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WHEN DOES A RING EXTENSION OF A GOING-DOWN DOMAIN SATISFY GOING-DOWN?

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Abstract. If R is a going-down domain and T is a commutative unital ring extension of R, then $R \subseteq T$ satisfies going-down if and only if the associated reduced ring of T is a torsion-free R-module.

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1. Introduction

All rings considered below are commutative with $1 \neq 0$; all ring extensions and subrings are unital. If A is a ring, then Spec(A) denotes the set of prime ideals of A; Max(A) the set of maximal ideals of A; $Z(A)$ the set of zero-divisors of A; $\mathrm{tq}(A) := A_{A \setminus Z(A)}$ the total quotient ring of A; $\sqrt{I}$ the radical of a proper ideal I of A; $\sqrt{A} := \sqrt{0}$ the nilradical of A; and $A_{\mathrm{red}} := A/\sqrt{A}$ the associated reduced ring of A. As in [9, page 28], GD, GU and LO denote the going-down, going-up and lying-over properties, respectively, of ring extensions (more generally, of ring homomorphisms).

Our interest in this note is in finding a torsion-theoretic characterization of the ring extensions of a going-down domain that satisfy GD. We devote the rest of this paragraph to some background on going-down domains, and then turn to the appropriateness of torsion-theoretic considerations. Let R be a (commutative integral) domain. As in [2], R is said to be a *going-down domain* if $R \subseteq T$ satisfies GD for each overring T of R. (As usual, an *overring* of a domain D is any ring contained between D and its quotient field.) The most familiar examples of going-down domains are arbitrary Prüfer domains and domains of Krull dimension at

Dedicated to the memory of my friend and collaborator, Jim (James A.) Huckaba.

most 1. It was shown in [6, Theorem 1] that if R is a going-down domain, then $R \subseteq T$ satisfies GD for each domain T that contains R as a subring. More generally, it was shown recently by Shapiro and the author [7, Theorem 3.6] that if R is a going-down domain, then $R \subseteq T$ satisfies GD for each ring extension T of R such that (i) T is a torsion-free R-module and (ii) $\mathrm{tq}(T)$ is a von Neumann regular ring.

In particular, many ring extensions of a going-down domain satisfy GD. Nevertheless, each going-down domain D which is not a field has some extensions B such that $D \subseteq B$ does not satisfy GD. In fact, for each ring A of positive Krull dimension, there exists a ring extension B of A such that $A \subseteq B$ does not satisfy GD. (In detail, if $P_2 \subset P_1$ are distinct prime ideals of A, embed A in $B := A \times A/P_1$ via $a \mapsto (a, a + P_1)$, and note that $Q := A \times 0 \in \mathrm{Spec}(B)$ is a minimal prime ideal of B such that $Q \cap A = P_1$.) The question thus arises whether there is a torsion-theoretic characterization of GD when the base ring is a going-down domain.

In order to avoid hypotheses such as (ii) concerning total quotient rings, we will focus here on torsion-theoretic criteria in the spirit of (i). Note that (i) has a significant pedigree. Indeed, (i) appears as an assumption in the classical going-down theorem of Cohen-Seidenberg [1, Theorem 5]. The above question (which is basically also the question that was raised in the title) is answered in Corollary 2.10: if R is a going-down domain and T is a ring extension of R, then: $R \subseteq T$ satisfies GD $\Leftrightarrow$ $(T_{R \setminus M})_{\mathrm{red}}$ is a torsion-free R_M-module for each $M \in \mathrm{Max}(R)$ $\Leftrightarrow$ T_{red} is a torsion-free R-module.

The proof of Corollary 2.10 depends in part on some results, Lemma 2.1 and Theorem 2.2, whose hypotheses involve reduced rings and locally divided domains, respectively. Recall that a ring A is called *reduced* if A has no nonzero nilpotent elements. Also, recall from [3] that if R is a domain, then R is called a *divided domain* if $PR_P = P$ for each $P \in \mathrm{Spec}(R)$; and that R is called a *locally divided domain* if R_Q is a divided domain for each prime (resp., maximal) ideal Q of R. A quasilocal locally divided domain is the same as a divided domain. Each divided domain is a quasilocal going-down domain [3, Proposition 2.1], but the converse is false [3, Example 2.9]. A key fact used in the proof of Theorem 2.7 (which is needed for the proof of Corollary 2.10) is that quasilocal going-down domains are characterized as the domains having an integral divided overring [3, Theorem 2.5]. Finally, we note that Remark 2.3 (a) gives an example using the idealization construction to show that the "reduced" assumption cannot be deleted from Lemma 2.1; and that our final result, Corollary 2.11, establishes that the torsion-theoretic condition from Corollary 2.10 actually characterizes going-down domains.

2. Results

We begin with an important case where GD implies the "torsion-free" condition.

Lemma 2.1. *Let R be a domain, and let T be a ring extension of R such that T is a reduced ring and $R \subseteq T$ satisfies GD. Then T is a torsion-free R-module.*

Proof. According to a characterization of GD in [9, Exercise 37, page 44], a ring extension $A \subseteq B$ satisfies GD if and only if, for each $P \in \mathrm{Spec(A)}$, the torsion submodule of B/PB as an (A/P)-module is contained in $\sqrt{PB}/PB$. Applying this characterization to our data and $P := 0 \in \mathrm{Spec(R)}$, we see that the torsion submodule of T as an R-module is contained in the nilradical of T. This ideal is 0 since T is reduced, and so the assertion follows. □

We next give some sufficient conditions for GD.

Theorem 2.2. *Let R be a locally divided domain, and let T be a ring extension of R such that T is a torsion-free R-module. Then $R \subseteq T$ satisfies GD.*

Proof. Suppose that the assertion fails. According to a characterization of GD in [9, Exercise 37, page 44], there exist prime ideals P of R and Q of T such that PT is not disjoint from $(R \setminus P)(T \setminus Q)$ and Q is minimal among the prime ideals of T that contain P. Hence, $\sum_{i=1}^{n} p_i t_i = rt$ for some elements $p_i \in P, t_i \in T, r \in R \setminus P$ and $t \in T \setminus Q$. Put $\mathfrak{P} := Q \cap R$. Note that $P \subset \mathfrak{P}$ are distinct; in fact, $r \in \mathfrak{P} \setminus P$ since Q is a prime ideal of T.

Since $R_{\mathfrak{P}}$ is a divided domain, $(PR_{\mathfrak{P}})(R_{\mathfrak{P}})_{PR_{\mathfrak{P}}} = PR_{\mathfrak{P}}$. The upshot is that $PR_P = PR_{\mathfrak{P}}$. (In fact, requiring this equality whenever $P \subseteq \mathfrak{P}$ are prime ideals is equivalent to the ambient domain R being a locally divided domain [4, Theorem 2.4].) Let K denote the quotient field of R, and let $L := \mathrm{tq}(T)$. Since T is a torsion R-module, the inclusion map $R \hookrightarrow T \hookrightarrow L$ extends to a (necessarily injective) R-algebra homomorphism $K \to L$, and so we may view $K \subseteq L$.

Working inside L, we see that for each i, $p_i/r \in PR_P = PR_{\mathfrak{P}}$. Thus, $p_i/r = \xi_i/z$ for some elements $\xi_i \in P$, $z \in R \setminus \mathfrak{P}$. Hence, $\sum \xi_i t_i \in PT \subseteq Q$, although

$$\sum \xi_i t_i = \sum z \frac{p_i}{r} t_i = tz,$$

with both t and z in $T \setminus Q$, contradicting $Q \in \mathrm{Spec}(T)$. □

Remark 2.3. (a) The assumption that T is reduced cannot be deleted from Lemma 2.1. To see this, let R be any domain that is not a field, choose $r \in R$ to be any nonzero nonunit element of R, and let T be the idealization $T := R(+)R/Rr$. (A convenient reference for background on the idealization construction is [8].)

View $R \subseteq T$ in the usual way, via the embedding $a \mapsto (a, 0)$. Note that $R \subseteq T$ satisfies GD. This can be seen by using the fact that $\mathrm{Spec}(T) = \{P(+)R/Rr \mid P \in \mathrm{Spec}(R)\}$; or by noticing that $T_{\mathrm{red}} = T/\sqrt{T} = T/(0(+)R/Rr) \cong R$ and appealing to [5, Lemma 3.2 (a)]. Of course, T is not reduced; and T is not a torsion-free R-module (since $r \cdot (0, 1 + Rr) = 0$).

(b) Another proof of Theorem 2.2 can be given for the case where R is quasilocal (that is, a divided domain). One need only apply the following special case of [7, Proposition 3.2]: if D is a divided domain and B is a ring extension of D such that B is a torsion-free D-module, then $D \subseteq B$ satisfies GD.

(c) The sufficient conditions for GD in Theorem 2.2 should be contrasted with the sufficient conditions for GD in the classical going-down theorem of Cohen-Seidenberg [1, Theorem 5]. While both results include the assumptions that the given base ring R is a domain and the given extension ring T is a torsion-free R-module, [1, Theorem 5] also assumes that R is integrally closed and that T is integral over R. Easy examples, such as $\mathbb{Z}[2i] \subseteq \mathbb{Q}(i)$ (where $i = \sqrt{-1} \in \mathbb{C}$), show that the hypotheses of Theorem 2.2 do not imply either of these assumptions from [1, Theorem 5].

Corollary 2.4. *Let R be a locally divided domain, and let T be a ring extension of R such that T is a reduced ring. Then $R \subseteq T$ satisfies GD if and only if T is a torsion-free R-module.*

Proof. Combine Lemma 2.1 and Theorem 2.2. □

We next characterize the going-down extensions of any locally divided domain.

Corollary 2.5. *Let R be a locally divided domain and let T be a ring extension of R. Then $R \subseteq T$ satisfies GD if and only if T_{red} is a torsion-free R-module.*

Proof. The inclusion map $R \hookrightarrow T$ induces a ring homomorphism $R_{\mathrm{red}} = R/\sqrt{R} \cong R \to T/\sqrt{T} = T_{\mathrm{red}}$. The latter map is an injection since $\sqrt{T} \cap R = \sqrt{R}$, and so we may view $R_{\mathrm{red}} \subseteq T_{\mathrm{red}}$. In view of Corollary 2.4, it now suffices to recall that a given ring extension (more generally, ring homomorphism) $A \subseteq B$ satisfies GD if and only if the induced ring extension $A_{\mathrm{red}} \subseteq B_{\mathrm{red}}$ satisfies GD [5, Lemma 3.2 (a)]. □

Lemma 2.6. *Let R be a domain and T an extension ring of R such that T is a torsion-free R-module. Then T_{red} is a torsion-free R-module.*

Proof. Suppose that $r\bar{t} = 0$ for some elements $r \in R \cong R_{\mathrm{red}}$ and $\bar{t} \in T_{\mathrm{red}}$. We will show that either $r = 0$ or $\bar{t} = 0$. Choose $t \in T$ such that $\bar{t} = t + \sqrt{T}$. Then

$rt \in \sqrt{T}$ (since $r\bar{t} = rt + \sqrt{T}$), and so $r^n t^n = 0$ for some positive integer n. Since T is torsion-free over R, either $r^n = 0$ or $t^n = 0$. Without loss of generality, $r \neq 0$. Hence, $r^n \neq 0$, and so $t^n = 0$. Thus, $t \in \sqrt{T}$, whence $\bar{t} = 0 \in T_{\text{red}}$. □

We next obtain a companion for Corollary 2.5.

Theorem 2.7. *Let R be a quasilocal going-down domain and let T be a ring extension of R. Then $R \subseteq T$ satisfies GD if and only if T_{red} is a torsion-free R-module.*

Proof. Recall from [5, Lemma 3.2 (a)] that $R \subseteq T$ satisfies GD if and only if $R \cong R_{\text{red}} \subseteq T_{\text{red}}$ satisfies GD. Moreover, $(T_{\text{red}})_{\text{red}} \cong T_{\text{red}}$. Thus, without loss of generality, we may suppose that T is a reduced ring. Under this assumption, we will prove that $R \subseteq T$ satisfies GD if and only if T is a torsion-free R-module.

The "only if" assertion is immediate from Lemma 2.1. Conversely, suppose that (the reduced ring) T is a torsion-free R-module. We must prove that $R \subseteq T$ satisfies GD. To that end, consider $P_2 \subseteq P_1$ in $\text{Spec}(R)$ and Q_1 in $\text{Spec}(T)$ such that $Q_1 \cap R = P_1$. Our task is to find Q_2 in $\text{Spec}(T)$ such that $Q_2 \subseteq Q_1$ and $Q_2 \cap R = P_2$. Since R is a quasilocal going-down domain, [3, Theorem 2.5] provides a divided integral overring D of R; necessarily, $R \subseteq D$ is unibranched, in the sense that the canonical map $\text{Spec}(D) \to \text{Spec}(R)$, $I \mapsto I \cap R$, is a bijection. Let K denote the quotient field of R, and let $L := \text{tq}(T)$. As T is a torsion-free R-module, we may view $K \subseteq L$. Since D is integral over R, $R \subseteq D$ satisfies LO and GU (cf. [9, Theorem 44]), and so there exist $p_2 \subseteq p_1$ in $\text{Spec}(D)$ such that $p_i \cap R = P_i$, for $i = 1, 2$. Next, working inside L, consider $E := DT$, the subring of L generated by $D \cup T$. Note that $T \subseteq E$ satisfies LO since $T \subseteq E$ inherits integrality from $R \subseteq D$, and so there exists $q_1 \in \text{Spec}(E)$ such that $q_1 \cap T = Q_1$. As $(q_1 \cap D) \cap R = q_1 \cap R = (q_1 \cap T) \cap R = Q_1 \cap R = P_1$, the fact that $R \subseteq D$ is unibranched implies that $q_1 \cap D = p_1$. We claim that $D \subseteq E$ satisfies GD.

To prove the above claim, we will apply Theorem 2.2. In order to legitimize that application, we must verify that E is a torsion-free D-module. Suppose that $\delta e = 0$ for some $\delta \in D, e \in E = DT$. We can write $e = \sum_{i=1}^{n} d_i t_i$ for some $d_i \in D$, $t_i \in T$. Also, $d_i = \frac{r_i}{s}$ and $\delta = \frac{r}{s}$ for some $r_1, \ldots, r_n, r \in R$ and $s \in R \setminus \{0\}$. Put $f := \sum_{i=1}^{n} r_i t_i$. Note that $rf = s^2 \delta e = 0$ (in L and, hence, also in T). As T is torsion-free over R, either $r = 0$ or $f = 0$. Hence, either $\delta = r/s = 0$ or $e = f/s = 0$, thus proving the claim.

By the above claim, there exists $q_2 \in \text{Spec}(E)$ such that $q_2 \subseteq q_1$ and $q_2 \cap D = p_2$. Then $Q_2 := q_2 \cap T$ has the required properties, namely, $Q_2 \subseteq q_1 \cap T = Q_1$ and $Q_2 \cap R = q_2 \cap T \cap R = q_2 \cap R = q_2 \cap D \cap R = p_2 \cap R = P_2$.

□

Corollary 2.8. *Let R be a quasilocal going-down domain, and let T be a ring extension of R such that T is a torsion-free R-module. Then $R \subseteq T$ satisfies GD.*

Proof. Combine Lemma 2.6 and Theorem 2.7. □

The next technical lemma will simplify matters.

Lemma 2.9. *Let R be a domain and T a ring extension of R. Then:*

(a) For each prime ideal P of R, the rings $(T_{R\setminus P})_{\mathrm{red}}$ and $(T_{\mathrm{red}})_{R\setminus P}$ are isomorphic as R_P-modules.

(b) The following three conditions are equivalent:

(1) $(T_{R\setminus P})_{\mathrm{red}}$ is a torsion-free R_P-module for every prime ideal P of R;

(2) $(T_{R\setminus M})_{\mathrm{red}}$ is a torsion-free R_M-module for every maximal ideal M of R;

(3) T_{red} is a torsion-free R-module.

Proof. (a) Let $A := (T_{R\setminus P})_{\mathrm{red}}$ and $B := (T_{\mathrm{red}})_{R\setminus P}$. It is easy to use universal mapping properties to verify that the (well-defined) functions $f : A \to B$ given by $t/z + \sqrt{T_{R\setminus P}} \mapsto (t + \sqrt{T})/z$ (for $t \in T$ and $z \in R \setminus P$) and $g : B \to A$ given by $(t + \sqrt{T})/z \mapsto t/z + \sqrt{T_{R\setminus P}}$ are ring homomorphisms, in fact R_P-algebra homomorphisms, such that $f \circ g$ and $g \circ f$ are the appropriate identity functions.

(b) It is easy to verify that if E is an R-module, then: E is a torsion-free R-module $\Leftrightarrow$ $E_{R\setminus M}$ is a torsion-free R_M-module for each $M \in \mathrm{Max}(R)$ $\Leftrightarrow$ $E_{R\setminus P}$ is a torsion-free R_P-module for each $P \in \mathrm{Spec}(R)$. Hence, (b) follows from (a) by letting $E := T_{\mathrm{red}}$. □

Since GD is a local property of ring extensions, Theorem 2.7 and Lemma 2.9 lead to the following characterization of the going-down extensions of any going-down domain.

Corollary 2.10. *Let R be a going-down domain and let T be a ring extension of R. Then the following conditions are equivalent:*

(1) $(T_{R\setminus P})_{\mathrm{red}}$ is a torsion-free R_P-module for every prime ideal P of R;

(2) $(T_{R\setminus M})_{\mathrm{red}}$ is a torsion-free R_M-module for every maximal ideal M of R;

(3) T_{red} is a torsion-free R-module;

(4) $R \subseteq T$ satisfies GD.

As noted in the proof of Lemma 2.1, [9] already contains a torsion-theoretic characterization of GD (for arbitrary ring extensions). We close by pointing out that the torsion-theoretic condition that we have identified in this note is specific to, and in fact serves to characterize, going-down domains. One upshot of the reasoning in Corollary 2.11 is that the assumptions of "locally divided domain" and "(quasilocal) going-down domain" in the earlier results cannot be deleted.

Corollary 2.11. *Let R be a domain. Then the following conditions are equivalent:*

(1) If T is a ring extension of R, then $R \subseteq T$ satisfies GD if and only if, for every prime ideal P of R, $(T_{R\setminus P})_{\text{red}}$ is a torsion-free R_P-module;

(2) If T is a ring extension of R, then $R \subseteq T$ satisfies GD if and only if, for every maximal ideal M of R, $(T_{R\setminus M})_{\text{red}}$ is a torsion-free R_M-module;

(3) If T is a ring extension of R, then $R \subseteq T$ satisfies GD if and only if T_{red} is a torsion-free R-module;

(4) If T is a ring extension of R and $(T_{R\setminus P})_{\text{red}}$ is a torsion-free R_P-module for every prime ideal P of R, then $R \subseteq T$ satisfies GD;

(5) If T is a ring extension of R and $(T_{R\setminus M})_{\text{red}}$ is a torsion-free R_M-module for every maximal ideal M of R, then $R \subseteq T$ satisfies GD;

(6) If T is a ring extension of R and T_{red} is a torsion-free R-module, then $R \subseteq T$ satisfies GD;

(7) R is a going-down domain.

Proof. Lemma 2.9 (b) gives that conditions (1), (2), and (3) are equivalent. Moreover, (7) implies these equivalent conditions, by Corollary 2.10. Lemma 2.9 (b) gives that conditions (4), (5), and (6) are equivalent; and it is trivial that (3) $\Rightarrow$ (6). Finally, if (6) holds, it follows that $R \subseteq T$ satisfies GD for each overring T of R, and so R is a going-down domain, thus giving (7). □

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