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# DIVISION $\mathbb{Z}_3$-ALGEBRAS

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Abstract. Our main purpose is to classify the finite dimensional central simple associative division $\mathbb{Z}_3$-algebras over a field $K$ of characteristic 0, and then study the existence of $\mathbb{Z}_3$-involutions on $\mathbb{Z}_3$-algebra $\mathcal{A} = M_{p+q+p}(\mathcal{D})$, where $\mathcal{D}$ is a central division algebra over a field $K$ of characteristic 0 and $p, q > 0$.

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## 1. Introduction

An associative $\mathbb{Z}_n$-ring $R = \bigoplus_{i=0}^{n-1} R_i$ is nothing but a $(\mathbb{Z}/n\mathbb{Z})$-graded associative ring. A $(\mathbb{Z}/n\mathbb{Z})$-graded ideal $I = \bigoplus_{i=0}^{n-1} I_i$ of an associative $\mathbb{Z}_n$-ring $R$ is called a $\mathbb{Z}_n$-ideal of $R$. An associative $\mathbb{Z}_n$-ring $R$ is simple if it has no non-trivial $\mathbb{Z}_n$-ideals. Let $R$ be an associative $\mathbb{Z}_n$-ring with $1 \in R_0$, then $R$ is said to be a division $\mathbb{Z}_n$-ring if all nonzero homogeneous elements are invertible, i.e., every $0 \neq r_\alpha \in R_\alpha$ has an inverse $r_\alpha^{-1}$, necessarily in $R_{n-\alpha}$.

Let $K$ be a field of characteristic 0 (not necessarily algebraically closed). An associative $(\mathbb{Z}/n\mathbb{Z})$-graded $K$-algebra $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{A}_i$ is a finite dimensional central simple $\mathbb{Z}_n$-algebra over a field $K$, if $Z(\mathcal{A}) \cap \mathcal{A}_0 = K$, where $Z(\mathcal{A}) = \{a \in \mathcal{A} \mid ab = ba \ \forall\ b \in \mathcal{A}\}$ is the center of $\mathcal{A}$, and the only $\mathbb{Z}_n$-ideals of $\mathcal{A}$ are (0) and $\mathcal{A}$ itself. An associative $(\mathbb{Z}/2\mathbb{Z})$-graded $K$-algebra $\mathcal{A}$ is called associative superalgebra (see [3,1,5]).

## 2. Examples of $\mathbb{Z}_n$-algebras

**Example 2.1.** Let $\mathcal{A} = K(\sqrt[n]{a})$ be an algebraic field extension of the field $K$ of degree $n$, that is $[\mathcal{A} : K] = n$. We can make $\mathcal{A}$ into a $\mathbb{Z}_n$-algebra by setting

$$\mathcal{A}_0 = K,\ \mathcal{A}_1 = K.\sqrt[n]{a},\ \ldots,\ \mathcal{A}_i = K.\sqrt[n]{a^i},\ \ldots,\ \mathcal{A}_{n-1} = K.\sqrt[n]{a^{n-1}}.$$

Note that $\mathcal{A}$ is a central simple $\mathbb{Z}_n$-algebra, since $\mathcal{A}$ is a field and $\mathcal{A} \cap \mathcal{A}_0 = K$.

**Example 2.2.** Let $\omega$ be a fixed primitive $n$-th root of unity. For $a, b \in K^{\times}$, let $\mathcal{A} =< a, b >_\omega$ be the $K$-algebra which is generated by $\{i, j\}$ which satisfy $\{i^n = a \ , \ j^n = b \ , \ ij = \omega ji\}$. Then $\mathcal{A}$ is a vector space over $K$ with basis $\{i^r j^s \ : \ 0 \le r, s < n\}$. So $\mathcal{A}$ has dimension $n^2$ as a $K$-algebra. (See [4, Section 15.4] and [2, Exercise 4.28]). This is a generalization of the quaternion algebras. We can make $\mathcal{A}$ into $\mathbb{Z}_n$-algebra by setting $\mathcal{A}_l =< i^k j^m \ : \ k+m \equiv l( \mod n) >_K$ .

**Example 2.3.** A $\mathbb{Z}_n$-space over a field $K$ is a left $K$-vector space $V$ which is $\mathbb{Z}_n$-graded $V = \bigoplus_{i=0}^{n-1} V_i$. The associative algebra $\mathrm{End}_K V = \bigoplus_{i=0}^{n-1} \mathrm{End}_i V$, where

$$\mathrm{End}_i V := \{a \in \mathrm{End}_K V \ : \ v_j a \in V_{i+j}\},$$

is an associative $\mathbb{Z}_n$-algebra.

**Example 2.4.** Let $\mathcal{D} = \bigoplus_{i=0}^{n-1} \mathcal{D}_i$ be a $\mathbb{Z}_n$-division algebra then $\mathcal{A} = M_k(\mathcal{D})$ can be made into $\mathbb{Z}_n$-algebra by setting

$$\mathcal{A}_0 = M_k(\mathcal{D}_0),\ \mathcal{A}_1 = M_k(\mathcal{D}_1), \ldots,\ \mathcal{A}_{n-1} = M_k(\mathcal{D}_{n-1}).$$

**Example 2.5.** Let $\mathcal{D}$ be a central division algebra over a field $K$ and let $\mathcal{A} = M_3(\mathcal{D})$. If $\mathcal{A}_0 = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$, $\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$, $\mathcal{A}_2 = \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}$. Then $\mathcal{A}$ is a $\mathbb{Z}_3$-algebra written by $\mathcal{A} = M_{1+1+1}(\mathcal{D})$.

**Theorem 2.6.** *Let $\mathcal{D}$ be a central division algebra over a field $K$ and let $\mathcal{A} = M_n(\mathcal{D})$, then $\mathcal{A}$ can be made into $\mathbb{Z}_n$-algebra by setting*

$$\mathcal{A}_0 = \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix},\ \mathcal{A}_1 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & * \\ * & 0 & \cdots & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & * & 0 \\ 0 & 0 & \cdots & \cdots & 0 & * \\ * & 0 & 0 & \cdots & \cdots & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 & 0 \end{bmatrix}, \ldots,\ \mathcal{A}_{n-1} = \begin{bmatrix} 0 & * & 0 & \cdots & 0 \\ \vdots & 0 & * & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & \vdots & \ddots & * \\ * & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

**Proof.** Note that $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{A}_i$ and $\mathcal{A}_1^2 \subseteq \mathcal{A}_2$, $\mathcal{A}_1^3 \subseteq \mathcal{A}_3$, $\ldots$, $\mathcal{A}_1^{n-1} \subseteq \mathcal{A}_{n-1}$, $\mathcal{A}_1^n \subseteq \mathcal{A}_0$. Therefore $\mathcal{A}_i \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ where the subscripts are taken modulo $n$. $\square$

In the next theorem we will show that any matrix $\mathcal{A} = M_n(\mathcal{D})$, where $\mathcal{D}$ is a division algebra, and where $n = p + q + r$ such that $p, q, r > 0$, can be made into $\mathbb{Z}_3$-algebra.

**Theorem 2.7.** *Let $\mathcal{D}$ be a division algebra and let $\mathcal{A} = M_n(\mathcal{D})$ where $n = p+q+r$ such that $p, q, r > 0$. If $\mathcal{A}_0 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & z \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$, $\mathcal{A}_2 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & g \\ h & 0 & 0 \end{pmatrix}$ where $a \in M_p(\mathcal{D})$, $b \in M_q(\mathcal{D})$, $c \in M_r(\mathcal{D})$, $z \in M_{p\times r}(\mathcal{D})$, $x \in M_{q\times p}(\mathcal{D})$, $y \in M_{r\times q}(\mathcal{D})$, $f \in M_{p\times q}(\mathcal{D})$, $g \in M_{q\times r}(\mathcal{D})$, $h \in M_{r\times p}(\mathcal{D})$, then $\mathcal{A}$ is a $\mathbb{Z}_3$-algebra written by $\mathcal{A} = M_{p+q+r}(\mathcal{D})$.*

**Proof.** Note that $\mathcal{A} = \bigoplus_{i=0}^{2} \mathcal{A}_i$, and $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ where the subscripts are taken modulo 3. Therefore $\mathcal{A} = M_{p+q+r}(\mathcal{D})$ is a $\mathbb{Z}_3$-algebra. □

**Definition 2.8.** Let $\mathcal{A} = \bigoplus_{i=0}^{2} \mathcal{A}_i$ be a $\mathbb{Z}_3$-algebra, then the $\mathbb{Z}_3$-additive map $\sigma : \mathcal{A} \to \mathcal{A}$ such that for $a_i \in \mathcal{A}_i$, $b_j \in \mathcal{A}_j$ and $r \equiv ij \bmod 3$

$$\sigma(a_i b_j) = (-1)^r \sigma(b_j)\sigma(a_i)$$

is called a $\mathbb{Z}_3$-antiautomorphism on $\mathcal{A}$.

**Definition 2.9.** A $\mathbb{Z}_3$-involution on a $\mathbb{Z}_3$-algebra $\mathcal{A}$ is a $\mathbb{Z}_3$-antiautomorphism on $\mathcal{A}$ of order 2.

Let $V = \bigoplus_{i=0}^{n-1} V_i$ be a left $\mathbb{Z}_n$-space over a field $K$. A symmetric $\mathbb{Z}_n$-form on $V$ is a $\mathbb{Z}_n$-bilinear form

$$(\ ,\ ) : V \times V \to K, \quad V = V_0 \perp V_1 \perp \ldots \perp V_{n-1},$$

which is symmetric on $V_{2r}$ and skew-symmetric on $V_{2r+1}$.
The symmetric $\mathbb{Z}_n$-form $(\ ,\ )$ on $V$ is nondegenerate if

$$(v_i, V) = \{0\} \Rightarrow v_i = 0 \quad \text{and} \quad (V, v_i) = \{0\} \Rightarrow v_i = 0.$$

**Theorem 2.10.** *A nondegenerate symmetric $\mathbb{Z}_3$-form $(\ ,\ )$ on a finite dimensional $\mathbb{Z}_3$-space $V$ over a field $K$, induces a $\mathbb{Z}_3$-involution $*$ on $\mathrm{End}_K V$ via*

$$(v_i a_k, v_j) = (-1)^{kj}(v_i, v_j a_k^*)\ \forall v_i, v_j \in V.$$

**Proof.** Let $a_\alpha, b_\beta \in \mathrm{End}_K V$, then

$$\begin{aligned} (v_i a_\alpha b_\beta, v_j) &= (-1)^{(\alpha+\beta)j}(v_i, v_j(a_\alpha b_\beta)^*) \\ &= (-1)^{\beta j}(v_i a_\alpha, v_j b_\beta^*) \\ &= (-1)^{\alpha(\beta+j)}(-1)^{\beta j}(v_i, v_j b_\beta^* a_\alpha^*). \end{aligned}$$

Which implies that $(-1)^{(\alpha+\beta)j}(v_i, v_j(a_\alpha b_\beta)^*) = (-1)^{\alpha(\beta+j)}(-1)^{\beta j}(v_i, v_j b_\beta^* a_\alpha^*)$. We will show that

$$\alpha(\beta+j) \mod 3 + \beta j \mod 3 = (\alpha+\beta)j \mod 3 + \alpha\beta \mod 3 \qquad (1)$$

case by case on $\alpha$.
If $\alpha = 0$, then (1) becomes $\beta j \mod 3 = \beta j \mod 3$.
If $\alpha = 1$, then (1) becomes

$$(\beta+j) \mod 3 + \beta j \mod 3 = (1+\beta)j \mod 3 + \beta \mod 3 \qquad (2)$$

we show that (2) is true case by case on $j$.
If $j = 0$, then (2) becomes $\beta \mod 3 = \beta \mod 3$.
If $j = 1$, then (2) becomes $(\beta+1) \mod 3 + \beta \mod 3 = (\beta+1) \mod 3 + \beta \mod 3$.
If $j = 2$, then (2) becomes

$$(\beta+2) \mod 3 + 2\beta \mod 3 = (2+2\beta) \mod 3 + \beta \mod 3 \qquad (3)$$

and we will show that (3) is true case by case on $\beta$.
If $\beta = 0$, then (3) becomes $2 \mod 3 = 2 \mod 3$.
If $\beta = 1$, then (3) becomes $2 \mod 3 = 1 \mod 3 + 1 \mod 3$.
If $\beta = 2$, then (3) becomes $1 \mod 3 + 1 \mod 3 = 2 \mod 3$.
If $\alpha = 2$, then (1) becomes

$$2(\beta+j) \mod 3 + \beta j \mod 3 = (2+\beta)j \mod 3 + 2\beta \mod 3 \qquad (4)$$

and we will show that (4) is true case by case on $j$.
If $j = 0$, then (4) becomes $2\beta \mod 3 = 2\beta \mod 3$.
If $j = 2$, then (4) becomes $2(\beta+2) \mod 3 + 2\beta \mod 3 = 2(\beta+2) \mod 3 + 2\beta \mod 3$.
If $j = 1$, then (4) becomes

$$2(\beta+1) \mod 3 + \beta \mod 3 = (2+\beta) \mod 3 + 2\beta \mod 3 \qquad (5)$$

and we will show that (5) is true case by case on $\beta$.
If $\beta = 0$, then (5) becomes $2 \mod 3 = 2 \mod 3$.
If $\beta = 1$, then (5) becomes $1 \mod 3 + 1 \mod 3 = 2 \mod 3$.
If $\beta = 2$, then (5) becomes $2 \mod 3 = 1 \mod 3 + 1 \mod 3$.
Therefore, in all cases we have

$$\begin{aligned}(-1)^{(\alpha+\beta)j}(v_i, v_j(a_\alpha b_\beta)^*) &= (-1)^{\alpha(\beta+j)}(-1)^{\beta j}(v_i, v_j b_\beta^* a_\alpha^*) \\ &= (-1)^{(\alpha+\beta)j}(-1)^{\alpha\beta}(v_i, v_j b_\beta^* a_\alpha^*),\end{aligned}$$

which implies that $(v_i, v_j(a_\alpha b_\beta)^*) = (-1)^{\alpha\beta}(v_i, v_j b_\beta^* a_\alpha^*)$, for all $v_i \in V_i$, and hence

$$(v_i, v_j(a_\alpha b_\beta)^* - (-1)^{\alpha\beta} v_j b_\beta^* a_\alpha^*) = 0 \quad \forall v_i \in V_i$$

and because of the nondegenerancy of the $\mathbb{Z}_3$-form ( , ) on $V$, we have

$$v_j(a_\alpha b_\beta)^* = (-1)^{\alpha\beta} v_j b_\beta^* a_\alpha^*.$$

□

## 3. Division $\mathbb{Z}_3$-algebras

We start this section by proving a structure theorem on $\mathbb{Z}_3$-division algebras which is a restate of Division Superalgebra Theorem, see [5, P. 438], but first we need the following lemma. The proof of this lemma is exactly the same as the proof of [6, Lemmata 3,5].

**Lemma 3.1.** *If $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$ is a central simple unital $\mathbb{Z}_3$-algebra over $K$ then either $\mathcal{A}$ is simple as an algebra or $\mathcal{A}_0$ is simple and $\mathcal{A}_1 = \mathcal{A}_0 u$ and $\mathcal{A}_2 = \mathcal{A}_0 u^2$, with $u \in Z(\mathcal{A}) \cap \mathcal{A}_1$ and $u^3 = 1$.*

**Theorem 3.2** (Division $\mathbb{Z}_3$-algebra Theorem)**.** *If $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2$ is a finite dimensional central division $\mathbb{Z}_3$-algebra over the field $K$ of characteristic 0, then exactly one of the following holds where throughout $\mathcal{C}$ denotes a central division algebra over $K$ and $\omega \in K$ denotes a primitive third root of unity.*
*(i) $\mathcal{D} = \mathcal{D}_0 = \mathcal{C}$, i.e., $\mathcal{D}_1 = \{0\}$, $\mathcal{D}_2 = \{0\}$.*
*(ii) $\mathcal{D} = \mathcal{C} \otimes_K K[u]$, $u^3 = \lambda \in K^\times$, $\mathcal{D}_0 = \mathcal{C} \otimes_K K$, $\mathcal{D}_1 = \mathcal{C} \otimes_K Ku$, $\mathcal{D}_2 = \mathcal{C} \otimes_K Ku^2$.*
*(iii) $\mathcal{D} = \mathcal{C}$, $\mathcal{D}_0 = C_{\mathcal{D}}(u)$, the centralizer of $u$ in $\mathcal{C}$,*

$$\mathcal{D}_1 = \{c \in \mathcal{D} \ : \ cu = \sigma(u)c\},$$

$$\mathcal{D}_2 = \{c \in \mathcal{D} \ : \ cu = \sigma^2(u)c\},$$

*for some Galois extension $K[u] \subset \mathcal{C}$ of order 3 with Galois automorphism $\sigma$.*
*(iv) $\mathcal{D} = M_3(\mathcal{C}) = \mathcal{C} \otimes_K M_3(K)$,*
*$\mathcal{D}_0 = \mathcal{C} \otimes_K K[u]$, $\mathcal{D}_1 = \mathcal{C} \otimes_K K[u]W_1$, $\mathcal{D}_2 = \mathcal{C} \otimes_K K[u]W_1^{-1}$,*
*where $u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix}$, $W_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$, $W_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \in M_3(K)$, $\lambda \notin K^3$.*

**Proof.** Let $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2$ be a central division $\mathbb{Z}_3$-algebra over $K$, and let $\mathcal{D}_1 \neq \{0\}$, then $\mathcal{D}_2 \neq \{0\}$. If $0 \neq v \in \mathcal{D}_1$, then $\mathcal{D}_0 v \subseteq \mathcal{D}_1 = \mathcal{D}_1 v^{-1} v \subseteq \mathcal{D}_0 v$, and $\mathcal{D}_0 v^{-1} \subseteq \mathcal{D}_2 = \mathcal{D}_2 v v^{-1} \subseteq \mathcal{D}_0 v^{-1}$. Therefore $\mathcal{D}_1 = \mathcal{D}_0 v$ and $\mathcal{D}_2 = \mathcal{D}_0 v^{-1}$ for any $0 \neq v \in \mathcal{D}_1$.

For any $a \in \mathcal{D}_0$, $va = a^{\psi_v} v$, where $a^{\psi_v} = vav^{-1}$, and $\psi_v|_{\mathcal{D}_0}$ is an automorphism of $\mathcal{D}_0$ as an algebra over $K = Z(\mathcal{D}) \cap \mathcal{D}_0$. Since any element of $\mathcal{D}_1$ is of the form $c_0 v$, $c_0 \in \mathcal{D}_0$, the restriction of $\psi_v$ to $Z(\mathcal{D}_0)$ does not depend on the particular choice of $v \in \mathcal{D}_1$.

Assume first that $\psi_v|_{\mathcal{D}_0}$ is an inner automorphism of $\mathcal{D}_0$, say $\psi_v|_{\mathcal{D}_0} = \psi_c$, where $c \in \mathcal{D}_0$ (up to multiplication by an element of $Z(\mathcal{D}_0)$). Therefore $vav^{-1} = cac^{-1}$ implies that $c^{-1}vav^{-1}c = a$ for all $a \in \mathcal{D}_0$. Letting $u = c^{-1}v \in \mathcal{D}_1$ then $u^{-1} \in \mathcal{D}_2$ and $uau^{-1} = a$ for all $a \in \mathcal{D}_0$, so $u$ centralizes $\mathcal{D}_0$. Since $\mathcal{D}_1 = \mathcal{D}_0 u$, $\mathcal{D}_2 = \mathcal{D}_0 u^{-1}$, $u$ centralizes $\mathcal{D}_1$ and $\mathcal{D}_2$. Thus $u \in Z(\mathcal{D})$ and $u^3 \in Z(\mathcal{D}) \cap \mathcal{D}_0$, say $u^3 = \lambda \in K^\times$. Letting $\mathcal{C} = \mathcal{D}_0$, $\mathcal{D} = \mathcal{C} \otimes_K K[u]$. Note that $\mathcal{D}$ is simple as an algebra if and only if $\lambda \notin K^3$. If $\lambda \in K^3$, we may assume that $\lambda = 1$. This is the only case where $\mathcal{D}$ is not simple as an algebra.

Assume next that $\sigma = \psi_v|_{\mathcal{D}_0}$ is not an inner automorphism of $\mathcal{D}_0$ over $K$. If $\sigma$ is not the identity then $K$ is the fixed subfield of $Z(\mathcal{D}_0)$, which implies that $Z(\mathcal{D}_0)$ is a Galois extension of $K$ of order 3 with Galois automorphism $\sigma$. We may choose $u \in Z(\mathcal{D}_0)$ such that $Z(\mathcal{D}_0) = K[u]$, $u^3 = \lambda \notin K^3$ with $\sigma(u) \neq u \in K[u]$. Now, $(av)u = a\sigma(u)v = \sigma(u)(av)$ implies that $\sigma(vu) = v\sigma(u) = \sigma(\sigma(u)v) = \sigma^2(u)v$ and hence $av^2u = av\sigma(u)v = a(v\sigma(u))v = a(\sigma^2(u)v)v = \sigma^2(u)(av^2)$ for all $a \in \mathcal{D}_0$. Therefore $\mathcal{D}_0 = C_{\mathcal{D}}(u)$, the centralizer of $u$ in $\mathcal{D}$, and $\mathcal{D}_1 = \{c \in \mathcal{D} : cu = \sigma(u)c\} = \mathcal{D}_0 v$, $\mathcal{D}_2 = \{c \in \mathcal{D} : cu = \sigma^2(u)c\} = \mathcal{D}_0 v^2$. If $\mathcal{D}$ is a division algebra then $\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2$ as above.

If $\mathcal{D}$ is not a division algebra then since $\mathcal{D}_0$ is not central simple over $K = Z(\mathcal{D}) \cap \mathcal{D}_0$ then, by Lemma 3.1, $\mathcal{D}$ is a central simple algebra over $K$. Let $J \neq \{0\}$ be a right ideal of $\mathcal{D}$. If $0 \neq a_0 + a_1 + a_2 \in J$ then at least one of $a_i \neq 0$ and multiplying by $a_i^{-1}$ on the right, $1 + b_1 + b_2 \in J$ for some $b_1 \in \mathcal{D}_1$, $b_2 \in \mathcal{D}_2$. Hence $(1 + b_1 + b_2)\mathcal{D} \subseteq J$. If $J$ contains an element $0 \neq a'_0 + a'_1 + a'_2 \notin (1 + b_1 + b_2)\mathcal{D}$ then arguing as above, we obtain an element $1 + b'_1 + b'_2 \in J$, $b'_1 + b'_2 \neq b_1 + b_2$. In that case $0 \neq b_1 - b'_1 + b_2 - b'_2 \in J$, where $b_1 - b'_1 \in \mathcal{D}_1$ and $b_2 - b'_2 \in \mathcal{D}_2$. If $b_1 - b'_1 = 0$ or $b_2 - b'_2 = 0$, then $1 \in J$ and hence $J = \mathcal{D}$. If $b_1 - b'_1 \neq 0$ and $b_2 - b'_2 \neq 0$, then multiplying $b_1 - b'_1 + b_2 - b'_2$ by $(b_1 - b'_1)^{-1}$, $1 + c_1 \in J$ and $c_1 \neq 0$. If $J$ contains an element $0 \neq a''_0 + a''_1 + a''_2 \notin (1 + c_1)\mathcal{D}$, then arguing as above, we obtain an element $1 + c'_1 \in J$, $c'_1 \neq c_1$. In that case $0 \neq c_1 - c'_1 \in J$ and hence $1 \in J$ which must be the whole of $\mathcal{D}$. Therefore a descending chain of nonzero right ideals in $\mathcal{D}$ has length at most 3 and $\mathcal{D}$ is isomorphic to $M_3(\mathcal{C})$, where $\mathcal{C}$ is a central division algebra over $K$. If $K[u]$ were to embed in $\mathcal{C}$ then $\mathcal{D}_0 = C_{\mathcal{D}}(u) \supseteq M_3(\mathcal{C})$ which is not a division algebra. Therefore $K[u]$ does not embed in $\mathcal{C}$ but rather the algebraic extension $K[u]$ of order 3 embeds in $M_3(K)$ and $u$, $W_1$, $W_2$ can be chosen as

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{pmatrix},\ W_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},\ W_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} = W_1^{-1},\ \lambda \notin K^3$$

where

$$\mathcal{D}_0 = \mathcal{C} \otimes K[u],\ \mathcal{D}_1 = \mathcal{C} \otimes K[u]W_1,\ \mathcal{D}_2 = \mathcal{C} \otimes K[u]W_2. \qquad \square$$

## 4. $\mathbb{Z}_3$-Involution

In this section we will obtain more information on the $\mathbb{Z}_3$-involutions of the central simple $\mathbb{Z}_3$-algebra $\mathcal{A} = M_{p+q+r}(\mathcal{D})$, where $\mathcal{D}$ is a central division algebra over a field $K$ of characteristic 0, and $p, q, r > 0$.

**Example 4.1.** Let $\mathcal{A} = M_{1+1+1}(K)$ be a $\mathbb{Z}_3$-algebra then the $\mathbb{Z}_3$-additive map $\sigma : \mathcal{A} \to \mathcal{A}$ defined by

$$\sigma(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}) = \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \sigma(\begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 & -c \\ b & 0 & 0 \\ 0 & a & 0 \end{pmatrix}, \quad \sigma(\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & a \\ -c & 0 & 0 \end{pmatrix}$$

is a $\mathbb{Z}_3$-involution on $\mathcal{A}$.

**Theorem 4.2.** *Let $\mathcal{D}$ be a division algebra, and let $\mathcal{A} = M_{p+q+p}(\mathcal{D})$, $p, q > 0$ be a $\mathbb{Z}_3$-algebra with $\mathcal{A}_0 = M_p(\mathcal{D}) \oplus M_q(\mathcal{D}) \oplus M_p(\mathcal{D})$ and*

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{pmatrix}$$

*with $a, y \in M_{q\times p}(\mathcal{D})$, $b, x \in M_{p\times q}(\mathcal{D})$, $c, z \in M_{p\times p}(\mathcal{D})$. If $\mathcal{D}$ has an involution $\bar{\ }$, then $*$ defined by*

$$\begin{pmatrix} f & x & c \\ a & g & y \\ z & b & h \end{pmatrix}^* = \begin{pmatrix} \tilde{h} & \tilde{y} & -\tilde{c} \\ \tilde{b} & \tilde{g} & \tilde{x} \\ -\tilde{z} & \tilde{a} & \tilde{f} \end{pmatrix}$$

*is a $\mathbb{Z}_3$-involution on $\mathcal{A}$, where for any matrix $a$ over $\mathcal{D}$, $\tilde{a} = \bar{a}^t$, $t$ the transpose.*

**Proof.** If $\mathcal{D}$ has an involution $\bar{\ }$, then for any $a \in M_p(\mathcal{D})$ or $a \in M_q(\mathcal{D})$, $\tilde{a} = \bar{a}^t$, $t$ the transpose, defines involutions on $M_p(\mathcal{D})$ and on $M_q(\mathcal{D})$. Moreover if $a \in M_{p\times q}(\mathcal{D})$ $(M_{q\times p}(\mathcal{D}))$, then $\tilde{a} \in M_{q\times p}(\mathcal{D})$ $(M_{p\times q}(\mathcal{D}))$.

Let $\begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & z \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}$ be two matrices in $\mathcal{A}_1$, then

$$\left[\begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & z \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}\right]^* = \begin{pmatrix} 0 & cy & 0 \\ 0 & 0 & az \\ bx & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & \widetilde{az} & 0 \\ 0 & 0 & \widetilde{cy} \\ -\widetilde{bx} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \tilde{z}\tilde{a} & 0 \\ 0 & 0 & \tilde{y}\tilde{c} \\ -\tilde{x}\tilde{b} & 0 & 0 \end{pmatrix}.$$

And

$$-\begin{pmatrix}0&0&z\\x&0&0\\0&y&0\end{pmatrix}^*\begin{pmatrix}0&0&c\\a&0&0\\0&b&0\end{pmatrix}^* = -\begin{pmatrix}0&0&-\tilde{z}\\\tilde{y}&0&0\\0&\tilde{x}&0\end{pmatrix}\begin{pmatrix}0&0&-\tilde{c}\\\tilde{b}&0&0\\0&\tilde{a}&0\end{pmatrix}$$
$$= -\begin{pmatrix}0&-\tilde{z}\tilde{a}&0\\0&0&-\tilde{y}\tilde{c}\\\tilde{x}\tilde{b}&0&0\end{pmatrix} = \begin{pmatrix}0&\tilde{z}\tilde{a}&0\\0&0&\tilde{y}\tilde{c}\\-\tilde{x}\tilde{b}&0&0\end{pmatrix}.$$

Which implies that $(XY)^* = -Y^*X^*$ for all $X, Y \in \mathcal{A}_1$. Moreover

$$[\begin{pmatrix}0&a&0\\0&0&b\\c&0&0\end{pmatrix}\begin{pmatrix}0&y&0\\0&0&z\\x&0&0\end{pmatrix}]^* = \begin{pmatrix}0&0&az\\bx&0&0\\0&cy&0\end{pmatrix}^* = \begin{pmatrix}0&0&-\widetilde{az}\\\widetilde{cy}&0&0\\0&\widetilde{bx}&0\end{pmatrix}$$
$$= \begin{pmatrix}0&0&-\tilde{z}\tilde{a}\\\tilde{y}\tilde{c}&0&0\\0&\tilde{x}\tilde{b}&0\end{pmatrix}.$$

And

$$-\begin{pmatrix}0&y&0\\0&0&z\\x&0&0\end{pmatrix}^*\begin{pmatrix}0&a&0\\0&0&b\\c&0&0\end{pmatrix}^* = -\begin{pmatrix}0&\tilde{z}&0\\0&0&\tilde{y}\\-\tilde{x}&0&0\end{pmatrix}\begin{pmatrix}0&\tilde{b}&0\\0&0&\tilde{a}\\-\tilde{c}&0&0\end{pmatrix}$$
$$= \begin{pmatrix}0&0&-\tilde{z}\tilde{a}\\\tilde{y}\tilde{c}&0&0\\0&\tilde{x}\tilde{b}&0\end{pmatrix}.$$

Which implies that $(XY)^* = -Y^*X^*$ for all $X, Y \in \mathcal{A}_2$.
Finally, let $X = \begin{pmatrix}0&0&c\\a&0&0\\0&b&0\end{pmatrix}$ be a general matrix in $\mathcal{A}_1$ and $Y = \begin{pmatrix}0&y&0\\0&0&z\\x&0&0\end{pmatrix}$ a general matrix in $\mathcal{A}_2$, then

$$(XY)^* = \begin{pmatrix}cx&0&0\\0&ay&0\\0&0&bz\end{pmatrix}^* = \begin{pmatrix}\widetilde{bz}&0&0\\0&\widetilde{ay}&0\\0&0&\widetilde{cx}\end{pmatrix}$$
$$= \begin{pmatrix}\tilde{z}\tilde{b}&0&0\\0&\tilde{y}\tilde{a}&0\\0&0&\tilde{x}\tilde{c}\end{pmatrix}$$

and $Y^*X^* = \begin{pmatrix} 0 & \tilde{z} & 0 \\ 0 & 0 & \tilde{y} \\ -\tilde{x} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\tilde{c} \\ \tilde{b} & 0 & 0 \\ 0 & \tilde{a} & 0 \end{pmatrix} = \begin{pmatrix} \tilde{z}\tilde{b} & 0 & 0 \\ 0 & \tilde{y}\tilde{a} & 0 \\ 0 & 0 & \tilde{x}\tilde{c} \end{pmatrix} = (XY)^*.$

Similarly, $(YX)^* = X^*Y^*$. □

**Theorem 4.3.** *Let $\mathcal{D}$ be a division algebra. If $\mathcal{A} = M_{p+q+r}(\mathcal{D})$, $p, q, r > 0$ is a $\mathbb{Z}_3$-algebra with $\mathcal{A}_0 = M_p(\mathcal{D}) \oplus M_q(\mathcal{D}) \oplus M_r(\mathcal{D})$ and*

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{pmatrix}$$

*with $a \in M_{q\times p}(\mathcal{D})$, $y \in M_{q\times r}(\mathcal{D})$, $x \in M_{p\times q}(\mathcal{D})$, $b \in M_{r\times q}(\mathcal{D})$, $c \in M_{p\times p}(\mathcal{D})$, $z \in M_{r\times p}(\mathcal{D})$. Let $A = M_p(\mathcal{D}) \oplus \{0\} \oplus M_r(\mathcal{D})$ if $*$ is a $\mathbb{Z}_3$-involution on $\mathcal{A}$ with $(A, *|_A)$ is simple then $p = r$, $\mathcal{D}$ has an involution $^-$, and $(\mathcal{A}, *)$ is isomorphic to $M_{p+q+p}(\mathcal{D})$ with the $\mathbb{Z}_3$-involution $*$ given by*

$$\begin{pmatrix} f & x & c \\ a & g & y \\ z & b & h \end{pmatrix}^* = \begin{pmatrix} \tilde{h} & \frac{\tilde{y}}{\bar{\alpha}} & -\mu\tilde{c} \\ \frac{\tilde{b}}{\alpha} & \tilde{g} & \alpha\tilde{x} \\ -\tilde{\mu}\tilde{z} & \bar{\alpha}\tilde{a} & \tilde{f} \end{pmatrix}, \tag{6}$$

*for $\mu, \alpha \in K$ such that $\mu\tilde{\mu} = 1$ and $\frac{\alpha}{\bar{\alpha}} = \mu$, where $\tilde{a} = \bar{a}^t$ for any matrix $a$ over $\mathcal{D}$, $t$ the transpose. If $\tilde{}$ is of the first kind then $\mu$ and $\alpha$ may be chosen equal to 1. Conversely if $\mathcal{D}$ has an involution $^-$ then (6) defines a $\mathbb{Z}_3$-involution on the simple $\mathbb{Z}_3$-algebra $M_{p+q+p}(\mathcal{D})$.*

**Proof.** In recent work on the primitive $\mathbb{Z}_3$-algebras which has yet to appear, we prove that a $\mathbb{Z}_3$-algebra $M_n(\mathcal{D})$ has a $\mathbb{Z}_3$-involution if and only if $\mathcal{D}$ has. In this case since $\mathcal{D} = \mathcal{D}_0$, $\mathcal{D}$ has an involution $^-$ then $\tilde{a} = \bar{a}^t$ for any matrix $a$ over $\mathcal{D}$, $t$ the transpose, extends to involutions on $M_p(\mathcal{D})$ and on $M_q(\mathcal{D})$. Since $(A, *|_A)$ is simple by assumption, $M_r(\mathcal{D})$ is anti-isomorphic to $M_p(\mathcal{D})$ and $r = p$. Up to isomorphism, $(A, *|_A)$ is given by $(M_p(\mathcal{D}) \oplus \{0\} \oplus M_p(\mathcal{D}), *)$ with $(a, 0, b)^* = (\widetilde{b}, 0, \widetilde{a})$. The proof for $p > q$ goes along the same lines we may let $p \le q$. Letting

$$f_{11} = \sum_{i=1}^{p} e_{ii} \qquad f_{22} = \sum_{i=p+1}^{p+q} e_{ii} \qquad f_{33} = \sum_{i=p+q+1}^{p+q+p} e_{ii}$$

$$f_{12} = \sum_{i=1}^{p} e_{i\ p+i} \qquad f_{13} = \sum_{i=1}^{p} e_{i\ p+q+i}$$

$$f_{21} = \sum_{i=1}^{p} e_{p+i\ i} \qquad f_{23} = \sum_{i=1}^{p} e_{p+i\ p+q+i}$$

$$f_{31} = \sum_{i=1}^{p} e_{p+q+i\ i} \qquad f_{32} = \sum_{i=1}^{p} e_{p+q+i\ p+i}$$

We have

$$\mathcal{A}_0 = M_p(\mathcal{D})f_{11} \oplus M_q(\mathcal{D})f_{22} \oplus M_p(\mathcal{D})f_{33},$$

$$\mathcal{A}_1 = M_p(\mathcal{D})f_{13} \oplus (M_q(\mathcal{D})f_{21} + f_{21}M_p(\mathcal{D})) \oplus (M_p(\mathcal{D})f_{32} + f_{32}M_q(\mathcal{D}))$$

$$\mathcal{A}_2 = M_p(\mathcal{D})f_{31} \oplus (M_p(\mathcal{D})f_{12} + f_{12}M_q(\mathcal{D})) \oplus (M_q(\mathcal{D})f_{23} + f_{23}M_p(\mathcal{D}))$$

$$f_{11}^* = f_{33},\ f_{33}^* = f_{11},\ f_{22}^* = f_{22}.$$

Hence

$$f_{13}^* = (f_{11}f_{13}f_{33})^* = f_{11}f_{13}^*f_{33},$$

and

$$f_{13}^* = cf_{13}, \qquad \text{for some } c \in M_p(\mathcal{D}).$$

For any $a \in M_p(\mathcal{D})$,

$$(af_{13})^* = ((af_{11})f_{13})^* = cf_{13}\widetilde{a}f_{33} = c\widetilde{a}f_{13}$$

While

$$(af_{13})^* = (f_{13}(af_{33}))^* = \widetilde{a}f_{11}cf_{13} = \widetilde{a}cf_{13}.$$

Therefore $c \in Z(M_p(\mathcal{D}))$. Moreover $f_{13} = f_{13}^{**} = (cf_{13})^* = \widetilde{c}cf_{13}$ implies $\widetilde{c}c = I_p$. So $c = -\mu \in K$ with $\mu\widetilde{\mu} = 1$. Similarly $f_{31}^* = df_{31}$, $d \in Z(M_p(\mathcal{D}))$. But

$$f_{33} = f_{11}^* = (f_{13}f_{31})^* = f_{31}^*f_{13}^* = (df_{31})(cf_{13}) = dcf_{33}$$

which implies that $dc = 1$, and hence $d = c^{-1} = \widetilde{c} = -\widetilde{\mu}$. Therefore

$$(af_{13})^* = -\widetilde{a}\mu f_{13} \ \text{ and } \ (af_{31})^* = -\widetilde{a}\widetilde{\mu}f_{31}.$$

Moreover

$$f_{12}^* = \alpha f_{23}, \quad f_{23}^* = \beta f_{12}$$

for some $\alpha, \beta \in K$ with $\alpha\beta = \mu$, since $(f_{12})^{**} = (\alpha f_{23})^* = \bar{\alpha}(f_{23})^* = \bar{\alpha}\beta f_{12} = f_{12}$, then $\bar{\alpha}\beta = 1$, so $\beta = \frac{1}{\bar{\alpha}}$ and $\frac{\alpha}{\bar{\alpha}} = \mu$.
Similarly, $f_{21}^* = \gamma f_{32},\ f_{32}^* = \delta f_{21}$ for some $\gamma, \delta \in K$ with $\gamma\delta = \widetilde{\mu}$ which implies that $\overline{\gamma\delta} = \mu = \alpha\beta$, so we may take $\gamma = \bar{\alpha}$ and $\delta = \bar{\beta} = \frac{1}{\alpha}$. Therefore

$$\begin{pmatrix} f & x & c \\ a & g & y \\ z & b & h \end{pmatrix}^* = \begin{pmatrix} \tilde{h} & \frac{\tilde{y}}{\bar{\alpha}} & -\mu\tilde{c} \\ \frac{\tilde{b}}{\alpha} & \tilde{g} & \alpha\tilde{x} \\ -\tilde{\mu}\tilde{z} & \bar{\alpha}\tilde{a} & \tilde{f} \end{pmatrix},$$

for $a, y \in M_{q\times p}(\mathcal{D})$, $x, b \in M_{p\times q}(\mathcal{D})$, $c, z \in M_{p\times p}(\mathcal{D})$. The converse of the theorem is proved in Theorem 4.2. □

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