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RELATIVE ASCENT AND DESCENT IN A DOMAIN EXTENSION

Tariq Shah

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Abstract. In this study we continue to investigate the ascent and descent of valuation domains, *PVD*s, *GCD*-domains, $*$-domains, $**$-domains, locally $*$-domains, *URDs, UFD*s, *RBFD*s, *CK*-domains, *BVD*s, *CHFD*s, and a particular case of *LHFD*s for domain extensions $A \subseteq B$ relative to the Condition 1: "Let $A \subseteq B$ be a unitary commutative ring extension. For each $b \in B$ there exist $u \in U(B)$ and $a \in A$ such that $b = au$" and with the further assumption that the conductor ideal $A : B$ is a maximal ideal in A.

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1. Introduction

Following Cohn [9], an integral domain D is said to be *atomic* if each nonzero nonunit element of D is a product of a finite number of irreducible elements (atoms) of D. UFDs and Noetherian domains are well-known examples of atomic domains. An integral domain D satisfies the *ascending chain condition on principal ideals* ($ACCP$) if every ascending chain of principal ideals of D becomes stationary. An integral domain D satisfies ACCP if and only if $D[\{X_\alpha\}]$ satisfies ACCP for any family of indeterminates $\{X_\alpha\}$ (cf. [3, page 5]) but the polynomial extension $D[X]$ is not necessarily an atomic domain when D is an atomic domain [17]. It is well-known that any domain satisfying $ACCP$ is an atomic, but the converse does not hold (cf. [10], see also [23]).

By [3], an atomic domain D is a *bounded factorization domain* (BFD) if for each nonzero nonunit element x of D, there is a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of D, then $n \leq N(x)$.

Krull and Noetherian domains are BFDs ([3, Proposition 2.2]). Also in general a BFD satisfies ACCP but the converse is not true (cf. [3, Example 2.1]).

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Following Zaks [22], an atomic domain D is a *half-factorial domain* (HFD) if for each nonzero nonunit element x of D, if $x = x_1 \cdots x_m = y_1 \cdots y_n$ with each x_i, y_j irreducible in D, then $m = n$. UFDs are examples of HFDs and Krull domains having divisor class group isomorphic to 0 or $\mathbb{Z}_2$ are HFDs. An HFD is a BFD (cf. [3]). By [3, Page 11], if $D[X]$ is an HFD, then certainly D is an HFD. However, $D[X]$ need not be an HFD if D is an HFD. For example the domain $D = \mathbb{R}+X\mathbb{C}[X]$ is an HFD, but $D[Z]$ is not an HFD, as $(X(1+iZ))(X(1-iZ)) = X^2(1+Z^2)$ are decompositions into atoms of different lengths (cf. [3, Page 11]).

By [2, page 217], the *elasticity* of an atomic domain D is defined as

$$\rho(D) = \sup\{m/n : \ x_1 \cdots x_m = y_1 \cdots y_n, \ \text{each } x_i, y_j \in D \text{ is irreducible}\}.$$

Thus $1 \leq \rho(D) \leq \infty$ and $\rho(D) = 1$ if and only if D is an HFD. Infact, the elasticity measures how far an atomic domain D is being an HFD.

By [3], an integral domain D is known as an *idf-domain* if each nonzero element of D has atmost a finite number of non-associate irreducible divisors. UFDs are examples of idf-domains. But there are idf-domains which are not even atomic. Moreover, the Noetherian domain $D = \mathbb{R}+X\mathbb{C}[X]$ is an HFD but not an idf-domain (cf. [3, Example 4.1(a)]).

By [3], an atomic domain D is a *finite factorization domain* (FFD) if each nonzero nonunit element of D has a finite number of non-associate divisors. Hence it has only a finite number of factorizations up to order and associates. An FFD is not an HFD and vice versa. Further, an integral domain D is an FFD if and only if D is an atomic idf-domain (cf. [3, Theorem 5.1]).

In general,

$$\begin{array}{ccccccc}
\textbf{HFD} & \Rightarrow & \textbf{BFD} & \Rightarrow & \textbf{ACCP} & \Rightarrow & \textbf{Atomic} \\
\Uparrow & & \Uparrow & & & & \\
\textbf{UFD} & \Rightarrow & \textbf{FFD} & \Rightarrow & \textbf{idf-domain} & &
\end{array}$$

but none of the above implication is reversible.

Following Cohn [9], an element x of an integral domain D is said to be *primal* if x divides a product a_1a_2 ; $a_1, a_2 \in D,$ then x can be written as $x = x_1x_2$ such that x_i divides a_i, $i = 1, 2$. An element whose divisors are primal elements is called completely primal. A domain D is called a *pre-Schreier* if every nonzero element x of D is primal. An integrally closed pre-Schreier domain is called a *Schreier* domain. By [9], any GCD-domain (an integral domain in which every pair of elements has a greatest common divisor) is a Schreier domain but the converse is not true.

Following Zafrullah [19], an element x of an integral domain D is said to be *rigid* if whenever r, $s \in D$ and r, s divide x, then s divides r or r divides s. Also, D is said to be a *semirigid* domain if every nonzero element of D can be expressed as a product of a finite number of rigid elements.

The ascent and descent of factorization properties for atomic domains, domains satisfying ACCP, BFDs, HFDs, pre-Schreier, semirigid domains, FFDs and idf-domains were studied in [16] and [18] for domain extension $A \subseteq B$ where the conductor ideal $A : B$ is maximal in A and which satisfy Condition 1: For each $b \in B$ there exist $u \in U(B)$ and $a \in A$ such that $b = ua$, where $U(B)$ is the group of units of B.

The purpose of this study is to continue the investigations started in [16] and [18] for ascent and descent of unique factorization domains (UFDs). Specific case is of locally half-factorial domains (LHFDs), congruence half-factorial domains (CHFDs), boundary valuation domains (BVDs), rationally bounded factorization domains (RBFDs), Cohen-Kaplansky domains (CK-domains), valuation domains, GCD-domains, pseudo-valuation domains (PVDs), $*$-domains, $**$-domains, locally $*$-domains and unique representation domains (URDs) relative to Condition 1. Also we have compared it with the pullbacks considered in [7] and [12] to observe the ascent and descent for some of these properties of domains.

2. Preliminaries

We restate the established results regarding ascent and descent of factorization properties for domain extension $A \subseteq B$ relative to the Condition 1 and under the assumption that the conductor ideal $A : B$ is a maximal ideal in A.

Condition 1: *Let $A \subseteq B$ be a unitary (commutative) ring extension and let $U(B)$ represents the set of units of B. For each $b \in B$ there exist $u \in U(B)$ and $a \in A$ such that $b = ua$.*

Recall that for a unitary (commutative) ring extension $A \subseteq B$, the conductor of A in B is the largest common ideal $A : B = \{a \in A : aB \subseteq A\}$ of A and B.

The followings are a few examples of unitary (commutative) ring extensions which satisfy Condition 1.

Example 2.1. [18, Example 1] (a) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1.

(b) If B is a fraction ring of A, then the ring extension $A \subseteq B$ satisfies Condition 1. Hence the ring extension $A \subseteq B$ satisfying Condition 1 is a generalization of localization.

(c) If the ring extensions $A \subseteq B$ and $B \subseteq C$ satisfy Condition 1, then so does the ring extension $A \subseteq C$.

(d) If the ring extension $A \subseteq B$ satisfies Condition 1, then the extensions of rings $A + XB[X] \subseteq B[X]$ and $A + XB[[X]] \subseteq B[[X]]$ satisfy Condition 1.

The following remark provides the examples of domain extensions $A \subseteq B$ satisfying Condition 1 where the conductor ideal $A : B$ is a maximal ideal of A.

Remark 2.2. (i) Let $F \subset K$ be any field extension, the domain extension $A = F + XK[X] \subseteq K[X] = B$ and $C = F + XK[[X]] \subseteq K[[X]] = D$ satisfy Condition 1 where the conductor ideals $A : B$ and $C : D$ are maximal ideals in A and C respectively.

(ii) Let $F \subset K$ be a field extension, where K is a root extension of F and $K(Y)$ is the quotient field of $K[Y]$; then $A = F + XK(Y)[[X]] \subseteq K + XK(Y)[[X]] = B$ satisfies Condition 1 and $A : B = XK(Y)[[X]]$ is the maximal ideal in A.

There are a number of examples of domain extensions $A \subseteq B$ satisfying Condition 1, where the conductor ideal $A : B$ is not a maximal ideal of A. This becomes clear from the following remark.

Remark 2.3. (i) Following [4, Example 5.3], let V be a valuation domain such that its quotient field K is the countable union of an increasing family $\{V_i\}_{i \in I}$ of valuation overrings of V. Let L be a proper field extension of K with L^*/K^* is infinite.

(a) The domain extension $V_i + XL[[X]] \subseteq L[[X]]$ satisfies *Condition 1* since the extension $V_i \subseteq L$ satisfies the Condition 1. But $XL[[X]]$ is not a maximal ideal of $V_i + XL[[X]]$. Also note that $U(V_i + XL[[X]]) \neq U(L[[X]])$.

(b) The domain extension $V_i + XL[[X]] \subseteq K + XL[[X]]$ satisfies Condition 1, but $XL[[X]]$ is not a maximal ideal in $V_i + XL[[X]]$. Also, $U(V_i + XL[[X]]) \neq U(K + XL[[X]])$.

(ii) The domain extension $A = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{Q} + X\mathbb{R}[[X]] = B$ satisfies Condition 1, but the conductor ideal $A : B$ is not a maximal ideal in A.

(iii) The domain extension $A = \mathbb{Z}_{(2)} + X\mathbb{R}[[X]] \subseteq \mathbb{R}[[X]] = E$, satisfies Condition 1, but the conductor ideal $A : E$ is not a maximal ideal in A.

In the following we restate some results from [16] and [18].

Theorem 2.4. [16, Proposition 2.6] *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A.*

(a) *A is atomic if and only if B is atomic.*
(b) *If A is atomic, then $\rho(A) = \rho(B)$.*
(c) *A satisfies ACCP if and only if B satisfies ACCP.*
(d) *A is a BFD if and only if B is a BFD.*
(e) *A is an HFD if and only if B is an HFD.*

Theorem 2.5. [18, Theorem 1] *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is an idf-domain, then B is an idf-domain.*

Theorem 2.6. [18, Theorem 2] *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is an FFD, then B is an FFD.*

Proposition 2.7. [16, Proposition 2.7] *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is a pre-Schreier ring, then B is a pre-Schreier ring.*

Theorem 2.8. [16, Theorem 2.10] *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is a semirigid domain, then B is a semirigid domain.*

3. Ascent and Descent of Atomic Domains

This section is devoted to UFDs, LHFDs, CHFDs, BVDs, RBFDs and CK-domains. We begin with the following remark.

Remark 3.1. (i) If B is a field, then the ring extension $A \subseteq B$ satisfies Condition 1. So in this case if A is UFD, then B is obviously UFD.

(ii) If B is a fraction ring of A, then the ring extension $A \subseteq B$ satisfies Condition 1. Obviously A is a UFD implies that B is a UFD.

(iii) If the ring extension $A \subseteq B$ satisfies Condition 1, then the ring extension $A + XB[X] \subseteq B[X]$ satisfies Condition 1. Now $A + XB[X]$ is a UFD if and only if $A = B$ and B is a UFD, or equivalently $B[X]$ is a UFD.

The observations in Remark 3.1 conclude the following

Proposition 3.2. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is a UFD , then B is a UFD.*

Proof. As A is a UFD, A is atomic and pre-Schreier. Using [16, Proposition 2.6(a) and Proposition 2.7], we get B to be atomic and pre-Schreier. Hence B is a UFD (cf. [21, Page 1895]). □

Remark 3.3. The converse of Proposition 3.2 is not true. For example, the domain extension $A = \mathbb{R} + X\mathbb{C}[X] \subseteq \mathbb{C}[X] = B$ satisfies Condition 1 and $A : B = X\mathbb{C}[X]$ is a maximal ideal in A. $\mathbb{C}[X]$ is a UFD, but $\mathbb{R} + X\mathbb{C}[X]$ is not a UFD. Moreover as $\mathbb{R} + X\mathbb{C}[X]$ is an atomic domain but not pre-Schreier, the descent of a pre-Schreier ring is not necessarily pre-Schreier.

By [5], an integral domain D is a *locally half-factorial domain* ($LHFD$) if each of its localization D_S is an HFD. By the same [5], there is an example of Dedekind HFD D with divisor class group $\mathbb{Z}_6$, but with a non HFD localization.

Now, if we put the restriction the multiplicative system is $S = D - P$, where P is a prime ideal in D. We may represent the ring D_S with one maximal ideal as D_P and with the further assumption that the domain D is an HFD.

Proposition 3.4. *Let B be a domain extension of an HFD A such that $A \subseteq B$ satisfies Condition 1 and $A : B$ is a maximal ideal in A. If B_{PB} is an HFD, then A_P is an HFD, for each prime ideal P of A.*

Proof. Let P be a prime ideal of A. By [16, Proposition 2.2(c)], PB is a prime ideal in B with $PB \bigcap A = P$. Suppose that B_{PB} is an HFD. We first note that $A_P \subseteq B_{PB}$ satisfies Condition 1. Let $x/s \in B_{PB}$ where $x \in B$ and $s \in B - PB$. Choose $u, v \in U(B)$ with $xu, sv \in A$. Then $sv \in A - P$; for $sv \in P$ implies $s = svv^{-1} \in PB$. So $(x/s)(u/v) \in A_P$ where $u/v \in U(B) = U(B_{PB})$. Note that this also shows that $B_{PB} = A_P B = B_{A-P}$. So $A_P \subseteq B_{PB}$ satisfies Condition 1. By [16, Proposition 2.6(e)] A_P is an HFD if $A_P : B_{PB}$ is a maximal ideal of A_P, that is, if $A_P : B_{PB} = PA_P$. If $A_P = B_{PB}$, then A_P is an HFD. So we can assume that $A_P \neq B_{PB}$ and hence $A_P : B_{PB} \subseteq PA_P$. Now $P = A : B$ implies $PA_P = (A : B)_P \subseteq A_P : B_{A-P} = A_P : B_{PB}$. So we have $A_P : B_{PB} = PA_P$. □

By [8], an atomic domain D is a *congruence half-factorial domain* ($CHFD$) of order r, where $1 < r \in \mathbb{Z}^+$, if the equality $\prod_{i=1}^{m} x_i = \prod_{i=1}^{n} y_i$, where x_i and y_i are irreducible elements in D, implies $n \equiv m(mod r)$. Obviously an HFD is a CHFD.

Theorem 3.5. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If B is a CHFD of order r, then A is a CHFD of order r.*

Proof. Assume that B is a CHFD of order r. This means B is atomic and by [16, Proposition 2.6(a)], A is atomic. Now for any $0 \neq a \in A - U(A)$ we have $a = \prod_{i=1}^{m} x_i$, where the x_i's are irreducible elements in A. Suppose that there exists another factorization $\prod_{i=1}^{n} y_i$ of a, where y_i's are irreducible elements in A. Since the x_i's and y_i's are irreducible elements in B (cf. [16, Theorem 2.5(d)], we have two irreducible factorizations of a in B. Since B is a CHFD, we have $n \equiv m(mod \mathrm{r})$, where $r > 1$. Hence A is a CHFD. □

By [14, Definition 12], the domain extension $A \subseteq B$ is said to satisfy the property $(*)$ if for any $0 \neq b \in B$ (1) $b = ua$, where $u \in U(B)$ and $a \in A$, and (2) $b = ua = u_1a_1$ ($u, u_1 \in U(B)$ and $a, a_1 \in A$) implies that $u/u_1 \in U(A)$.

Now, if our Condition 1 is replaced by property $(*)$ then the converse of Theorem 3.5 is obtained.

Theorem 3.6. *Let $A \subseteq B$ be a domain extension which satisfies $(*)$ and $M = A : B$ is a maximal ideal in A. If A is a CHFD of order r, then B is a CHFD of order r.*

Proof. Assume that A is a CHFD of order r. This means A is atomic and by [16, Proposition 2.6(a)], B is atomic. For any $0 \neq z \in B - U(B)$, let $z = \prod_{i=1}^{m} x_i = \prod_{i=1}^{n} y_i$, where the x_i's and y_i's are irreducible elements in B. Then by (*) and [16, Theorem 2.5(c)] $z = \prod_{i=1}^{m} a_i u_i = \prod_{i=1}^{n} c_i v_i$, where u_i's, v_i's are units in B and a_i's, c_i's are irreducible elements in A. Say $\prod_{i=1}^{m} u_i = u, \prod_{i=1}^{n} v_i = v \in U(B)$, so we have $u\prod_{i=1}^{m} a_i = v\prod_{i=1}^{n} c_i$ and therefore $(u/v)\prod_{i=1}^{m} a_i = \prod_{i=1}^{n} c_i$. Since $u/v \in U(A)$, hence $((u/v)a_1 \prod_{i=2}^{m} a_i = \prod_{i=1}^{n} c_i$ implies $n \equiv m(mod \mathrm{r})$, as A is a CHFD of order r. Thus B is a CHFD. □

Let D be an HFD with quotient field K. If $D \neq K$, then by [15], we define the boundary map $\delta_D : K^* \to \mathbb{Z}$ by $\delta_D(\alpha) = t - s$, where $\alpha = (x_1 \cdots x_t)/(y_1 \cdots y_s)$ with x_i's, y_j's irreducible elements in D.

By [15], an integral domain D with quotient field K, is called *boundary valuation domain* (BVD) if D is an HFD and for any $\alpha \in K^*$ with $\delta_D(\alpha) \neq 0$, either $\alpha \in D$ or $\alpha^{-1} \in D$, where δ_D is a boundary map defined on D.

Theorem 3.7. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A, K_A and K_B denote the quotient fields of A and B respectively. If A is a BVD, then B is a BVD.*

Proof. A is a BVD implies A is an HFD. Then by [16, Theorem 2.6(e)] B is an HFD. For $a \in K_B^*$, we have $a = (b/d)$, where $b, d \in B - 0$ This implies $a = (b/d) = ((ab')/(cd'))$, where $a, c \in A$, $b', d' \in U(B)$. This means $\alpha = (b/d) = ((a_1 \cdots a_s b')/(c_1 \cdots c_t d'))$, where $a_1, ..., a_s, c_1, ..., c_t$ are irreducible elements in A and by [16, Theorem 2.5(d)] are also irreducible elements in B. Obviously $u = ((b')/(d')) \in U(B)$ and therefore $a = (b/d) = (a/c)u = ((a_1 \cdots a_s)/(c_1 \cdots c_t))u$. Since $\delta_B(\alpha) \neq 0$ implies $\delta_A(a/c) \neq 0$, therefore either $a/c \in A$ or $c/a \in A$. This implies $(a/c)u \in B$ or $(c/au) \in B$. Hence B is a BVD. □

By [2], an integral domain D is called a *rationally bounded factorization domain* ($RBFD$) if D is an atomic domain and $\rho(D) < \infty$.

Proposition 3.8. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. Then A is an RBFD if and only if B is an RBFD.*

Proof. Follows from [16, Proposition 2.6(a,b,d)]. □

By [1], an atomic domain D is said to be a *Cohen-Kaplansky domain* (CK-domain) if it has finitely many non-associate irreducible elements. A semilocal PID is an example of CK-domain. Hence a DVR is a CK-domain. But $\mathbb{R} + X\mathbb{C}[[X]]$ is a one dimensional local domain which is not a CK-domain (cf.[1, page 27]). This means for a domain extension $A \subseteq B$ satisfying Condition 1 such that $M = A : B$ is a maximal ideal in A, the descent is not possible for a CK-domain. Although if $K \subset F$ is a finite field extension, then $K + XL[[X]]$ is a local CK-domain (cf.[1, page 18]).

4. Ascent and Descent of Non-atomic Domains

In this section we discuss the ascent and descent of valuation domains, PVDs, GCD-domains, URDs, $*$-domains, $**$-domains and locally $*$-domains in domain extensions $A \subseteq B$ relative to the Condition 1.

By [13, page 12], an integral domain D with quotient field K, is said to be a *valuation domain* if it satisfies either of the (equivalent) condition:

(1) For any two elements $x, y \in D$, either x divides y or y divides x.

(2) For any element $x \in K$, either $x \in D$ or $x^{-1} \in D$.

Theorem 4.1. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1. If A is a valuation domain, then B is a valuation domain.*

Proof. Suppose A be a valuation domain. Let $x, y \in B$, then there exist $x_1, y_1 \in A, u, v \in U(B)$ such that $x = ux_1, y = vy_1$. As A is a valuation domain, x_1 divides y_1 or y_1 divides x_1. If x_1 divides y_1 in A, then there exists $a \in A$ such that $y_1 = ax_1$. We may write $y = vy_1 = vau^{-1}ux_1 = cx$, where $vau^{-1} = c \in B$. Hence x divides y in B. Similarly, if y_1 divides x_1 in A, then y divides x in B. Thus B is a valuation domain. □

Following [7], let A and B be any commutative rings with $A \subseteq B$, and $I = A : B$ be the common nonzero conductor ideal of B into A. Setting $D = A/I$ and $E = B/I$, we obtain the natural surjections $\pi_1 : B \longrightarrow E$ and $\pi_2 : A \longrightarrow D$ and the inclusions $i_1 : D \hookrightarrow E$ and $i_2 : A \hookrightarrow B$. These maps yield a commutative diagram

$$\begin{array}{ccc} A = \pi_1^{-1}(D) & \xrightarrow{\pi_2} & D \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_1} \\ B & \xrightarrow{\pi_1} & E \end{array}$$

called *a conductor square* □, which defines A as a pullback of π_1 and i_1.

The conductor square □ of Boynton [7] is a special case of the pullback of type □ of Houston and Taylor [12] in which I is any nonzero common ideal of A and B.

The pullbacks considered in this study are in fact a Boynton's [7] conductor square where we assume that the conductor ideal $I = A : B$ is a maximal ideal in A.

Remark 4.2. The converse of Theorem 4.1 is not true, as in the pullback

$$\begin{array}{ccc} A = \mathbb{R} + X\mathbb{C}[[X]] & \rightarrow & A/(A : B) \simeq \mathbb{R} \\ \downarrow & & \downarrow \\ B = \mathbb{C}[[X]] & \rightarrow & B/(A : B) \simeq \mathbb{C} \end{array}$$

$C[[X]]$ is a valuation domain, but $R + XC[[X]]$ is not a valuation domain and obviously the domain extension $A \subseteq B$ satisfies Condition 1. Moreover as $qf(A/(A : B)) \neq qf(B/(A : B))$, it does not satisfy the assumption of [12, Theorem 1.3], though $A : B$ is a maximal and hence a prime ideal in B.

In [11], an integral domain D with quotient field K, is said to be a *pseudo-valuation domain* (PVD) if, whenever P is a prime ideal in D and $xy \in P$, where $x, y \in K$, then $x \in P$ or $y \in P$. A valuation ring is a PVD, but converse is not necessarily true.

An element d of a ring R is said to be a proper divisor of an element b of R if $b = md$ for some non unit m of R. A commutative ring R with identity is said to be a PVR if and only if for every $a, b \in R$, either a divides b or d divides a for every proper divisor d of b (cf. [6, Proposition 4]).

Theorem 4.3. *Let $A \subseteq B$ be a (unitary) commutative ring extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A. If A is a PVR, then B is a PVR.*

Proof. Let $b_1, b_2 \in B$ such that $b_1 = a_1u_1, b_2 = a_2u_2$, where $a_1, a_2 \in A$ and $u_1, u_2 \in U(B)$. Since A is a PVR, by [6, Proposition 4(d)] either a_1 divides a_2 in A or if c divides a_2 in A (where c is the proper divisor) implies c divides a_1 in A. Obviously a_1 divides a_2 implies b_1 divides b_2 in B. Now suppose b_1 does not divide b_2 in B (then of course a_1 does not divide a_2 in A), so c divides a_1 in A implies c divides b_1 in B, whereas c divides b_2 in B. Hence B is a PVR. □

We do not know the ascent of GCD-domains for domain extensions $A \subseteq B$ which satisfies Condition 1 such that $M = A : B$ is a maximal ideal in A. The domain extension $A = \mathbb{Z} + X\mathbb{Q}[X] \subseteq \mathbb{Q}[X] = B$ satisfies Condition 1, but the conductor ideal $X\mathbb{Q}[X]$ is not a maximal ideal in A. Of course A and B both are GCD-domains. On the other hand the descent is not possible, as: $A = \mathbb{Q} + X\mathbb{R}[X] \subseteq \mathbb{R}[X] = B$ satisfies Condition 1 and $A : B = X\mathbb{R}[X]$ is maximal ideal in A. Obviously B is a GCD-domain, but A is not a GCD-domain.

By [20], a *unique representation domain* (URD) is a GCD-domain whose nonzero nonunit elements are expressible as a product of finitely many packets (by a *packet* x, we mean for every factorization $x = ab$; a divides b^2 or b divides a^2(cf. [20, Lemma 4])). The finite product of mutually coprime packets resembles the canonical representation $up_1^{a1} \cdots p_n^{an}$ of an element in a UFD, where u is a unit and p_i's are non-associate primes and $a_1, a_2, .., a_n \in \mathbb{Z}^+$(see [20, Page 19]).

Remark 4.4. Let $A \subseteq B$ be a domain extension such that $x \in A$ is a packet. So for every factorization $x = x_1x_2$; x_1 divides x_2^2 in A or x_2 divides x_1^2 in A. This implies x_1 divides x_2^2 in B or x_2 divides x_1^2 in B. Hence x is a packet in B.

Proposition 4.5. *Let $A \subseteq B$ be a domain extension which satisfies Condition 1 such that B is a GCD-domain. If A is a URD, then B is a URD.*

Proof. For $b \in B$, we have $b = au$, where $u \in U(B)$ and $a \in A$. Now A is a URD, so $a = vx_1^{a1} \cdots x_n^{an}$, where $v \in U(A)$ and the x_i's are packets in A. Thus

$b = (vu)x_1^{a1} \cdots x_n^{an}$, where $vu \in U(B)$ and the x_i's are packets in B. Hence B is a URD. □

Remark 4.6. Let A be an integral domain and S be a multiplicative system in A.

(i) If A is a URD, then $B = S^{-1}A$ is URD (cf. [20, Propositions 6]). So the domain extension $A \subseteq B$ satisfies Condition 1.

(ii) Let A be a URD such that $B = A + XS^{-1}A[X]$ is a GCD-domain, then $B = A + XS^{-1}A[X]$ is a URD (cf. [20, Propositions 7]). But the domain extension $A \subseteq B$ does not satisfy Condition 1.

(iii) Let A be a UFD, then $B = A + XS^{-1}A[X]$ is a URD for every multiplicative system S (cf. [20, Corollary 3]). But the domain extension $A \subseteq B$ does not satisfy Condition 1.

(iv) If $B = A + XS^{-1}A[X]$ is URD, then A is URD (cf. [20, Propositions 7]). But the domain extension $A \subseteq B$ does not satisfy Condition 1.

We recall the following from [21].

Let D be an integral domain.

property-∗ : $(\cap_i(a_i))(\cap_j(b_j)) = \cap_{i,\,j}(a_ib_j)$ for all a_i, $b_j \in D$, where $i = 1, .. , m$ and $j = 1, .., n$.

property-∗∗ : $((a) \cap (b))((c) \cap (d)) = (ac) \cap (ad) \cap (bc) \cap (bd)$, where a, b, c, $d \in D^*$.

An integral domain D is called *∗-domain* (respectively *∗∗-domain*) if it satisfies property-∗ (respectively property-∗∗). D is said to be a *locally ∗-domain* if for each maximal ideal M, D_M has property-∗.

Theorem 4.7. *Let A be a PVD which is not a valuation domain. Let $A \subseteq B$ is a domain extension which satisfies Condition 1 and $M = A : B$ is a maximal ideal in A.*

(1) If A is a ∗-domain, then B is a ∗-domain.

(2) If A is a locally ∗-domain, then B is a locally ∗-domain.

(3) If A is a ∗∗-domain, then B is a ∗∗-domain.

Proof. (1) By [21, Theorem 4.4] A is pre-Schreier and B is a pre-Schreier (cf. [16, Proposition 2.7]). Thus the result follows by [21, page 1896].

(2) By [21, Theorem 2.1], A is a ∗-domain. From part (1), B is a ∗-domain. Hence B is a locally ∗-domain by [21, Theorem 2.1].

(3) Let A be a $**$-domain. This implies A is a pre-Schreier ring, by [21, Theorem 4.4] and hence B is a pre-Schreier ring due to [16, Proposition 2.7]. By Theorem 4.3 B is a PVD and [21, Theorem 4.4] yields B is a $**$-domain. □

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Tariq Shah
Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
e-mail: stariqshah@gmail.com