)$, we have $M \in \mathcal{N}(L)$. Thus $nFrat(L) \subseteq \bigcap_{M/N\in\mathcal{N}(L/N)} M$ and $(nFrat(L) + N)/N \subseteq nFrat(L/N)$.

(2) Since $N \subseteq nFrat(L)$, $N \subseteq M$ for all $M \in \mathcal{N}(L)$. Also, $M/N \in \mathcal{N}(L/N)$ if and only if $M \in \mathcal{N}(L)$. Then, $nFrat(L)/N = (\bigcap_{M\in\mathcal{N}(L)} M)/N = \bigcap_{M\in\mathcal{N}(L)} M/N = nFrat(L/N)$. □

In group theory, if G is finite then $Frat(G)$ nilpotent. In Lie algebras, it is known that the Frattini ideal is nilpotent.

Theorem 3.3. *Let L be a Lie algebra, then $\phi(L)$ is nilpotent.*

Proof. Let $x \in \phi(L)$. Then $L_1(x) \subseteq \phi(L)$ since $\phi(L) \trianglelefteq L$. Thus $L_0(x) + \phi(L) = L$ and $L_0(x) + F(L) = L$. Thus $L_0(x) = L$ and x is nilpotent. Hence $\phi(L)$ is nilpotent. □

As in group theory, it is not always true that $nFrat(L)$ or $R(L)$ are nilpotent.

Example 3.4. *Let $L = gl(n, F)$. If $char(F) = 0$, then there are two maximal ideals L' and $Z(L)$. Then $L' \cap Z(L) = 0$. However, if $char(F) = p$, where $p \neq 2$ and $p \mid n$ then , $Z(L) \subseteq L' = sl(n, F)$. Thus the only maximal ideal is $sl(n, F)$ and so $nFrat(L) = sl(n, F)$ and therefore, $nFrat(L)$ is not nilpotent. In this example, $R(L) = nFrat(L) = sl(n, F)$. Thus $R(L)$ is also not nilpotent.*

It is known in group theory that if N is a normal subgroup in G then $Frat(N) \subseteq Frat(G)$. This result does not hold in complete generality for Lie algebras. However, it does hold in the following case.

Theorem 3.5. *Let N be an ideal of L over a field F, then if char$(F) = 0$ or if L is solvable, then $F(N) \subseteq F(L)$.*

Proof. If char$(F) = 0$, then $F(N)$ is an ideal in L by [22]. If L is solvable, then $F(N)$ is an ideal in L by [4]. Thus $F(N) \subseteq F(L)$. See Theorem 3.7. □

If L is over a field of characteristic$p$ it is possible for $F(N) \nsubseteq F(L)$ when N is an ideal in L. The following is an example based off an similar example given by Jacobson in [9].

Example 3.6. *Let A be a vector space over a field of characteristic p and let $\{x_0, x_1, \ldots, x_{p-1}\}$ be a basis for A. Define linear transformations R and S of A by*

$$R(x_i) = x_{i+1},\ i < p-1$$
$$R(x_{p-1}) = x_0$$
$$S(x_i) = ix_{i-1},\ i(mod\ p) > 0$$
$$S(x_0) = 0$$

Then $[R, S] = I$ where I is the identity linear transformation. Let B be the three dimensional Lie algebra with basis $\{R, S, I\}$. B is a Heisenberg Lie algebra. Let L be the semi direct sum of A and B with multiplication given by $[b, a] = b(a)$ for $b \in B$, $a \in A$. Let $K = A + \langle S\rangle$. K is nilpotent since S is a nilpotent linear transformation. Hence $F(K) = K^2$ and K is not an ideal in L. Also $F(K) = K^2 = \langle x_0, x_1, \ldots, x_{p-2}\rangle$ is not an ideal in L. Now let $H = A + \langle S, I\rangle$. H is not nilpotent since $[I, x_0] = x_0$. H is solvable and K is an ideal in H. $F(K) = K^2$ is also an ideal in H. Hence $F(K) \subseteq F(H)$. Also $F(H) \subsetneq H^2$ and $dimH^2 = dimK^2 + 1$. Thus $F(K) = K^2 \subseteq F(H) \subsetneq H^2$ yields that $F(K) = F(H)$. Therefore, $F(H)$ is not an ideal in L even though H is ideal in L. Furthermore, $F(L) = 0$ as we now will show. B acts irreducibly on A, hence B is maximal in L. Thus $F(L) \subseteq B$. Since L is solvable, $F(L)$ is an ideal of L and therefore, $[F(L), A] \subseteq F(L) \cap A \subseteq B \cap A = 0$. Since $F(L)$ consists of linear transformations of A, $F(L) = 0$. Hence $F(H) \subsetneq F(L)$ and $F(L) = 0$ does not imply that $F(H) = 0$ even though H is an ideal in L.

Under certain conditions in Lie algebras, we get that $F(N) \subseteq F(L)$. This property also carries over to the Frattini ideal, $nFrat(L)$, and $R(L)$.

Theorem 3.7. *If $N \subseteq L$ and $F(N)$ is an ideal of L, then $F(N) \subseteq F(L)$.*

Proof. Suppose not. Then there exists a maximal subalgebra M in L such that $F(N) \nsubseteq M$. Then $F(N) + M = L$. So $(F(N) + M) \cap N = F(N) + (N \cap M) = N$.

Thus $N \cap M = N$ which implies that $F(N) \subset N \subseteq M$ which is a contradiction. Therefore, $F(N) \subseteq F(L)$. $\square$

Theorem 3.8. *If $N \subseteq L$ and $\phi(N)$ is an ideal of L, then $\phi(N) \subseteq \phi(L)$.*

Proof. Suppose not. $\phi(N)+\phi(L)$ is an ideal in L. So if $\phi(N)+\phi(L) \subseteq F(L)$ then $\phi(N) \subseteq \phi(L)$ which is a contradiction. So there exists a maximal subalgebra M of L such that $\phi(N) \nsubseteq M$. Then $\phi(N)+M = L$. This implies that $\phi(N)+(N\cap M) = N$ as above. This implies that $N \cap M = N$. Thus $N \subseteq M$ and hence $\phi(N) \subseteq M$ which is a contradiction. Therefore, $\phi(N) \subseteq \phi(L)$. $\square$

Theorem 3.9. *If $N \subseteq L$ and $nFrat(N)$ is an ideal of L, then $nFrat(N) \subseteq nFrat(L)$.*

Proof. Suppose not. Then there exists a maximal ideal M of L such that $nFrat(N) \nsubseteq M$. Then $nFrat(N) + M = L$. Then $nFrat(N) + (M \cap N) = N$ which implies $M \cap N = N$. Thus $nFrat(N) \subseteq N \subseteq M$ which is a contradiction. Therefore, $nFrat(N) \subseteq nFrat(L)$. $\square$

Theorem 3.10. *If $N \subseteq L$ and $R(N)$ is an ideal of L, then $R(N) \subseteq R(L)$.*

Proof. Suppose not. Then there exists a maximal ideal M of L such that $R(N) \nsubseteq M$. Then $R(N)+M = L$. Then $R(N)+(M\cap N) = N$ which implies $M\cap N = N$. Thus $R(N) \subseteq N \subseteq M$ which is a contradiction. Therefore, $R(N) \subseteq R(L)$. $\square$

Here we consider what happens when L is the direct sum of ideals. Unlike in group theory, we do not get equality for the Frattini subalgebra. However, we do get equality for the Frattini ideal, $nFrat(L)$, and $R(L)$. The proofs of the first two are shown by Towers in [22].

Lemma 3.11. *If $L = L_1 \oplus \cdots \oplus L_n$, then $F(L) = F(L_1) \oplus \cdots \oplus F(L_n)$.*

Theorem 3.12. *If $L = L_1 \oplus \cdots \oplus L_n$, then $\phi(L) = \phi(L_1) \oplus \cdots \oplus \phi(L_n)$.*

Theorem 3.13. *If $L = L_1 \oplus \cdots \oplus L_n$, then $nFrat(L) = nFrat(L_1) \oplus \cdots \oplus nFrat(L_n)$.*

Proof. Since $L = L_1 \oplus \cdots \oplus L_n$, $nFrat(L_i) \subsetneq L_i$ for each i. Let $M_j \in \mathcal{N}(L_j)$. Then we have $L_1 \oplus \cdots \oplus L_{j-1} \oplus L_{j+1} \oplus \cdots \oplus L_n \oplus M_j \in \mathcal{N}(L)$. Thus, $nFrat(L) \subseteq L_1 \oplus\cdots\oplus L_{j-1} \oplus L_{j+1} \oplus\cdots\oplus L_n \oplus nFrat(M_j)$. Therefore, $nFrat(L) \subseteq nFrat(L_1) \oplus \cdots \oplus nFrat(L_n)$.

Now consider L_j. Since L_j is an ideal of L, $nFrat(L_j) \subseteq nFrat(L)$ by Theorem 3.9. Thus, $nFrat(L_1) \oplus \cdots \oplus nFrat(L_n) \subseteq nFrat(L)$. $\square$

Theorem 3.14. *If $L = L_1 \oplus \cdots \oplus L_n$, then $R(L) = R(L_1) \oplus \cdots \oplus R(L_n)$.*

Proof. The proof is similar to that of $nFrat(L)$. $\square$

4. Characterizations of nilpotency

In this section, relations between $\phi(L)$, $nFrat(L)$, and $R(L)$ are investigated. We also find a characterization of nilpotency in terms of the equality of these ideals.

Lemma 4.1. *In any Lie algebra L, $\phi(L) \subseteq R(L)$ and $nFrat(L) \subseteq R(L)$.*

Theorem 4.2. *$\phi(L) \subseteq nFrat(L)$.*

Proof. For each $N \in \mathcal{N}$, $\phi(L) + N \trianglelefteq L$. Thus $\phi(L) + N = N$ or $\phi(L) + N = L$. If $\phi(L) + N = L$ then $N = L$ as $\phi(L)$ cannot be supplemented, which is a contradiction. Thus $\phi(L) + N = N$, so $\phi(L) \subseteq N$ for all $N \in \mathcal{N}$. Therefore, $\phi(L) \subseteq nFrat(L)$. $\square$

Corollary 4.3. *For a Lie algebra L, $\phi(L) \subseteq nFrat(L) \subseteq R(L)$.*

The following is an example of Corollary 4.3.

Example 4.4. *Let L be the nonabelian two dimensional Lie algebra, $L = span\{x, y\}$ with $[x, y] = y$. Then y is the only maximal ideal of L. So $\phi(L) = 0$ and $nFrat(L) = R(L) = y$. Thus $\phi(L) \subset nFrat(L) \subseteq R(L)$.*

Lemma 4.5. *If L is a solvable Lie algebra, then $R(L) = nFrat(L)$.*

Proof. Let L be a solvable Lie algebra. Let N be a maximal ideal of L. Then $\dim(L/N) = 1$ for any $N \in \mathcal{N}$. This is true if and only if $N \in \mathcal{R}$ by Lemma 1.2. But then the set of all maximal ideals is equal to the set of all maximal subalgebras that are ideals, $\mathcal{N} = \mathcal{R}$. So $nFrat(L) = R(L)$. $\square$

Lemma 4.5 is not true if L is not solvable. The following is an example of a non-solvable Lie algebra with $R(L) \neq nFrat(L)$.

Example 4.6. *Let $L = sl(2, F)$. L is not solvable. Since the only maximal ideal is $\{0\}$ then $nFrat(L) = \{0\}$. However, L contains no maximal subalgebras that are ideals, so $\mathcal{R} = \emptyset$ which implies $R(L) = sl(2, F)$. Thus $R(L) \neq nFrat(L)$.*

Theorem 4.7. *Let L be a Lie algebra. Then L is nilpotent if and only if $\phi(L) = nFrat(L) = R(L)$.*

Proof. If L is nilpotent then L is solvable. Hence $\phi(L) = nFrat(L) = R(L)$.

So suppose $\phi(L) = nFrat(L) = R(L)$. Let M be a maximal ideal of L such that $M = \oplus M_i$. Consider the Lie algebra homomorphism $\Pi : L \longrightarrow \bigoplus(L/M_i)$, where $\Pi(x) = (x + M_1, x + M_2, \ldots, x + M_n)$ for $x \in L$. So each L/M_i is a 1-dimensional subalgebra of L and hence an abelian subalgebra. The $Ker\Pi = \cap M_i = R(L)$. But then $L/Ker\Pi$ is abelian since it is the direct sum of abelian subalgebras. Thus $L/Ker\Pi = L/R(L)$ which equals $L/\phi(L)$ is nilpotent and hence L is nilpotent. $\square$

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