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UNITS OF THE GROUP ALGEBRA $\mathbb{F}_{5^k}(C_5 \rtimes C_4)$

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Abstract. The Structure of the Unit Groups of the Group Algebra of the group $C_5 \rtimes C_4$ over any field of characteristic 5 is established.

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1. Introduction

Let RG denote the group ring of the group G over the ring R and $\mathcal{U}(RG)$ denote the unit group of RG. The homomorphism $\varepsilon : KG \longrightarrow K$ given by $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$ is called the augmentation mapping of KG. The normalized unit group of KG denoted by $V(KG)$ consists of all the invertible elements of KG of augmentation 1. It is a well known fact that $\mathcal{U}(KG) \cong \mathcal{U}(K) \times V(KG)$.

We are interested in establishing the structure of $\mathcal{U}(KG)$ when K is a field of characteristic p and G is a finite group of order ap^m where $(a, p) = 1$ and p is a prime. For further details on group rings, see [7].

The order of $\mathcal{U}(\mathbb{F}_{p^k} D_{2p^m})$ is determined in [2] and the structure of $\mathcal{U}(\mathbb{F}_{3^k} D_6)$ is established in [1] where $\mathbb{F}_{p^k}$ is the Galois field of p^k elements and D_6 is the dihedral group of order 6. It is shown in [8] that V_1 and $V_1/Z(V_1)$ are elementary abelian 3-groups where $V_1 = 1 + J(\mathbb{F}_{3^k} D_6)$, $J(\mathbb{F}_{3^k} D_6)$ is the Jacobson Radical of $\mathbb{F}_{3^k} D_6$ and $Z(V_1)$ is the center of V_1. Also it is shown that $\mathcal{U}(\mathbb{F}_{5^k} D_{10})/\mathcal{V}_2 \cong {C_{5^k-1}}^2$, $\mathcal{V}_2$ is nilpotent of class 4 and $Z(\mathcal{V}_2) \cong {C_5}^{3k}$ where $\mathcal{V}_2 = 1+J(\mathbb{F}_{5^k} D_{10})$ in [6]. Additionally in [3,4], the structures of $\mathcal{U}(\mathbb{F}_{5^k} D_{10})$ and $\mathcal{U}(\mathbb{F}_{5^k} D_{20})$ are established in terms of split extensions of cyclic groups. Our main result is (see Theorem 2.9):

Theorem. $V(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) \cong [(({C_5}^{10k} \rtimes {C_5}^{4k}) \rtimes {C_5}^{k}) \rtimes {C_5}^{k}] \rtimes ({C_{5^k-1}}^3)$ *where* $C_5 \rtimes C_4 \cong \langle x, y \mid x^4 = y^5 = 1, yxy = x\rangle$.

Denote by $\widehat{g} = \sum_{h\in\langle g\rangle} h \in RG$. The next two results can be found in [5].

Theorem 1.1. *Let F be an arbitrary field of characteristic $p > 0$, let G be a p-solvable group and c be the sum of all p-elements of G including 1, then*

$$J(FG) = lann(c)$$

where $J(FG)$ is the Jacobson radical of FG and $lann(c)$ is the left annihilator of c.

Theorem 1.2. *Let N be a normal subgroup of G such that G/N is p-solvable. If $|G/N| = np^a$ where $(n,p) = 1$, then*

$$J(FG)^{p^a} \subseteq FG \cdot J(FN) \subseteq J(FG)$$

where F is a field of characteristic $p > 0$. In particular, if G is p-solvable of order np^a where $(n,p) = 1$, then $J(FG)^{p^a} = 0$.

The next result can be found in [7].

Proposition 1.3. *Let G be an abelian group of order n and K a field such that the characteristic of the field doesn't divide n. If K contains a primitive root of unity of order n, then*

$$KG \cong \underbrace{K \oplus \cdots \oplus K}_{n-times}$$

2. Proof of Main Theorem

Let $G \cong C_5 \rtimes C_4$. Define the group epimorphism

$$\theta : \mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) \longrightarrow \mathcal{U}(\mathbb{F}_{5^k}C_4)$$

by

$$\sum_{i=1}^{5}(\alpha_i + \alpha_{i+5}x + \alpha_{i+10}x^2 + \alpha_{i+15}x^3)y^{i-1} \mapsto \sum_{i=1}^{5}\alpha_i + \alpha_{i+5}\overline{x} + \alpha_{i+10}\overline{x}^2 + \alpha_{i+15}\overline{x}^3$$

where $\overline{x}$ generates C_4. Define the group homomorphism

$$\psi : \mathcal{U}(\mathbb{F}_{5^k}C_4) \longrightarrow \mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4))$$

by

$$a + b\overline{x} + c\overline{x}^2 + d\overline{x}^3 \mapsto a + bx + cx^2 + dx^3$$

where $a, b, c, d \in \mathbb{F}_{5^k}$. Clearly $\theta \circ \psi = 1$, therefore $\mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4))$ is a split extension of $\mathcal{U}(\mathbb{F}_{5^k}C_4)$ by $ker(\theta)$.

Therefore $\mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) \cong H \rtimes \mathcal{U}(\mathbb{F}_{5^k}C_4)$ where $H \cong ker(\theta)$. Let

$$\alpha = \sum_{i=1}^{5}(\alpha_i + \alpha_{i+5}x + \alpha_{i+10}x^2 + \alpha_{i+15}x^3)y^{i-1} \in \mathbb{F}_{5^k}(C_5 \rtimes C_4),$$

then $\alpha \in H$ if and only if $\sum_{i=1}^{5}\alpha_i = 1$, $\sum_{j=1}^{5}\alpha_{j+5} = 0$, $\sum_{l=1}^{5}\alpha_{l+10} = 0$ and $\sum_{m=1}^{5}\alpha_{m+15} = 0$ where $\alpha_i \in \mathbb{F}_{5^k}$. Therefore $|H| = \left(5^{4k}\right)^4 = 5^{16k}$.

Lemma 2.1. *H has exponent 5.*

Proof. $C_5 \rtimes C_4$ is solvable and hence p-solvable. Clearly $|C_5 \rtimes C_4| = 4 \times 5$ and by Theorem 1.2, $J(\mathbb{F}_{5^k}(C_5 \rtimes C_4))^5 = 0$ and $1 + J(\mathbb{F}_{5^k}(C_5 \rtimes C_4))$ has exponent 5. Now by Theorem 1.1, $J(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) = \{\alpha \in \mathbb{F}_{5^k}(C_5 \rtimes C_4) \,|\, \alpha\widehat{y} = 0\}$. Let

$$\alpha = \sum_{i=1}^{5}(\alpha_i + \alpha_{i+5}x + \alpha_{i+10}x^2 + \alpha_{i+15}x^3)y^{i-1} \in \mathbb{F}_{5^k}(C_5 \rtimes C_4),$$

then

$$\alpha \in J(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) \Longleftrightarrow \sum_{i=1}^{5}\alpha_i = \sum_{j=1}^{5}\alpha_{j+5} = \sum_{l=1}^{5}\alpha_{l+10} = \sum_{m=1}^{5}\alpha_{m+15} = 0$$

where $\alpha_i \in \mathbb{F}_{5^k}$. Therefore $H \cong 1 + J(\mathbb{F}_{5^k}((C_5 \rtimes C_4)))$. □

Lemma 2.2. *Let V be the set of elements H of the form $1 + \sum_{i=0}^{4} isx^3y^i$ where $s \in \mathbb{F}_{5^k}$. Then $V \cong {C_5}^k$.*

Proof. Let $v_1 = 1 + \sum_{i=0}^{4} isx^3y^i \in V$ and $v_2 = 1 + \sum_{i=0}^{4} irx^3y^i \in V$ where $r, s \in \mathbb{F}_{5^k}$. Then $v_1v_2 = 1 + \sum_{i=0}^{4} i(s+r)x^3y^i \in V$. Therefore V is closed under multiplication. It can easily be shown that V is abelian. Therefore $V \cong {C_5}^k$. □

Lemma 2.3. $|N_H(V)| = 5^{14k}$ *where $N_H(V)$ is the normalizer of V in H.*

Proof. $N_H(V) = \{h \in H \,|\, V^h = V\}$. Let $v = 1 + \sum_{i=0}^{4} isx^3y^i \in V$ and

$$h = 1 - \sum_{i=1}^{4}(\alpha_i + \alpha_{i+4}x + \alpha_{i+8}x^2 + \alpha_{i+12}x^3) + \sum_{i=1}^{4}(\alpha_i + \alpha_{i+4}x + \alpha_{i+8}x^2 + \alpha_{i+12}x^3)y^i \in H$$

where $\alpha_i, s \in \mathbb{F}_{5^k}$, then

$$v^h = 1 + \sum_{i=0}^{4}[\gamma_1 x + (\gamma_2 + i)sx^3]y^i$$

where $\gamma_1 = 3s\left(\sum_{i=1}^{4} i\alpha_{i+8}\right)$, $\gamma_2 = 3s\left(\sum_{i=1}^{4} i\alpha_i\right)$ and $\alpha_i, s \in \mathbb{F}_{5^k}$. Then $h \in N_H(V)$ iff $\left(\sum_{i=1}^{4} i\alpha_{i+8}\right) = 0$ and $\left(\sum_{i=1}^{4} i\alpha_i\right) = 0$. Thus every element of $N_H(V)$ has the form:

$$1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+7}x^2) + (\alpha_i + \alpha_{i+7}x^2)y^i + i(\alpha_i + \alpha_{i+7}x^2)y^4] - \sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+10}x^3) + \sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+10}x^3)y^i$$

where $\alpha_i \in \mathbb{F}_{5^k}$. Therefore $|N_H(V)| = 5^{14k}$. □

Let S_1 be the subset of H consisting of elements of the form

$$1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+5}x^2) + (\alpha_i + \alpha_{i+5}x^2)y^i + i(\alpha_i + \alpha_{i+5}x^2)y^4] + 3(\alpha_4 x + \alpha_5 x + \alpha_9 x^3 + \alpha_{10}x^3) + (\alpha_4 x + \alpha_9 x^3)(y + y^4) + (\alpha_5 x + \alpha_{10}x^3)(y^2 + y^3)$$

where $\alpha_i \in \mathbb{F}_{5^k}$. It can be shown that S_1 is a group and $S_1 \cong {C_5}^{10k}$. Also let S_2 be the subset of H consisting of elements of the form

$$1 + 3(\alpha_1 + \alpha_2 + \alpha_5 x^2 + \alpha_6 x^2) + (\alpha_1 + \alpha_5 x^2)(y + y^4) + (\alpha_2 + \alpha_6 x^2)(y^2 + y^3) + (\alpha_3 x + \alpha_7 x^3)(1 - y^4) + (\alpha_4 x + \alpha_8 x^3)(y - y^3)$$

where $\alpha_i \in \mathbb{F}_{5^k}$. Again, it can be shown that S_2 is a group and $S_2 \cong {C_5}^{8k}$.

Lemma 2.4. $N_H(V) \cong {C_5}^{10k} \rtimes {C_5}^{4k}$.

Proof. Clearly $S_1 < N_H(V)$ and $S_2 < N_H(V)$. Let

$$s_1 = 1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+5}x^2) + (\alpha_i + \alpha_{i+5}x^2)y^i + i(\alpha_i + \alpha_{i+5}x^2)y^4] + 3(\alpha_4 x + \alpha_5 x + \alpha_9 x^3 + \alpha_{10}x^3) + (\alpha_4 x + \alpha_9 x^3)(y + y^4) + (\alpha_5 x + \alpha_{10}x^3)(y^2 + y^3) \in S_1$$

and

$$s_2 = 1 + 3(\beta_1 + \beta_2 + \beta_5 x^2 + \beta_6 x^2) + (\beta_1 + \beta_5 x^2)(y + y^4) + (\beta_2 + \beta_6 x^2)(y^2 + y^3) + (\alpha_3 x + \alpha_7 x^3)(1 - y^4) + (\alpha_4 x + \alpha_8 x^3)(y - y^3) \in S_2$$

where $\alpha_i, \beta_j \in \mathbb{F}_{5^k}$. Then

$$s_1^{s_2} = 1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+5}x^2) + ((\alpha_i + i\delta_1) + (\alpha_{i+5} + i\delta_3)x^2)y^i + i(\alpha_i + \alpha_{i+5}x^2)y^4]$$
$$+ 4(\delta_1 + \delta_3 x^2)y^4 + 3(\alpha_4 x + \alpha_5 x + 2\delta_2 x + \alpha_9 x^3 + \alpha_{10}x^3 + 2\delta_4 x^3)$$
$$+ ((\alpha_4 + \delta_2)x + (\alpha_9 + \delta_4)x^3)(y + y^4) + ((\alpha_5 + \delta_2)x + (\alpha_{10} + \delta_4)x^3)(y^2 + y^3)$$

where

$$\delta_1 = (3\beta_7 + 4\beta_8)(\alpha_4 - \alpha_5) + (3\beta_3 + 4\beta_4)(\alpha_9 - \alpha_{10})$$
$$\delta_2 = (\alpha_4 - \alpha_5)(3\beta_3\beta_8 + \beta_3\beta_7 + 4\beta_4\beta_8 + 3\beta_4\beta_7) + (\alpha_2 - \alpha_3)(3\beta_3 + 4\beta_4)$$
$$+ (\alpha_7 - \alpha_8)(3\beta_3 + 4\beta_4) + (\alpha_4 - \alpha_5)(3{\beta_3}^2 + 2{\beta_4}^2 + 3{\beta_7}^2 + 2{\beta_8}^2 + 3\beta_3\beta_4 + 3\beta_7\beta_8)$$
$$\delta_3 = (3\beta_3 + 4\beta_4)(\alpha_4 - \alpha_5) + (3\beta_7 + 4\beta_8)(\alpha_9 - \alpha_{10})$$
$$\delta_4 = (\alpha_4 - \alpha_5)(3{\beta_3}^2 + 2{\beta_4}^2 + 3{\beta_7}^2 + 2{\beta_8}^2 + 3\beta_3\beta_4 + 3\beta_7\beta_8) + (\alpha_2 - \alpha_3)(3\beta_7 + 4\beta_8)$$
$$+ (\alpha_7 - \alpha_8)(3\beta_3 + 4\beta_4) + (\alpha_4 - \alpha_5)(\beta_3\beta_7 + 3\beta_3\beta_8 + 4\beta_4\beta_8 + 3\beta_4\beta_7)$$

and $\alpha_i, \beta_j \in \mathbb{F}_{5^k}$. Clearly $s_1^{s_2} \in S_1$ and S_2 normalizes S_1. Let

$$M = S_1 \cap S_2 = \{1 + 3(\beta_1 + \beta_2 + \beta_3 x^2 + \beta_4 x^2) + (\beta_1 + \beta_3 x^2)(y + y^4) + (\beta_2 + \beta_4)(y^2 + y^3)\}$$

where $\beta_i \in \mathbb{F}_{5^k}$. By the second Isomorphism Theorem $S_1S_2/S_1 \cong S_2/S_1 \cap S_2$. Now $|S_1 \cap S_2| = 5^{4k}$. Therefore $|S_1S_2| = 5^{14k} = N_H(V)$. Clearly S_2 is an elementary abelian 5-group and therefore S_2 completely reduces. Let $S_2 \cong M \times W \cong {C_5}^{4k} \times {C_5}^{4k}$. Clearly $W \cap S_1 = \{1\}$ and W normalizes S_1. Thus $N_H(V) \cong {C_5}^{10k} \rtimes {C_5}^{4k}$. $\square$

Lemma 2.5. *Let S_3 be the subset of H consisting of elements of the form*

$$1 + 3(\alpha_1 + \alpha_2) + \alpha_1(y + y^4) + \alpha_2(y^2 + y^3) + \alpha_3 x^2(y - y^4) + \alpha_4 x^2(y^2 - y^3)$$

where $\alpha_i \in \mathbb{F}_{5^k}$. Then S_3 is a group and $S_3 \cong {C_5}^{4k}$.

Proof. Let

$$a = 1 + 3(\alpha_1 + \alpha_2) + \alpha_1(y + y^4) + \alpha_2(y^2 + y^3) + \alpha_3 x^2(y - y^4) + \alpha_4 x^2(y^2 - y^3) \in S_3$$

and

$$b = 1 + 3(\beta_1 + \beta_2) + \beta_1(y + y^4) + \beta_2(y^2 + y^3) + \beta_3 x^2(y - y^4) + \beta_4 x^2(y^2 - y^3) \in S_3$$

where $\alpha_i, \beta_j \in \mathbb{F}_{5^k}$. Then

$$ab = 1 + 3(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2) + (\alpha_1 + \beta_1 + \gamma_1)(y + y^4)$$
$$+ (\alpha_2 + \beta_2 + \gamma_2)(y^2 + y^3) + (\alpha_3 + \beta_3 + \gamma_3)x^2(y - y^4) + (\alpha_4 + \beta_4 + 2\gamma_3)x^2(y^2 - y^3)$$

where

$$\gamma_1 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) + 4\alpha_4\beta_3 + \alpha_4\beta_4 + 4\alpha_3\beta_4$$

$$\gamma_2 = (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) + \alpha_3\beta_4 + \alpha_3\beta_3 + \alpha_4\beta_3$$

$$\gamma_3 = (\alpha_1 - \alpha_2)(3\beta_3 + \beta_4) + (\beta_1 - \beta_2)(3\alpha_3 + \alpha_4)$$

and $\alpha_i, \beta_j \in \mathbb{F}_{5^k}$. Therefore S_3 is closed under multiplication. It can easily be shown that S_3 is abelian. Therefore $S_3 \cong {C_5}^{4k}$. □

Lemma 2.6. *Let H_1 be the subset of H consisting of elements of the form*

$$1 + \sum_{i=1}^{3}[(4-i)\alpha_i + \alpha_i y^i + i\alpha_i y^4] - \sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+7}x^2 + \alpha_{i+11}x^3) +$$

$$\sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+7}x^2 + \alpha_{i+11}x^3)y^i$$

where $\alpha_i \in \mathbb{F}_{5^k}$. Then H_1 is a group and $H_1 \cong ({C_5}^{4k} \rtimes {C_5}^{10k}) \rtimes {C_5}^{k}$.

Proof. It can easily be shown that H_1 is closed under multiplication and clearly $S_3 < H_1$ and $N_H(V) < H_1$. Let

$$s = 1 + 3(\beta_1 + \beta_2) + \beta_1(y + y^4) + \beta_2(y^2 + y^3) + \beta_3 x^2(y - y^4) + \beta_4 x^2(y^2 - y^3) \in S_3$$

and

$$n = 1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+7}x^2) + (\alpha_i + \alpha_{i+7}x^2)y^i + i(\alpha_i + \alpha_{i+7}x^2)y^4]$$
$$- \sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+10}x^3) + \sum_{i=1}^{4}(\alpha_{i+3}x + \alpha_{i+10}x^3)y^i \in N_H(V),$$

where $\alpha_i, \beta_j \in \mathbb{F}_{5^k}$. Then

$$n^s = 1 + \sum_{i=1}^{3}[(4-i)(\alpha_i + \alpha_{i+7}x^2) + (\alpha_i + \alpha_{i+7}x^2)y^i + i(\alpha_i + \alpha_{i+7}x^2)y^4]$$
$$- \sum_{i=1}^{4}(\delta_{i+3} + \delta_{i+10}) + \sum_{i=1}^{4}(\delta_{i+3}x + \delta_{i+10}x^3)y^i$$

where the δ_l's are functions of the α_i's and the β_j's. Clearly $n^s \in N_H(V)$ and S_3 normalizes $N_H(V)$. Let $T = \{(1 + 3\alpha_1 + 3\alpha_2) + \alpha_1(y + y^4) + \alpha_2(y^2 + y^3) + \sum_{i=1}^{4}\alpha_3 i x^2 y^i\}$ where $\alpha_i \in \mathbb{F}_{5^k}$. Now $N_H(V)S_3/S_3 \cong S_3/N_H(V) \cap S_3$ and $|N_H(V) \cap S_3| = 5^{3k}$. Therefore $|N_H(V)S_3| = 5^{15k} = H_1$ and $S_3 \cong T \times R \cong {C_5}^{3k} \times {C_5}^{k}$. Clearly $R \cap N_H(V) = \{1\}$ and R normalizes $N_H(V)$. Thus $H_1 \cong ({C_5}^{10k} \rtimes {C_5}^{4k}) \rtimes {C_5}^{k}$. □

Lemma 2.7. *Let S_4 be the subset of H consisting of elements of the form*

$$1+(r+rx+rx^2+rx^3)(y+y^2-y^3-y^4)$$

where $r\in\mathbb{F}_{5^k}$. Then S_4 is a group and $S_4\cong {C_5}^k$.

Proof. Let

$$a=1+(r+rx+rx^2+rx^3)(y+y^2-y^3-y^4)\in S_4$$

and

$$b=1+(s+sx+sx^2+sx^3)(y+y^2-y^3-y^4)\in S_4$$

where $r,s\in\mathbb{F}_{5^k}$. Then

$$ab=1+((r+s)+(r+s)x+(r+s)x^2+(r+s)x^3)(y+y^2-y^3-y^4).$$

Therefore S_4 is closed under multiplication. It can easily be shown that S_4 is abelian. Therefore $S_4\cong {C_5}^k$. □

Lemma 2.8. $H\cong(({C_5}^{4k}\rtimes {C_5}^{10k})\rtimes {C_5}^k)\rtimes {C_5}^k$.

Proof. Let

$$\begin{aligned}h=1&+\sum_{i=1}^{3}[(4-i)\alpha_i+\alpha_i y^i+i\alpha_i y^4]-\sum_{i=1}^{4}(\alpha_{i+3}x+\alpha_{i+7}x^2+\alpha_{i+11}x^3)\\&+\sum_{i=1}^{4}(\alpha_{i+3}x+\alpha_{i+7}x^2+\alpha_{i+11}x^3)y^i\in H_1\end{aligned}$$

and $s=1+(r+rx+rx^2+rx^3)(y+y^2-y^3-y^4)\in S_4$ where $\alpha_i,r\in\mathbb{F}_{5^k}$. Then

$$\begin{aligned}h^s=1&\sum_{i=1}^{3}[(4-i)\alpha_i+(\alpha_i+i\gamma)y^i+i\alpha_i y^4]+4\gamma y^4-\sum_{i=1}^{4}(\delta_{i+3}x+\delta_{i+7}x^2+\delta_{i+11}x^3)\\&+\sum_{i=1}^{4}(\delta_{i+3}x+\delta_{i+7}x^2+\delta_{i+11}x^3)y^i\end{aligned}$$

where

$$\begin{aligned}\gamma=&r[(\alpha_5+\alpha_6+\alpha_{13}+\alpha_{14})+4(\alpha_4+\alpha_7+\alpha_{12}+\alpha_{15})]+r^2[(\alpha_4+\alpha_{11}+\alpha_{12})\\&+2(\alpha_5+\alpha_{10}+\alpha_{13})+3(\alpha_6+\alpha_{14}+\alpha_{15})+4(\alpha_7+\alpha_8+\alpha_{15})]\end{aligned}$$

and the δ_i's are functions of the α_j's and r.

Clearly $h^s\in H_1$ and $H_1\cap S_4=\{1\}$. Therefore $H\cong H_1\rtimes S_4\cong(({C_5}^{10k}\rtimes {C_5}^{4k})\rtimes {C_5}^k)\rtimes {C_5}^k$. □

Theorem 2.9. $V(\mathbb{F}_{5^k}(C_5\rtimes C_4)\cong[(({C_5}^{10k}\rtimes {C_5}^{4k})\rtimes {C_5}^k)\rtimes {C_5}^k]\rtimes({C_{5^k-1}}^3)$.

Proof. Recall that $\mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) \cong H \rtimes \mathcal{U}(\mathbb{F}_{5^k}C_4)$. By Propostion 1.3

$$\mathbb{F}_{5^k}C_4 \cong {\mathbb{F}_{5^k}}^4 \implies \mathcal{U}(\mathbb{F}_{5^k}C_4) \cong {C_{5^k-1}}^4.$$

Thus

$$\begin{aligned}\mathcal{U}(\mathbb{F}_{5^k}(C_5 \rtimes C_4)) &\cong [(({C_5}^{10k} \rtimes {C_5}^{4k}) \rtimes {C_5}^k) \rtimes {C_5}^k] \rtimes ({C_{5^k-1}}^4)\\ &\cong [(({C_5}^{10k} \rtimes {C_5}^{4k}) \rtimes {C_5}^k) \rtimes {C_5}^k] \rtimes ({C_{5^k-1}}^3) \times \mathcal{U}(\mathbb{F}_{5^k}).\end{aligned}$$

□

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