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NEW CRITERIA FOR p-NILPOTENCE OF FINITE GROUPS

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Abstract. A subgroup H is X-g pronormal in G if, for $H, X \leq G$ and $g \in G$, $H \cap X$ and $H^g \cap X$ are conjugate in $J = \langle H \cap X, H^g \cap X \rangle$. In this paper, we investigate the structure of a finite group G under the assumption that certain subgroups are X-g pronormal, where $X = F(G)$ is the Fitting subgroup of G.

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1. Introduction

All groups in this paper are finite. P. Hall introduced the conception of pronormality (see [9, p241]). Asaad in [1] proved that if G is a finite group of odd order n in which every minimal subgroup is pronormal in G, then G is supersoluble. D'Anielio in [7] introduced the notion of dualpronormality, and gave the structure of finite groups such that the n-maximal subgroups are dualpronormal. Bianchi etc in [5] introduced the notion of H-subgroups. H-subgroups were studied by Asaad in [2], and Csörgö and Herzog in [6]. Recently, X-g pronormality was introduced by Dark and Feldman in [8]. In this note, we will prove the following results:

Theorem 1. Let G be a group which has no section isomorphic to A_4, where A_4 is a alternating group of degree 4. Suppose that P is a Sylow p-subgroup of G, and that for every subgroup of P of order p or 4 (when $p = 2$) is $F(G)$-g pronormal in G, for all g in G. Then G is p-nilpotent.

Theorem 2. Let G be a group which has no section isomorphic to A_4, where A_4 is a alternating group of degree 4. Suppose that P is a Sylow p-subgroup of G, and that for every maximal subgroup of P is $F(G)$-g pronormal in G, for all g in G. Then G is p-nilpotent.

For some other notions and notations, the reader is referred to Robinson [13], and Gorenstein [10].

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2. Preliminaries

Definition 2.1. ([8, p. 780]) If $X, H \leq G$, and $g \in G$, we say that H is X-g *pronormal* if $H \cap X$ and $H^g \cap X$ are conjugate in $J = \langle H \cap X, H^g \cap X \rangle$, that is, there exists an $x \in J$ such that $H^g \cap X = H^x \cap X$.

Note that H is pronormal in G if H is G-g pronormal for all $g \in G$. So we have pronormality dose not implies X-g pronormality. Let $G = A_5$ and $X = F(G)$, then $X = 1$. If P is a Sylow 2-subgroup of G, P is pronormal in G by [10, p13, Exersices, 4(i)]. If P is X-g pronormal in G, then $P^g \cap X = (P \cap X)^x = P^x \cap X$, where $g \in G$ and $x \in J = \langle P^g \cap X, P \cap X \rangle$. But $J = 1$ for all $g \in G$. And so $x = 1$, a contradiction.

Lemma 2.2. *Let G be a group and let H, K, X, L be subgroups of G satisfying that H is X-g pronormal in G, $H \leq K$ and L is normal in G. Then the following hold:*

(1) H is X-g pronormal in K;

(2) If P is a Sylow p-subgroup of G, then $P \cap L$ is X-g pronormal in G;

(3) If $H \cap X \leq K \cap X \leq N_G(H \cap X)$, then $N_G(K \cap X) \leq N_G(H \cap X)$

(4) If H is subnormal in K, then H is normal in K;

(5) If $L \leq H$, then H/L is XL/L-g pronormal in G/L;

(6) If H is a p-group of G, and $(|H|, |L|) = 1$, then HL is X-g pronormal in G and HL/L is XL/L-g pronormal in G/L.

Proof. (1) By the definition of X-g pronormality, we can easily have the result.

(2) Since P is a Sylow p-subgroup, then P is pronormal in G. By [5, Corollary 4(1)], $P \cap L$ is pronormal in G and so is X-g pronormal in G.

(3) Let $g \in N_G(K \cap X)$, then $H \cap X, (H \cap X)^g \leq H \cap K \leq N_G(H \cap X)$, and since H is X-g pronormal in G, $H \cap X = (H \cap X)^g$. And so $g \in N_G(H \cap X)$.

(4) Let $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = K$ be a normal chain between H and K. If $n = 1$, the result is trivial. So we assume that $n > 1$ and $H \trianglelefteq H_{n-1} \trianglelefteq K$. Then we have $H \trianglelefteq H_1 \trianglelefteq N_K(H)$. Thus by [5, Lemma 5], $K = N_K(H_1) = N_K(H)$.

(5) Let $g \in G$. Since H is X-g pronormal in G, then $(H/L)^g \cap (XL/L) = (H^g \cap X)L/L$, we have H is X-g pronormal in G if and only if H/L is X-g pronormal in G/L.

(6) Since $(|H|, |L|) = 1$ and H is X-g pronormal in G, then, there exists an $x \in \langle H^g \cap X, H \cap X \rangle$, $(HL)^g \cap X = (H^g \cap X)(L^g \cap X) = (H^g \cap X)(L \cap X) = (H^x \cap X)(L^x \cap X) = (HL)^x \cap X$. This implies that HL is X-g pronormal. □

Lemma 2.3. ([14, Lemma 2.8]) *Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.*

(1) If N is normal in G of order p, then N is in $Z(G)$.

(2) If G has cyclic Sylow p-subgroups, then G is p-nilpotent.

(3) If $M \leq G$ and $|G : M| = p$, then $M \lhd G$.

Lemma 2.4. ([11, IV-5.4]) *Suppose that G is a group which is not nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.*

Lemma 2.5. ([11, III-5.2]) *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(1) G has a normal Sylow $p-$subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.

(2) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(3) If P is non-abelian and $p \neq 2$, then the exponent of P is p.

(4) If P is non-abelian and $p = 2$, then the exponent of P is 4.

(5) If P is abelian, then P is of exponent p.

(6) If $K \leq P$, then $[K, Q] = 1$ if and only if $K \leq \Phi(P)$.

3. The proof of the main results

In this section, we will give the proofs of the main theorems and some remarks.

The proof of the Theorem 1.

Proof. Assume that the result is not true and choose for G a counterexample of minimal order. Then we have the following steps.

Step 1. The hypotheses are inherited by all proper subgroups, thus G is a group which is not p-nilpotent but all proper subgroups are p-nilpotent.

Let K be a proper subgroup of G containing P. Since $K \cap F(G) \leq F(K)$, by Lemma 2.2(1), every subgroup of P of order p or 4 (when $p = 2$) is $F(K)$-g pronormal in K. The minimality of G implies that K is p-nilpotent. Then G is a group which is not p-nilpotent but all proper subgroups are p-nilpotent. By Lemma 2.4 and Lemma 2.5, G has a normal Sylow $p-$subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$ and P has the exponent at most 4 if $p = 2$ or p if p is odd.

Step 2. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then take $K = O_{p'}(G)$, every subgroup of PK/K of order p or 4 (when $p = 2$) is $F(G)K/K$-g pronormal in G/K. The minimal choice of G implies that G/K is p-nilpotent and so is G, a contradiction.

Step 3. $G/P \cap F(G)$ is p-nilpotent.

By step 2, $F(G) \neq 1$. Since $P \trianglelefteq G$ and $F(G) \trianglelefteq G$, $P \cap F(G) \trianglelefteq G$ and also $P \cap F(G) \trianglelefteq P$. $F(G/P \cap F(G)) = F(G)/P \cap F(G)$. Thus we have that every subgroup of $P/P \cap F(G)$ of order p or 4 (when $p = 2$) is $F(G)/P \cap F(G)$-g pronormal in $G/P \cap F(G)$ by Lemma 2.2(5). So $G/P \cap F(G)$ is p-nilpotent by the minimal choice of G.

Step 4. $P = F(G)$.

By step 3, $P \cap F(G) \neq 1$. Obviously $P \cap F(G) \leq P$. Otherwise $F(G)Q < G$, then by step 1, $F(G)Q$ is p-nilpotent. $F(G)Q = F(G) \times Q$ and so Q is normal in G which contradicts step 1.

If $P < F(G)$, then there exists some p'-subgroup of $F(G)$ which is normal in G. But this is impossible because $O_{p'}(G) = 1$.

Step 5. $P \cap \Phi(G) = 1$.

If $P \cap \Phi(G) \neq 1$. Let $K = P \cap \Phi(G)$, then G/K is p-nilpotent by step 3. This implies that $G/\Phi(G)$ is p-nilpotent and so is G, a contradiction.

Step 6. Final contradiction.

Since $P \cap \Phi(G) = 1$ and [12, Lemma 2.3], $F(P) = F(F(G)) = F(G) = P$ is the direct product of minimal normal subgroups of G which are contained in P. By step 1, P has the exponent p if p is odd and P is abelian or 4 if $p = 2$ and P is non-abelian. If p is odd or P is abelian, then $P = A_1 \times A_2 \times \cdots A_s$, where A_i is of order p and A_i is normal in G. Obviously, $C_G(A_i) < G$. Otherwise $C_G(A_i) = N_G(A_i)$ for all $i = 1, 2, \cdots s$, then G is p-nilpotent. So we have $G/C_G(A_i)$ is p-nilpotent for all i and $G/\cap_i C_G(A_i)$ is p-nilpotent. $C_G(A_i) = P$. Otherwise $C_G(A_i)Q < G$, $C_G(A_i)Q$ is p-nilpotent by step 1, and $C_G(A_i)Q = C_G(A_i) \times Q$. This implies that Q is normal in G, a contradiction. $P = C_G(A_i) \leq C_G(P)$. It follows $P = Z(P)$ and so G is p-nilpotent, a contradiction. Then $p = 2$, and P is non-abelian. Namely, every subgroup of P is a group in which every subgroup is of order 4, then by [16, Theorem 17, p. 149], P is cyclic or a generalized quaternion group. If P is cyclic, G is p-nilpotent by Lemma 2.3(2), a contradiction. Then P is a generalized quaternion group and $G \cong S_4$. Since $S_4 = C_2 A_4$, $G/C_2 \cong A_4$. This contradicts the hypotheses of the theorem.

The final contradiction completes the proof. □

Remark 3.1. The condition "G has no-section isomorphic to A_4" of Theorem 1 can't be removed. Let $G = S_4$, the Symmetric group of degree 4. The subgroups of the Sylow 3-subgroup of G of order 3 are $F(G)$-g pronormal in G, but G is not 3-nilpotent.

Remark 3.2. For any non-abelian simple groups G, the Sylow p-subgroup of G are pronormal in G, but not $F(G)$-g pronormal in G since $F(G) = 1$.

The proof of the Theorem 2.

Proof. Assume that the result is not true and choose for G a counterexample of minimal order. Then we have the following:

Step 1. Let M be a proper maximal subgroup of G containing P, then M is p-nilpotent.

Let P be a Sylow p-subgroup of G. $M \cap F(G) \leq F(M)$, By Lemma 2.2(1) every maximal subgroup of P is $F(M)$-g pronormal in M. Then the minimal choice of G implies that M is p-nilpotent.

Step 2. $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$, and take $K = O_{p'}(G)$. Then PK/K is a Sylow p-subgroup of G/K and $F(G/K) = F(G)K/K$ by [4, Lemma 3.1]. By Lemma 2.2(6), every maximal subgroup of PK/K is $F(G)K/K$-g pronormal in G/K. The minimal choice of G implies that G/K is p-nilpotent. And so G is p-nilpotent, a contradiction.

Step 3. $G/O_p(G)$ is p-nilpotent, G is p-solvable.

By step 2, $O_p(G) \neq 1$. Obviously, $O_p(G) \leq P$. If $P = O_p(G)$, then $G = N_G(P)$ is p-nilpotent. And so $O_p(G) \lneqq P$ and $F(G/O_p(G)) = F(G)O_p(G)/O_p(G)$ by [4, Lemma 3.1]. $P/O_p(G)$ is a Sylow p-subgroup of $G/O_p(G)$. By hypotheses and Lemma 2.2(5), every maximal subgroup of $P/O_p(G)$ is $F(G)O_p(G)/O_p(G))$-g pronormal in $G/O_p(G)$, then the minimal choice of G implies that $G/O_p(G)$ is p-nilpotent, and so G is p-solvable.

Step 4. $G = PQ$, where Q is an elementary abelian Sylow q-subgroup of G for some $q \neq p$. Moreover, P is maximal in G and $QO_p(G)/O_p(G)$ is minimal normal in $G/O_p(G)$.

By step 3, G is p-solvable. Then by [10, Theorem 3.5, p. 229], G has a Hall subgroup G_1 with $\pi(G_1) = \{p, q\}$, where p, q are different primes. Hence if $G_1 < G$, then G_1 is p-nilpotent since the minimal choice of G. And so Q is normal in G_1, where Q is a Sylow q-subgroup of G_1 and also Q is a Sylow q-subgroup of G. Then $O_p(G)Q = O_p(G) \times Q$ and so Q is normal in G, a contradiction. Thus $G_1 = G = PQ$, and by [10, Theorem 3.3, p. 131], G is solvable. Now let $T/O_p(G)$ be a minimal normal subgroup of $G/O_p(G)$ contained in $O_{pp'}(G)/O_p(G)$. Then $T = O_p(G)(T \cap Q)$. If $T \cap Q < Q$, then $PT < G$ and, therefore, PT is p-nilpotent by step 1. It follows that $1 < T \cap Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Hence

$T = O_{pp'}(G)$ and $QO_p(G)/O_p(G)$ is an elementary abelian q-group complementing $P/O_p(G)$. It implies that P is maximal in G.

Step 5. Let S be a maximal p-subgroup of P, then $S = F(G) = O_p(G)$.

$O_p(G) \leq F(G)$. If $O_p(G) \lneqq F(G)$, then there exists a normal q-subgroup R of G such that $R \nleq O_p(G)$ but $R \leq F(G)$, $q \neq p$. By step 3, G/R is p-nilpotent. Then $G \cong G/1 \leq G/O_p(G) \times G/R$ is p-nilpotent since p-nilpotence of finite groups is a saturated formation by [13, 9.3.4], a contradiction. So $O_p(G) \leq C_G(F(G)) \leq C_G(O_p(G)) \leq O_p(G)$ by [10, Theorem 1.3, p. 218] and step 4. Thus $F(G) = O_p(G)$.

We will prove that S is normal in G.

Obviously $O_p(G) \leq S$. If not, then $P \leq N_G(S) \lneqq G$. Since $F(G) \cap N_G(S) \leq F(N_G(S))$. By Lemma 2.2(1), S is $F(N_G(S))$-g pronormal in $N_G(S)$, the minimal choice of G implies that $N_G(S)$ is p-nilpotent. Since P is maximal subgroup of G by step 4, we have $N_G(S) = P$ or G. If $N_G(S) = P$, then $N_G(P) = P$ and $S \trianglelefteq P$. Obviously $SQ < G$, and so S is a Sylow p-subgroup of SQ, And since S is $F(SQ)$-g pronormal in SQ by Lemma 2.2(1). Then the minimal choice of G implies that SQ is p-nilpotent. Thus $SQ = S \times Q$ and Q is normal in G which is impossible. Then $G = PQ = SQ$, a contradiction since S is proper maximal subgroup of P. So S is normal in G. Thus $S = O_p(G)$.

Step 6. S has exponent p if $p > 2$ and at most 4 if $p = 2$; S is either an elementary abelian or non-abelian with $S' = Z(S) = \Phi(S)$ an elementary abelian group.

By minimal choice of G and step 1, we have that G is a minimal non-p-nilpotent but all proper subgroup are p-nilpotent. And by step 4, G is solvable. Then by [15, Theorem 1.2], we have the results.

Step 7. Final contradiction.

$S \cap \Phi(G) = 1$.

If not, then there exists a normal subgroup W of $S \cap \Phi(G)$. By hypotheses, S/W is $F(G)/W$-g pronormal in G/W by Lemma 2.2(5), then minimality of G implies that G/W is p-nilpotent and $G/\Phi(G)$ is p-nilpotent. Thus G is p-nilpotent since p-nilpotence of finite groups is a saturated formation by [13, 9.3.4], a contradiction.

Since $S \cap \Phi(G) = 1$ and [12, Lemma 2.3], $F(S)$ is the direct product of minimal normal subgroups of G which are contained in S.

By step 6, $F(S) = F(G) = O_p(G)$ and S is abelian or non-abelian with $S' = Z(S) = \Phi(S)$ an elementary abelian. If the former, then $S = A_1 \times A_2 \times \cdots \times A_s$, where A_i are abelian and of order p. Then A_i is normal in G, and so $A_i \leq Z(S)$. G is p-nilpotent, a contradiction. Then $p = 2$, S is an non-abelian and has exponent

4. Namely, every maximal subgroup of P is a group in which every subgroup is of order 4, then by [16, Theorem 17, p. 149], P is cyclic or a generalized quaternion group. If P is cyclic, G is p-nilpotent by Lemma 2.3(2), a contradiction. Then P is a generalized quaternion group and $G \cong S_4$. Since $S_4 = C_2A_4$, $G/C_2 \cong A_4$. This contradicts the hypotheses of the theorem.

So the minimal counterexample doesn't exist. This completes the proof. □

Remark 3.3. The condition of the Theorem 2 "G has no-section isomorphic to A_4" can't be removed. Let $G = S_4$ and P be a Sylow 2-group, the maximal subgroup of the Sylow 2-subgroup is $F(G)$-g pronormal in G, but G is not 2-nilpotent.

Corollary 3.4. ([3, Theorem 1.1]) *Let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P belongs to $H(G)$*

Corollary 3.5. ([7, p. 83]) *If every maximal subgroups of G are dualpronormal in G, then G is nilpotent.*

Corollary 3.6. ([7, Theorem 1]) *If every 2-maximal subgroup of a group G is dualpronormal, then either:*
(1) G is nilpotent;
(2) G is minimal non-nilpotent, $|G| = pq^\beta$, $G_p \lhd G$, $q|p-1$.

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