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PERIODIC RESOLUTIONS FOR CERTAIN FINITE GROUPS

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Abstract. Let p and q be distinct odd primes, and let

$$G = \langle x, y \mid x^p y = y^{-1} x^p,\ xy = y^{2q-1} x^{-1} \rangle \cong \mathbb{Z}/pq \rtimes Q_8.$$

Using this deficiency zero presentation discovered by Bernard Neumann, we investigate the beginning of a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module in the case $p \not\equiv q \pmod 4$. This gives enough information to compute $H^*(G, M)$ for any $\mathbb{Z}G$-module M.

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1. A family of presentations

This paper is concerned with the calculation of the beginning of a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, where G belongs to a certain class of periodic groups. A finite group G has periodic cohomology if and only if every Sylow subgroup of G is cyclic or generalised quaternion. See for example Cartan and Eilenberg [2], Section XII.11. The classification of such groups can be found in Theorem 6.1.11 and Section 6.3 of Wolf [11], as well as on page 103 of Thomas and Wall [10].

In this paper, we concentrate on groups expressible as a semidirect product

$$\mathbb{Z}/pq \rtimes Q_8 = (\mathbb{Z}/p \times \mathbb{Z}/q) \rtimes Q_8.$$

Here, p and q are distinct odd primes, and Q_8 acts on $\mathbb{Z}/p$ and $\mathbb{Z}/q$ via two different quotients of order two, so that the kernel of the action is equal to the centre of Q_8.

We begin by investigating the group G defined by two generators x and y, and two relations

$$x^p y = y^{-1} x^p \qquad x^r y^s = y^{2q-s} x^{-r}.$$

Here, p, q, r and s are odd (but not necessarily prime), and $(p, r) = (q, s) = 1$. In the case $r = s = 1$, this presentation was investigated by B. H. Neumann [7].

Since $x^{2p} y = x^p y^{-1} x^p = y x^{2p}$, we have

$$x^{2p} \in Z(G),$$

where $Z(G)$ denotes the centre of G. Also,

$$x^r y^{2q} x^{-r} = (x^r y^s)(y^{2q-s} x^{-r}) = (y^{2q-s} x^{-r})(x^r y^s) = y^{2q}$$

so y^{2q} commutes with both x^r and x^{2p}. Since r is coprime to $2p$, it follows that

$$y^{2q} \in Z(G).$$

We set $z = y^{2q} \in Z(G)$. Since $zx^p = x^p z = x^p y^{2q} = y^{-2q} x^p = z^{-1} x^p$, we have $z^2 = 1$. This means we can rewrite the second relation in the form

$$z x^r y^s = y^{-s} x^{-r}.$$

We have, using the first relation,

$$x^{p-r}(x^r y^s) = (y^{-s} x^{-r}) x^{p+r} = (z x^r y^s) x^{p+r} = (z x^{2p})(x^r y^s) x^{-p+r}$$

and so

$$x^{p(p-r)}(x^r y^s) = (z x^{2p})^p (x^r y^s) x^{-p(p-r)}.$$

But also, since $p - r$ is even and $x^{2p} \in Z(G)$ we have

$$x^{p(p-r)}(x^r y^s) = (x^r y^s) x^{p(p-r)}.$$

It follows that $x^{2p(p-r)} = (z x^{2p})^p$, so $x^{2pr} = z^{-p} = z$. Set $v = x^{4p}$ and $w = x^r$. Then $v^r = 1$, $v \in Z(G)$, and $w^{2p} = z$. Furthermore, since $(4p, r) = 1$, we have $G = \langle v, w, y \rangle$, and

$$w^p y = y^{-1} w^p \qquad w y^s = z y^{-s} w^{-1}.$$

Since $w y^s w y^s = z \in Z(G)$, conjugating by w^p gives

$$w y^s w y^s = w^p (w y^s w y^s) w^{-p} = w y^{-s} w y^{-s}$$

so that $y^{2s} w = w y^{-2s}$. Since $y^{2q} = z \in Z(G)$ and $(q, s) = 1$ we deduce that $y^2 w = w y^{-2}$ so that w^2 and y^2 commute. Since $q - s$ is even, this gives

$$w y^q = w y^s y^{q-s} = z y^{-s} w^{-1} y^{q-s} = z y^{-s} y^{s-q} w^{-1} = z y^{-q} w^{-1} = y^q w^{-1}.$$

Set $i = y^q$, $j = w^{-p}$, $k = ij$ and

$$Q = \langle i, j \rangle \subseteq G.$$

Since we have $i^2 = j^2 = z$, $ij = ji^{-1}$, and z is central with $z^2 = 1$, it follows that Q satisfies the relations of a quaternion group of order eight. Writing $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$ for i^{-1}, j^{-1} and k^{-1}, we have the usual quaternionic relations

$$ij = \bar{\jmath} i = k, \quad jk = \bar{k} j = i, \quad ki = \bar{\imath} k = j,$$
$$i^2 = j^2 = k^2 = ijk = z \in Z(Q), \quad z^2 = 1.$$

Since $(4,p)=(4,q)=1$, we have

$$G = \langle v, w^4, y^4, i, j\rangle,$$

and these elements satisfy the quaternion relations together with

$$iy^4 = y^4 i, \qquad jw^4 = w^4 j, \qquad iw^4 = w^{-4} i, \qquad jy^4 = y^{-4} j,$$
$$w^4y^4 = y^4w^4, \qquad (w^4)^p = 1, \qquad (y^4)^q = 1, \qquad v \in Z(G), \qquad v^r = 1.$$

It follows that there is a surjective homomorphism

$$\mathbb{Z}/r \times (\mathbb{Z}/p \times \mathbb{Z}/q) \rtimes Q_8 \to G$$

sending the generator of $\mathbb{Z}/r$ to v, the generator of $\mathbb{Z}/p$ to w^4, the generator of $\mathbb{Z}/q$ to y^4, and the generators of Q_8 to i and j. Finally, expressing x and y in terms of these generators, it is not hard to check that the corresponding elements of the group on the left satisfy the defining relations, so that this map is an isomorphism and $|G| = 8pqr$.

One way of keeping track of this isomorphism is as follows. By replacing p with $-p$ if necessary, we can assume that $p \equiv 3 \pmod 4$. Similarly, we can assume that $q \equiv 1 \pmod 4$. Then we can set $a = w^{1+p}$ and $b = y^{1-q}$, so that $\langle a\rangle = \langle w^4\rangle$ and $\langle b\rangle = \langle y^4\rangle$, and write N for the normal subgroup $\langle a,b\rangle$ of G. Then $G = \langle v\rangle \times (N \rtimes Q)$, and in terms of these generators we have $w = ja$ and $y = ib$. These elements satisfy

$$ab = ba, \quad a^p = 1, \quad b^q = 1, \quad ia = a^{-1}i, \quad ja = aj, \quad ib = bi, \quad jb = b^{-1}j.$$

Remarks. (i) It follows from the above discussion that the isomorphism type of G depends only on p, q and r, and not on s.

(ii) It would be interesting to know whether the periodic groups of the form

$$\mathbb{Z}/m \rtimes (Q_{2^n} \times \mathbb{Z}/r)$$

with m and r odd and coprime, all have deficiency zero presentations.

2. The resolution

Let G be the group discussed in the last section. We assume from now on that $r = s = 1$, so that $w = x$ and $v = 1$. We shall also assume that p and q are distinct odd primes. Thus

$$G = \langle x, y \mid x^p y = y^{-1} x^p,\ xy = y^{2q-1} x^{-1}\rangle \cong (\mathbb{Z}/p \times \mathbb{Z}/q) \rtimes Q_8.$$

By the analysis of Section 1, every element of G has a unique expression of the form $x^\alpha y^\beta$ with $0 \le \alpha \le 4p-1$, $0 \le \beta \le 2q-1$. The element $x^{2p} = y^{2q} = z$ is central, and $x^{4p} = y^{4q} = z^2 = 1$. In order to multiply such expressions, one uses these facts together with the rules

$$y^n x^m = \begin{cases} x^m y^n & m, n \text{ even} \\ x^m y^{-n} & m \text{ odd}, \ n \text{ even} \\ x^{-m} y^n & m \text{ even}, \ n \text{ odd} \\ x^{-m} y^{-n} z & m, n \text{ odd}. \end{cases}$$

Using Fox's free differential calculus [4] on the relators of the above presentation, namely yx^pyx^{-p} and $xyxy^{1-2q}$, we can form the beginning of a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module. The following sequence is exact:

$$\mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma' \xrightarrow{d_2} \mathbb{Z}G\beta \oplus \mathbb{Z}G\beta' \xrightarrow{d_1} \mathbb{Z}G\alpha \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Here, the maps are given as follows:

$$\begin{aligned} \varepsilon(\alpha) &= 1, \quad d_1(\beta) = (x-1)\alpha, \quad d_1(\beta') = (y-1)\alpha \\ d_2(\gamma) &= \frac{\partial(yx^pyx^{-p})}{\partial x}\beta + \frac{\partial(yx^pyx^{-p})}{\partial y}\beta' \\ &= (y-1)(x^{p-1} + \cdots + x + 1)\beta + (yx^p + 1)\beta' \\ d_2(\gamma') &= \frac{\partial(xyxy^{1-2q})}{\partial x}\beta + \frac{\partial(xyxy^{1-2q})}{\partial y}\beta' \\ &= (xy+1)\beta + (x - (y^{2q-2} + \cdots + y + 1))\beta'. \end{aligned}$$

We begin by examining the degenerate case where $p = 1$ rather than a prime. In this case, the equation for $d_2(\gamma)$ reduces to

$$d_2(\gamma) = (y-1)\beta + (yx+1)\beta'$$

while the equation for $d_2(\gamma')$ remains unaltered. The kernel of d_2 is generated by the element

$$(x-1)\gamma + (y-1)\gamma'$$

whose annihilator is the group sum.

Similarly, in the degenerate case where $q = 1$ rather than a prime, the equation for $d_2(\gamma)$ remains unaltered, while the equation for $d_2(\gamma')$ reduces to

$$d_2(\gamma') = (xy+1)\beta + (x-1)\beta'.$$

The kernel of d_2 is generated by the element

$$(x-1)\gamma + (yx^{p-1} - 1)\gamma'$$

whose annihilator is the group sum.

In each of these two cases, we obtain an exact sequence of the form

$$0 \to \mathbb{Z} \to \mathbb{Z}G\delta \xrightarrow{d_3} \mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma' \xrightarrow{d_2} \mathbb{Z}G\beta \oplus \mathbb{Z}G\beta' \xrightarrow{d_1} \mathbb{Z}G\alpha \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $d_3(\delta)$ is equal to the given generator for the kernel of d_2. Our purpose here is to try to understand under what conditions on primes p and q we can get a sequence like this one. This will happen precisely when the kernel of d_2 is generated by a single element. We do not completely succeed in this goal. What we do succeed in doing is obtaining a pair of generators (Theorem 4.8) for the kernel of d_2. The next step will be to find conditions under which these may be reduced to a single generator. We describe in Section 5 why one generator will not always suffice; there are delicate congruence conditions on p and q in order for the Swan obstruction to vanish. It seems plausible that these are exactly the conditions for reducing to a single generator; necessity is clear, but sufficiency is not.

Theorem 4.8 gives enough information to compute $H^*(G, M)$ for any $\mathbb{Z}G$-module M. Namely, since $H^*(G, \mathbb{Z})$ is periodic with period four, it suffices to compute the Tate cohomology groups $\hat{H}^i(G, M)$ for $-1 \leq i \leq 2$. This can be achieved by applying $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ to the segment of a Tate resolution given by

$$\mathbb{Z}G\delta \oplus \mathbb{Z}G\delta' \xrightarrow{d_3} \mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma' \xrightarrow{d_2} \mathbb{Z}G\beta \oplus \mathbb{Z}G\beta' \xrightarrow{d_1} \mathbb{Z}G\alpha \xrightarrow{\varepsilon^* \circ \varepsilon}$$
$$\mathbb{Z}G\alpha^* \xrightarrow{d_1^*} \mathbb{Z}G\beta^* \oplus \mathbb{Z}G\beta'^* \xrightarrow{d_2^*} \mathbb{Z}G\gamma^* \oplus \mathbb{Z}G\gamma'^*$$

and taking the cohomology of the resulting complex. Here, δ and δ' are taken to be a pair of generators for the kernel of d_2. The maps ε^*, d_1^* and d_2^* are the $\mathbb{Z}$-duals of ε, d_1 and d_2, $\mathbb{Z}G\alpha^*$ denotes $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G\alpha, \mathbb{Z})$, a free $\mathbb{Z}G$-module of rank one with basis element α^*, and so on.

3. The kernel of d_2

In order to find the kernel of d_2, we write out the equations in the last section in greater detail. We shall assume that $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$. As in Section 1, we write $i = y^q$, $j = x^{-p}$, $k = ij$, $a = x^{1+p}$, $b = y^{1-q}$. Thus $x = ja$ and $y = ib$, and the generators i, j, a and b satisfy

$$i^2 = j^2 = k^2 = ijk = z \in Z(G), \qquad z^2 = 1,$$
$$ab = ba, \quad a^p = b^q = 1, \quad ia = a^{-1}i, \quad ja = aj, \quad ib = bi, \quad jb = b^{-1}j.$$

Each element of G may be written uniquely as a product of an element of Q and an element of N. Substituting ja for x and ib for y, we get

$$x^{p-1} + \cdots + x + 1 = a^{p-1} + ja^{p-2} + 1a^{p-3} + \bar{\jmath}a^{p-4} + za^{p-5} + \cdots + ja + 1$$

$$y^{2q-2} + \cdots + y + 1 = 1b^{2q-2} + \bar{\imath}b^{2q-3} + zb^{2q-4} + ib^{2q-5} + \cdots + ib + 1,$$

so that

$$yx^p + 1 = \bar{k}b^{-1} + 1, \qquad xy + 1 = \bar{k}a^{-1}b + 1,$$

$$\begin{aligned}(y-1)(x^{p-1} + \cdots + x + 1) = -1(a^{p-3} + a^{p-7} + \cdots + 1)\\ - z(a^{p-1} + a^{p-5} + \cdots + a^2) + i(a^{p-3} + a^{p-7} + \cdots + 1)b\\ + \bar{\imath}(a^{p-1} + a^{p-5} + \cdots + a^2)b - j(a^{p-2} + a^{p-6} + \cdots + a)\\ - \bar{\jmath}(a^{p-4} + a^{p-8} + \cdots + a^3) + k(a^{p-2} + a^{p-6} + \cdots + a)b^{-1}\\ + \bar{k}(a^{p-4} + a^{p-8} + \cdots + a^3)b^{-1},\end{aligned}$$

and

$$\begin{aligned}x - (y^{2q-2} + \cdots + y + 1) = -1(b^{2q-2} + b^{2q-6} + \cdots + 1) - z(b^{2q-4} + b^{2q-8} + \cdots + b^2)\\ - i(b^{2q-5} + b^{2q-9} + \cdots + b) - \bar{\imath}(b^{2q-3} + b^{2q-7} + \cdots + b^3) + ja.\end{aligned}$$

Write the general element $\phi\gamma - \phi'\gamma'$ of the kernel of d_2 in the form

$$\begin{aligned}(1\phi_1 + z\phi_z + i\phi_i + \bar{\imath}\phi_{\bar{\imath}} + j\phi_j + \bar{\jmath}\phi_{\bar{\jmath}} + k\phi_k + \bar{k}\phi_{\bar{k}})\gamma\\ - (1\phi'_1 + z\phi'_z + i\phi'_i + \bar{\imath}\phi'_{\bar{\imath}} + j\phi'_j + \bar{\jmath}\phi'_{\bar{\jmath}} + k\phi'_k + \bar{k}\phi'_{\bar{k}})\gamma',\end{aligned}$$

where each of the coefficients $\phi_1, \ldots, \phi_{\bar{k}}, \phi'_1, \ldots, \phi'_{\bar{k}}$ is an element of the group algebra $\mathbb{Z}[\mathbb{Z}/pq]$. Then the equations describing the kernel of d_2 are

$$\phi(y-1)(x^{p-1} + \cdots + x + 1) = \phi'(xy + 1) \tag{1}$$

$$\phi(yx^p + 1) = \phi'(x - (y^{2q-2} + \cdots + y + 1)) \tag{2}$$

These equations give 16 conjugate linear equations in these 16 variables. In fact, it turns out that by conjugating each of these equations by one of 1, i, j or k, these 16 equations become linear in the 16 new variables

$$\begin{gathered}\phi_1 = \phi_1, \quad \phi_2 = \phi_z, \quad \phi_3 = i\phi_i\bar{\imath}, \quad \phi_4 = \bar{\imath}\phi_{\bar{\imath}}i,\\ \phi_5 = j\phi_j\bar{\jmath}, \quad \phi_6 = \bar{\jmath}\phi_{\bar{\jmath}}j, \quad \phi_7 = k\phi_k\bar{k}, \quad \phi_8 = \bar{k}\phi_{\bar{k}}k,\\ \phi'_1 = \phi'_1, \quad \phi'_2 = \phi'_z, \quad \phi'_3 = i\phi'_i\bar{\imath}, \quad \phi'_4 = \bar{\imath}\phi'_{\bar{\imath}}i,\\ \phi'_5 = j\phi'_j\bar{\jmath}, \quad \phi'_6 = \bar{\jmath}\phi'_{\bar{\jmath}}j, \quad \phi'_7 = k\phi'_k\bar{k}, \quad \phi'_8 = \bar{k}\phi'_{\bar{k}}k.\end{gathered}$$

In order to solve these equations, we work first over the quotient of the group algebra $\mathbb{Z}[\mathbb{Z}/pq]$ given by setting $a^{p-1} + \cdots + a + 1 = 0$ and $b^{q-1} + \cdots + b + 1 = 0$. This is the ring $\mathcal{O}_K = \mathbb{Z}[\zeta_{pq}]$ of integers in the cyclotomic number field $K = \mathbb{Q}(\zeta_{pq})$, where $\zeta_{pq} = e^{2\pi i/pq}$ is a primitive pqth root of unity. Thus a and b are primitive pth and qth roots of unity respectively.

In this quotient, we may multiply the equations by any non-zero element of K, without altering the space of solutions. We multiply the equation (1) by $a^3 + a^2 + a + 1$, and the equation (2) by $b^2 + 1$, to obtain

$$\begin{multline} \phi(-1 - za^2 + ib + \bar{\imath}a^2 b - ja + \bar{\jmath}(a^2 + a + 1) + kab^{-1} - \bar{k}(a^2 + a + 1)b^{-1}) \\ = \phi'(\bar{k}a^{-1}b + 1)(a^3 + a^2 + a + 1) \end{multline} \tag{3}$$

$$\phi(\bar{k}b^{-1} + 1)(b^2 + 1) = \phi'(-1 + z + ib^{-1} + \bar{\imath}b + ja(b^2 + 1)). \tag{4}$$

We then conjugate as above to make the equations linear. The resulting 16 equations are as follows:

$$\begin{multlined} -\phi_1 - \phi_2 a^2 + \phi_3 a^2 b + \phi_4 b + \phi_5(a^2 + a + 1) - \phi_6 a \\ -\phi_7(a^2 + a + 1)b^{-1} + \phi_8 ab^{-1} = (\phi_1' + \phi_7' a^{-1} b)(a^3 + a^2 + a + 1) \end{multlined} \tag{5}$$

$$\begin{multlined} \phi_1 b + \phi_2 a^{-2} b - \phi_3 - \phi_4 a^{-2} + \phi_5 a^{-1} b^{-1} - \phi_6(1 + a^{-1} + a^{-2})b^{-1} \\ +\phi_7(1 + a^{-1} + a^{-2}) - \phi_8 a^{-1} = (\phi_3' + \phi_6' ab)(1 + a^{-1} + a^{-2} + a^{-3}) \end{multlined} \tag{6}$$

$$\begin{multlined} -\phi_1 a + \phi_2(a^2 + a + 1) - \phi_3(a^2+a + 1)b + \phi_4 ab - \phi_5 - \phi_6 a^2 \\ +\phi_7 b^{-1} + \phi_8 a^2 b^{-1} = (\phi_3' a^{-1} b^{-1} + \phi_5')(a^3 + a^2 + a + 1) \end{multlined} \tag{7}$$

$$\begin{multlined} \phi_1 a^{-1} b - \phi_2(1 + a^{-1} + a^{-2})b - \phi_3 a^{-1} + \phi_4(1 + a^{-1} + a^{-2}) + \phi_5 a^{-2} b^{-1} \\ +\phi_6 b^{-1} - \phi_7 - \phi_8 a^{-2} = (\phi_2' ab^{-1} + \phi_7')(1 + a^{-1} + a^{-2} + a^{-3}) \end{multlined} \tag{8}$$

$$\begin{multlined} -\phi_1 a^2 - \phi_2 + \phi_3 b + \phi_4 a^2 b - \phi_5 a + \phi_6(a^2 + a + 1) + \phi_7 ab^{-1} \\ -\phi_8(a^2 + a + 1)b^{-1} = (\phi_2' + \phi_8' a^{-1} b)(a^3 + a^2 + a + 1) \end{multlined} \tag{9}$$

$$\begin{multlined} \phi_1 a^{-2} b + \phi_2 b - \phi_3 a^{-2} - \phi_4 - \phi_5(1 + a^{-1} + a^{-2})b^{-1} + \phi_6 a^{-1} b^{-1} \\ -\phi_7 a^{-1} + \phi_8(1 + a^{-1} + a^{-2}) = (\phi_4' + \phi_5' ab)(1 + a^{-1} + a^{-2} + a^{-3}) \end{multlined} \tag{10}$$

$$\begin{multlined} \phi_1(a^2 + a + 1) - \phi_2 a + \phi_3 ab - \phi_4(a^2 + a + 1)b - \phi_5 a^2 - \phi_6 \\ +\phi_7 a^2 b^{-1} + \phi_8 b^{-1} = (\phi_6' + \phi_4' a^{-1} b^{-1})(a^3 + a^2 + a + 1) \end{multlined} \tag{11}$$

$$\begin{multlined} -\phi_1(1 + a^{-1} + a^{-2})b + \phi_2 a^{-1} b + \phi_3(1 + a^{-1} + a^{-2}) - \phi_4 a^{-1} + \phi_5 b^{-1} \\ +\phi_6 a^{-2} b^{-1} - \phi_7 a^{-2} - \phi_8 = (\phi_1' ab^{-1} + \phi_8')(1 + a^{-1} + a^{-2} + a^{-3}) \end{multlined} \tag{12}$$

$$\phi_1(b^2 + 1) + \phi_7(b + b^{-1}) = -\phi_1' + \phi_2' + \phi_3' b + \phi_4' b^{-1} + \phi_6' a(b^2 + 1) \tag{13}$$

$$\phi_3(b^2 + 1) + \phi_6(b + b^{-1}) = \phi_1' b^{-1} + \phi_2' b - \phi_3' + \phi_4' + \phi_8' a^{-1}(b^2 + 1) \tag{14}$$

$$\phi_3(b + b^{-1}) + \phi_5(1 + b^{-2}) = \phi_1' a(1 + b^{-2}) - \phi_5' + \phi_6' + \phi_7' b + \phi_8' b^{-1} \tag{15}$$

$$\phi_2(b + b^{-1}) + \phi_7(1 + b^{-2}) = \phi_3' a^{-1}(1 + b^{-2}) + \phi_5' b^{-1} + \phi_6' b - \phi_7' + \phi_8' \tag{16}$$

$$\phi_2(b^2+1)+\phi_8(b+b^{-1}) = \phi'_1 - \phi'_2 + \phi'_3 b^{-1} + \phi'_4 b + \phi'_5 a(b^2+1) \tag{17}$$

$$\phi_4(b^2+1)+\phi_5(b+b^{-1}) = \phi'_1 b + \phi'_2 b^{-1} + \phi'_3 - \phi'_4 + \phi'_7 a^{-1}(b^2+1) \tag{18}$$

$$\phi_4(b+b^{-1})+\phi_6(1+b^{-2}) = \phi'_2 a(1+b^{-2}) + \phi'_5 - \phi'_6 + \phi'_7 b^{-1} + \phi'_8 b \tag{19}$$

$$\phi_1(b+b^{-1})+\phi_8(1+b^{-2}) = \phi'_4 a^{-1}(1+b^{-2}) + \phi'_5 b + \phi'_6 b^{-1} + \phi'_7 - \phi'_8. \tag{20}$$

We simplify this set of equations as follows. Adding equations (13) and (17), and equations (14) and (18), and dividing by b^2+1, we get

$$\phi_1+\phi_2+\phi_7 b^{-1}+\phi_8 b^{-1} = \phi'_3 b^{-1} + \phi'_4 b^{-1} + \phi'_5 a + \phi'_6 a. \tag{21}$$

$$\phi_3+\phi_4+\phi_5 b^{-1}+\phi_6 b^{-1} = \phi'_1 b^{-1} + \phi'_2 b^{-1} + \phi'_7 a^{-1} + \phi'_8 a^{-1} \tag{22}$$

Similarly, adding equations (15) and (19), and equations (16) and (20), and dividing by $b+b^{-1}$, we get

$$\phi_3+\phi_4+\phi_5 b^{-1}+\phi_6 b^{-1} = \phi'_1 ab^{-1} + \phi'_2 ab^{-1} + \phi'_7 + \phi'_8 \tag{23}$$

$$\phi_1+\phi_2+\phi_7 b^{-1}+\phi_8 b^{-1} = \phi'_3 a^{-1}b^{-1} + \phi'_4 a^{-1}b^{-1} + \phi'_5 + \phi'_6. \tag{24}$$

Comparing equations (22) and (23), and using the fact that $a-1$ is non-zero, we see that each side of equation (22) is separately zero. Similarly, comparing equations (21) and (24), we see that each side of (21) is separately zero. So the last four equations are equivalent to the following four equations:

$$\phi_1 b + \phi_2 b + \phi_7 + \phi_8 = 0 \tag{25}$$

$$\phi_3 b + \phi_4 b + \phi_5 + \phi_6 = 0 \tag{26}$$

$$\phi'_1 a + \phi'_2 a + \phi'_7 b + \phi'_8 b = 0 \tag{27}$$

$$\phi'_3 + \phi'_4 + \phi'_5 ab + \phi'_6 ab = 0. \tag{28}$$

Dividing the equations (13)+(14)b and (13)$b-$(14) by b^2+1, and dividing equations (15) + (16)b and (15)b − (16) by $b+b^{-1}$, we get:

$$\phi_1 + \phi_3 b + \phi_6 + \phi_7 b^{-1} = \phi'_2 + \phi'_4 b^{-1} + \phi'_6 a + \phi'_8 a^{-1} b \tag{29}$$

$$\phi_1 b - \phi_3 - \phi_6 b^{-1} + \phi_7 = -\phi'_1 b^{-1} + \phi'_3 + \phi'_6 ab - \phi'_8 a^{-1} \tag{30}$$

$$\phi_2 b + \phi_3 + \phi_5 b^{-1} + \phi_7 = \phi'_1 ab^{-1} + \phi'_3 a^{-1} + \phi'_6 b + \phi'_8 \tag{31}$$

$$-\phi_2 + \phi_3 b + \phi_5 - \phi_7 b^{-1} = \phi'_1 a - \phi'_3 a^{-1} b^{-1} - \phi'_5 + \phi'_7 b \tag{32}$$

Our sixteen equations are now (5)–(12) and (25)–(32).

Next, we observe that we have the following linear relations between the equations:

$$(5) - (8)a^2 b + (9) - (12)a^2 b = (25)(a^2+1)(b-b^{-1})$$

$$(6)a^2b - (7) + (10)a^2b - (11) = (25)(a^2+1)(b - b^{-1})$$
$$(5) + (9) = (-(25)b^{-1} + (26) + (27)(1 + a^{-1}))(a^2+1)$$
$$(6) + (10) = (25)(1 + a^{-2}) - (26)(1 + a^{-2})b^{-1} + (28)(1 + a^{-1} + a^{-2} + a^{-3})$$
$$-(6)a^3 + (12)a^2 = (29)(a^3 + a^2 + a + 1)$$
$$(9) + (11)a = -(25)(a^2+1)b^{-1} - (26)(a^3 + a) + (30)(a^3 + a^2 + a + 1)$$
$$(5)a - (7) = (31)(a^3 + a^2 + a + 1)$$
$$(6)a^2 + (12) = -(25)(a^3 + a) - (26)(a^2+1)b^{-1} + (32)(a^3 + a^2 + a + 1)$$

These relations imply that equations (5)–(12) are redundant, so that we are left with equations (25)–(32). It is convenient to express these in matrix form as follows:

$$\left(\begin{array}{cccccccc|cccccccc}
b & b & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & b & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 & 0 & 0 & 0 & b & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & ab & ab & 0 & 0 \\
1 & 0 & b & 0 & 0 & 1 & b^{-1} & 0 & 0 & 1 & 0 & b^{-1} & 0 & a & 0 & a^{-1}b \\
b & 0 & -1 & 0 & 0 & -b^{-1} & 1 & 0 & -b^{-1} & 0 & 1 & 0 & 0 & ab & 0 & -a^{-1} \\
0 & b & 1 & 0 & b^{-1} & 0 & 1 & 0 & ab^{-1} & 0 & a^{-1} & 0 & 0 & b & 0 & 1 \\
0 & -1 & b & 0 & 1 & 0 & -b^{-1} & 0 & a & 0 & -a^{-1}b^{-1} & 0 & -1 & 0 & b & 0
\end{array}\right)$$

A basis for the null space of this matrix is given by the columns of the following matrix:

$$\left(\begin{array}{cccccccc}
-1 & 0 & 0 & 0 & ab^{-2} & 0 & 0 & ab^{-1} \\
-1 & 0 & 0 & 0 & -b^{-2} & (a-1)b & a-1 & -b^{-1} \\
0 & -1 & 0 & 0 & b^{-1} & 0 & (1-a)b & -ab^{-2} \\
0 & -1 & 0 & 0 & -ab^{-1} & 1-a & 0 & b^{-2} \\
0 & b & 0 & 0 & 0 & -b & -b^2 & 0 \\
0 & b & 0 & 0 & a-1 & ab & ab^2 & (a-1)b^{-1} \\
b & 0 & 0 & 0 & 0 & -ab^2 & b & 1-a \\
b & 0 & 0 & 0 & (1-a)b^{-1} & b^2 & -ab & 0 \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 1 & b^{-1} \\
0 & 0 & -1 & 0 & 0 & 0 & -b^2 & -b \\
0 & 0 & 0 & ab & ab^{-1} & a & 0 & 0 \\
0 & 0 & 0 & ab & -ab & -ab^2 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -b^{-1} & 0 & 0 \\
0 & 0 & 0 & -1 & -b^{-2} & b & 0 & 0 \\
0 & 0 & ab^{-1} & 0 & 0 & 0 & ab & -ab^{-2} \\
0 & 0 & ab^{-1} & 0 & 0 & 0 & -ab^{-1} & a
\end{array}\right)$$

Now $a-1$ and $a+1$ are coprime in $\mathcal{O}_K$, and $b^2+1 = (b^4-1)/(b^2-1)$ is a unit in $\mathcal{O}_K$. Using these facts, we easily see that the minor of the above matrix given by rows 1–4 and rows 9–12 has determinant a unit in $\mathcal{O}_K$. It follows that the kernel has the columns of the above matrix as an $\mathcal{O}_K$-basis.

The first column of this matrix, for example, tells us that the element

$$(-1 - z + kb^{-1} + \bar{k}b^{-1})\gamma = (1 + z)(yx^p - 1)\gamma$$

is in the kernel of d_2, modulo the cyclotomic relations. We write

$$\begin{aligned} N_a &= 1 + a + \cdots + a^{p-1}, \\ N_b &= 1 + b + \cdots + b^{q-1}, \\ N_x &= x^{4p-1} + \cdots + x + 1 = (1 + z + j + \bar{\jmath})N_a, \\ N_y &= y^{4q-1} + \cdots + y + 1 = (1 + z + i + \bar{\imath})N_b. \end{aligned}$$

We note that the relations given at the beginning of this section imply that N_a and N_b lie in the centre of $\mathbb{Z}G$, and that

$$N_a^2 = pN_a, \quad N_b^2 = qN_b, \quad N_aN_x = N_xN_a = pN_x, \quad N_bN_y = N_yN_b = qN_y. \tag{33}$$

We have

$$d_2(1 + z)(yx^p - 1)\gamma = (1 - y)N_x\beta. \tag{34}$$

The remaining columns of this matrix lift to the following equations:

$$d_2y^q(1 + z)(yx^p - 1)\gamma = y^q(1 - y)N_x\beta \tag{35}$$

$$d_2(1 + z)(xy - 1)\gamma' = (1 - x)N_y\beta' \tag{36}$$

$$d_2x^p(1 + z)(xy - 1)\gamma' = x^p(1 - x)N_y\beta' \tag{37}$$

$$d_2x^{-p}((1 - x^{1+p} + x^{-p}y^{-2} - xy^{-2}z)(y - z)\gamma + (1 + xy^{-1})(1 - y^2)\gamma') = 0 \tag{38}$$

$$d_2x^{1+p}y^q((1 - x^{1+p} + x^{-p}y^{-2} - xy^{-2}z)(y - z)\gamma + (1 + xy^{-1})(1 - y^2)\gamma') = 0 \tag{39}$$

$$d_2((1 - x^{1+p} + x^{-p}y^{-2} - xy^{-2}z)(y - z)\gamma + (1 + xy^{-1})(1 - y^2)\gamma') = 0 \tag{40}$$

$$d_2xy^q((1 - x^{1+p} + x^{-p}y^{-2} - xy^{-2}z)(y - z)\gamma + (1 + xy^{-1})(1 - y^2)\gamma') = 0 \tag{41}$$

Now equation (35) is a y^q times equation (34), equation (37) is x^p times equation (36), and equations (38), (39) and (41) are obtained by multiplying equation (40) by x^{-p}, $x^{1+p}y^q$ and xy^q respectively. So we may restrict our attention to equations (34), (36) and (40). We have proved the following.

Theorem 3.1. *The kernel of*

$$\bar{d}_2 \colon (\mathbb{Z}G/I)\gamma \oplus (\mathbb{Z}G/I)\gamma' \to (\mathbb{Z}G/I)\beta \oplus (\mathbb{Z}G/I)\beta'$$

is generated by the elements $(1 + z)(yx^p - 1)\gamma$, $(1 + z)(xy - 1)\gamma'$ *and*

$$(1 - x^{1+p} + x^{-p}y^{-2} - xy^{-2}z)(y - z)\gamma + (1 + xy^{-1})(1 - y^2)\gamma'.$$

Here, I is the ideal of $\mathbb{Z}G$ generated by the central elements N_a and N_b. $\square$

4. The switchback map

To make use of the above theorem, we examine the switchback map for the snake lemma applied to the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & I\gamma \oplus I\gamma' \arrow[r] \arrow[d, "\underline{d}_2"] & \mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma' \arrow[r] \arrow[d, "d_2"] & (\mathbb{Z}G/I)\gamma \oplus (\mathbb{Z}G/I)\gamma' \arrow[r] \arrow[d, "\bar{d}_2"] & 0 \\ 0 \arrow[r] & I\beta \oplus I\beta' \arrow[r] & \mathbb{Z}G\beta \oplus \mathbb{Z}G\beta' \arrow[r] & (\mathbb{Z}G/I)\beta \oplus (\mathbb{Z}G/I)\beta' \arrow[r] & 0 \end{tikzcd}$$

Lemma 4.1. *As a $\mathbb{Z}G$-module, the image of the switchback map* $\ker \bar{d}_2 \to \operatorname{coker} \underline{d}_2$ *for the above diagram is a direct sum $M_p \oplus M_q$. The submodule $M_p \subseteq I\beta$ is generated by $(1-y)N_x\beta$, killed by multiplication by p, while the submodule $M_q \subseteq I\beta'$ is generated by $(1-x)N_y\beta'$, killed by multiplication by q.*

Proof. We apply the switchback map to the generators of $\ker \bar{d}_2$ named in Theorem 3.1. Equations (34), (36) and (40) imply that d_2 sends these to $(1-y)N_x\beta \in I\beta$, $(1-x)N_y\beta' \in I\beta'$ and zero respectively. It follows that the image of the switchback map is $M_p \oplus M_q \subseteq I\beta \oplus I\beta'$.

As an element of $\operatorname{coker} \underline{d}_2$, $(1-y)N_x\beta$ is killed by p, since equations (33) and (34), and the fact that d_2 is a module homomorphism give

$$\underline{d}_2 N_a(1+z)(yx^p-1)\gamma = N_a(1-y)N_x\beta = p(1-y)N_x\beta.$$

Similarly, equations (33) and (36) show that $(1-x)N_y\beta'$ is killed by q as an element of $\operatorname{coker} \underline{d}_2$. $\square$

Lemma 4.2. (i) *The annihilator in $\mathbb{Z}G$ of $(1-y)N_x\beta$ is the left ideal generated by the elements $1-x^{1+p}$, yx^p+1 and $y^{2q-1}+\cdots+y+1$.* (ii) *The annihilator in $\mathbb{Z}G$ of $(1-x)N_y\beta'$ is the left ideal generated by the elements $1-y^2$, $xy+1$ and $x^{2p-1}+\cdots+x+1$.*

Proof. (i) We have

$$\begin{aligned}
(1-x^{1+p})(1-y)N_x\beta &= (1-x^{1+p}-y+yx^{-1-p})N_x\beta \\
&= (1+yx^{-1-p})(1-x^{1+p})N_x\beta = 0, \\
(yx^p+1)(1-y)N_x\beta &= (yx^p+1-x^p-y)N_x\beta \\
&= (1-y)(1-x^p)N_x\beta = 0 \\
(y^{2q-1}+\cdots+y+1)(1-y)N_x\beta &= (1-y^{2q})N_x\beta = (1-x^{2p})N_x\beta = 0,
\end{aligned}$$

so these elements are in the annihilator. Since $\mathbb{Z}G$ has a basis consisting of the elements $y^n x^m$ with $0 \le n \le 2q-1$ and $0 \le m \le 4p-1$, the elements y^n and $y^n x^p$ span $\mathbb{Z}G/\mathbb{Z}G(1-x^{1+p})$. Then the elements y^n with $0 \le n \le 2q-1$ span $\mathbb{Z}G/\mathbb{Z}G(1-x^{1+p}, yx^p+1)$ and finally the elements y^n with $0 \le n \le 2q-2$ span

$$\mathbb{Z}G/\mathbb{Z}G(1-x^{1+p}, yx^p+1, y^{2q-1}+\cdots+y+1).$$

Since the submodule of $\mathbb{Z}G\beta$ generated by $(1-y)N_x\beta$ has $\mathbb{Z}$-rank $2q-1$ by Lemma 4.1, these elements must generate the annihilator.

(ii) We have

$$\begin{aligned}
(1-y^2)(1-x)N_y\beta' &= (1-y^2-x+xy^{-2})N_y\beta' \\
&= (1+xy^{-2})(1-y^2)N_y\beta' = 0, \\
(xy+1)(1-x)N_y\beta' &= (xy+1-y^{2q-1}-x)N_y\beta' \\
&= (x-(y^{2q-2}+\cdots+y+1))(y-1)N_y\beta' = 0, \\
(x^{2p-1}+\cdots+1)(1-x)N_y\beta' &= (1-x^{2p})N_y\beta' = (1-y^{2q})N_y\beta' = 0,
\end{aligned}$$

so these elements are in the annihilator. The rest of the proof is parallel to the proof of (i). □

Proposition 4.3. *The kernel of d_2 on $\mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma'$ is generated as a $\mathbb{Z}G$-module by the elements*

$$(1-x^{1+p})(1+z)(yx^p-1)\gamma \tag{42}$$

$$(1-x^{1+p}+x^{-p}y^{-2}-xy^{-2}z)(y-z)\gamma + (1+xy^{-1})(1-y^2)\gamma' \tag{43}$$

together with possibly some further elements of $I\gamma \oplus I\gamma'$.

Proof. The image in $\mathbb{Z}G/I \oplus \mathbb{Z}G/I$ of the kernel of d_2 is the same as the kernel of the switchback map on the kernel of $\bar{d}_2$. The kernel of $\bar{d}_2$ is calculated in Theorem 3.1, and the switchback map is described in Lemma 4.1. Thus using Lemma 4.2, we see that the kernel is generated by the elements (42) and (43) together with

$$(yx^p+1)(1+z)(yx^p-1)\gamma \tag{44}$$

$$(y^{2q-1}+\cdots+y+1)(1+z)(yx^p-1)\gamma \tag{45}$$

$$(N_a-p)(1+z)(yx^p-1)\gamma \tag{46}$$

$$(1-y^2)(1+z)(xy-1)\gamma' \tag{47}$$

$$(xy+1)(1+z)(xy-1)\gamma' \tag{48}$$

$$(N_b-q)(1+z)(xy-1)\gamma' \tag{49}$$

$$(x^{2p-1} + \cdots + x + 1)(1 + z)(xy - 1)\gamma'. \tag{50}$$

The elements (44) and (48) are equal to zero, the elements (45) and (50) are in $I\gamma \oplus I\gamma'$, and the element (47) is equal to $(1 + yx^{p-1})$ times (42) minus $(1 + z)$ times (43). The element (46) is equal to $(1 + 2a + 3a^2 + \cdots + pa^{p-1})$ times (42) (note that $x^{1+p} = a$), and the element (49) is equal to $(1+2b^2+3b^4+\cdots+qb^{2(q-1)})$ times (47). □

In order to make use of Lemma 4.1 for rank calculations, it is necessary next to calculate $\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G/I)$.

Lemma 4.4. *We have* $\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G/I = 0$ *and* $\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G/I) = 0$.

Proof. The short exact sequence of $\mathbb{Z}G$-modules

$$0 \to I \to \mathbb{Z}G \to \mathbb{Z}G/I \to 0$$

gives rise to an exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G/I) \to \mathbb{Z} \otimes_{\mathbb{Z}G} I \to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \to \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G/I \to 0.$$

Now I is generated as a submodule of $\mathbb{Z}G$ by the elements $u_1 = N_a$ and $u_2 = N_b$, which satisfy the relations $(a-1)u_1 = 0$, $(b-1)u_2 = 0$ and $N_b u_1 = N_a u_2$. A count of $\mathbb{Z}$-ranks shows that this is a presentation of the $\mathbb{Z}G$-module I.

Tensoring over $\mathbb{Z}G$ with $\mathbb{Z}$ via the augmentation map, we obtain generators $\bar{u}_1 = 1 \otimes N_a$ and $\bar{u}_2 = 1 \otimes N_b$ subject only to the relation $q\bar{u}_1 = p\bar{u}_2$. Choose integers r and s so that $rp + sq = 1$, and set $\bar{u}_3 = r\bar{u}_1 + s\bar{u}_2$. Then $p\bar{u}_3 = \bar{u}_1$ and $q\bar{u}_3 = \bar{u}_2$. So $\mathbb{Z} \otimes_{\mathbb{Z}G} I$ is a copy of $\mathbb{Z}$ generated by $\bar{u}_3$. The image in $\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G \cong \mathbb{Z}$ of $\bar{u}_1$ is p and of $\bar{u}_2$ is q, so the image of $\bar{u}_3$ is $rp+sq = 1$. It follows that the middle arrow in the above exact sequence is an isomorphism, and the two end terms are zero. □

Proposition 4.5. *The sequence*

$$\mathbb{Z}G/I \oplus \mathbb{Z}G/I \xrightarrow{\bar{d}_2} \mathbb{Z}G/I \oplus \mathbb{Z}G/I \xrightarrow{\bar{d}_1} \mathbb{Z}G/I \to 0$$

is exact.

Proof. This follows directly from the lemma, since the homology of this sequence calculates $\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G/I$ and $\mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G/I)$. □

Corollary 4.6. *The kernel and cokernel of* $\bar{d}_2$ *both have* $\mathbb{Z}$*-rank* $8(p-1)(q-1)$.

Proof. Since $\mathcal{O}_K$ has $\mathbb{Z}$-rank $(p-1)(q-1)$ and $\mathbb{Z}G/I$ has $\mathcal{O}_K$-rank 8, it follows that $\mathbb{Z}G/I$ has $\mathbb{Z}$-rank $8(p-1)(q-1)$. □

Corollary 4.7. *The kernel and cokernel of d_2 as a map from $I \oplus I$ to $I \oplus I$ both have torsion-free $\mathbb{Z}$-rank equal to $8p+8q-9$. The kernel is torsion-free, while the cokernel has a p-torsion summand M_p and a q-torsion summand M_q.*

Proof. According to Lemma 4.1 the image of the switchback map is a sum of a p-torsion module M_p and a q-torsion module M_q. So the torsion-free rank of the cokernel of d_2 on $I \oplus I$ is given by subtracting the torsion-free rank of the cokernel of $\bar{d}_2$ on $\mathbb{Z}G/I \oplus \mathbb{Z}G/I$ from the torsion-free rank of the cokernel of d_2 on $\mathbb{Z}G \oplus \mathbb{Z}G$. By Corollary 4.6, this gives a torsion-free rank of

$$(8pq-1) - 8(p-1)(q-1) = 8p + 8q - 9.$$

The torsion-free rank of the kernel is the same as that of the cokernel. □

Theorem 4.8. *The kernel of d_2 on $\mathbb{Z}G\gamma \oplus \mathbb{Z}G\gamma'$ is generated as a $\mathbb{Z}G$-module by the elements* (42) *and* (43), *together with $N_aN_b[(j-1)\gamma + (\bar{\imath}-1)\gamma']$. The latter element may be added into element* (42) *or into element* (43) *to give just two generators.*

Proof. Multiplying N_a by the element (43) gives

$$N_a(k - \bar{k} - j + \bar{\jmath})b^{-2}\gamma + N_a(kb^{-1} + 1 - \bar{k}b - zb^2)\gamma'$$

which generates a $\mathbb{Z}G$-submodule M_1 of $I \oplus I$ of $\mathbb{Z}$-rank $8q-4$ annihilated by $N_b(1+z)$. Multiplying N_b by the element (43) gives

$$N_b(i - z + k - j - ai + az - a\bar{k} + a\bar{\jmath})\gamma + N_b(1 + ak - z - a\bar{k})\gamma',$$

which generates a $\mathbb{Z}G$-submodule M_2 of $I \oplus I$ of $\mathbb{Z}$-rank $6p-3$ annihilated by $(1+z)(1-j)$ and by $N_a(1+z)$. The intersection $M_1 \cap M_2$ consists of the $\mathbb{Z}G$-submodule generated by N_aN_b times (43), annihilated by $(1+z)$, of $\mathbb{Z}$-rank 4, so $M_1 + M_2$ has $\mathbb{Z}$-rank $6p + 8q - 11$.

Multiplying N_b by the element (42) gives

$$N_b(1-a)(-1 - z + k + \bar{k})\gamma.$$

This is annihilated by $(1-z)$, and $(1+j)$ times this element equals $(1+z)N_b$ times (43). Letting M_3 be the $\mathbb{Z}G$-module generated by this element, we see that $(M_1+M_2+M_3)/(M_1+M_2)$ is annihilated by $(1-z)$ and $(1+j)$ and has $\mathbb{Z}$-rank $2p-2$. So $M_1+M_2+M_3$ has $\mathbb{Z}$-rank $8p+8q-13$. It is not hard to verify that it is a $\mathbb{Z}$-summand of $I \oplus I$, so it follows from Corollary 4.7 that $M_1+M_2+M_3$ is equal to the kernel of d_2 on $I \oplus I$ modulo multiples of N_aN_b. The $\mathbb{Z}G$-submodule generated by $N_aN_b[(j-1)\gamma + (\bar{\imath}-1)\gamma']$ is easily checked to be in the kernel, and has $\mathbb{Z}$-rank 7. Its intersection with $M_1+M_2+M_3$ has $\mathbb{Z}$-rank 3, so we are up to $8p+8q-9$ as required. The theorem now follows from Proposition 4.3. □

5. The Swan obstruction

Swan [8] has proved that a finite group G acts freely and cellularly on some finitely dominated CW complex with the homotopy type of an $(n-1)$-sphere S^{n-1} if and only if the integral cohomology of G is periodic with period dividing n. This means that $H^n(G,\mathbb{Z}) \cong \mathbb{Z}/|G|$, and that cup product with an additive generator of $H^n(G,\mathbb{Z})$ induces an isomorphism on Tate cohomology $\hat{H}^i(G,\mathbb{Z}) \to \hat{H}^{i+n}(G,\mathbb{Z})$ for all $i \in \mathbb{Z}$. The complex may be taken to be finite if and only if an obstruction called the Swan invariant vanishes. This invariant lies in a quotient of $\tilde{K}_0(\mathbb{Z}G)$ by a subgroup T_G called the Swan subgroup. This section contains an explanation of the Swan invariant, based on Jon Carlson's modules L_ζ.

Suppose that G is a periodic group. For a finitely generated $\mathbb{Z}G$-module M, we write $\Omega^n M$ for the nth kernel in a resolution of M by finitely generated free $\mathbb{Z}G$-modules. This is only well defined up to adding and removing finitely generated free summands (by the extended version of Schanuel's lemma). If $\zeta \in \hat{H}^n(G,\mathbb{Z})$, we can represent ζ by a cocycle $\hat{\zeta}\colon \Omega^n\mathbb{Z} \to \mathbb{Z}$. By adding a free summand to $\Omega^n\mathbb{Z}$ if necessary, we may assume that this map is surjective (even if $\zeta = 0$!), and we write $L_\zeta = \mathrm{Ker}(\hat{\zeta})$. So there is a short exact sequence

$$0 \to L_\zeta \to \Omega^n\mathbb{Z} \xrightarrow{\hat{\zeta}} \mathbb{Z} \to 0. \tag{51}$$

An element $\zeta \in \hat{H}^n(G,\mathbb{Z})$ lies in the multiplicative group $\hat{H}^*(G,\mathbb{Z})^\times$ of invertible elements if and only if L_ζ is projective. In this case, we get a well defined element $[L_\zeta] \in \tilde{K}_0(\mathbb{Z}G)$. Here, $\tilde{K}_0(\mathbb{Z}G)$ denotes the quotient of the Grothendieck ring $K_0(\mathbb{Z}G)$ of finitely generated projective $\mathbb{Z}G$-modules by the subgroup of finitely generated free $\mathbb{Z}G$-modules, so that $K_0(\mathbb{Z}G) \cong \mathbb{Z} \times \tilde{K}_0(\mathbb{Z}G)$ and $\tilde{K}_0(\mathbb{Z}G)$ is finite [9].

The modules L_ζ with $\zeta \in \hat{H}^0(G,\mathbb{Z})^\times$ are called Swan modules. The Swan module corresponding to $m \in (\mathbb{Z}/|G|)^\times \cong \hat{H}^0(G,\mathbb{Z})^\times$ can be described as the (projective) submodule of $\mathbb{Z}G$ generated by the sum of the group elements and by the elements divisible by n where n is an integer satisfying $mn \equiv 1$ modulo $|G|$. In other words, the Swan modules are exactly the projective modules which are generated by a single element after quotienting out the image of the sum of the group elements.

Lemma 5.1. *If ζ and η are elements of $\hat{H}^*(G,\mathbb{Z})^\times$ of degrees n and m, then $[L_\zeta][L_\eta] = 0$ and $[L_\zeta] + [L_\eta] = [L_{\zeta\eta}]$ in $\tilde{K}_0(\mathbb{Z}G)$.*

Proof. As elements of $\tilde{K}_0(\mathbb{Z}G)$, we have $[\Omega L_\eta] = -[L_\eta]$. Tensoring the exact sequence (51) with L_η we get

$$0 \to L_\zeta \otimes L_\eta \to \Omega^n L_\eta \to L_\eta \to 0.$$

This is an exact sequence of projective modules, so it splits. Using the fact that an invertible element of Tate cohomology has to lie in even degree, we get

$$[L_\eta] = [\Omega^n L_\eta] = [L_\eta] + [L_\zeta \otimes L_\eta]$$

so that

$$[L_\zeta][L_\eta] = [L_\zeta \otimes L_\eta] = 0.$$

The diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \Omega^n L_\eta \arrow[d] & & \\
0 \arrow[r] & L_{\zeta\eta} \arrow[r] \arrow[d] & \Omega^{n+m}\mathbb{Z} \arrow[r, "\widehat{\zeta\eta}"] \arrow[d, "\Omega^n(\eta)"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L_\zeta \arrow[r] \arrow[d] & \Omega^n\mathbb{Z} \arrow[r, "\hat{\zeta}"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

gives us a short exact sequence

$$0 \to \Omega^n L_\eta \to L_{\zeta\eta} \to L_\zeta \to 0$$

so that

$$[L_{\zeta\eta}] = [L_\zeta] + [\Omega^n L_\eta] = [L_\zeta] + [L_\eta].$$

□

This lemma implies that there is a well defined homomorphism

$$\begin{aligned} \sigma\colon \hat{H}^*(G, \mathbb{Z})^\times &\to \tilde{K}_0(\mathbb{Z}G) \\ \zeta &\mapsto -[L_\zeta] \end{aligned}$$

called the Swan homomorphism. In positive degrees, it has the following interpretation. We form the beginning of a free resolution and form the pushout

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
 & L_\zeta \arrow[r, equal] \arrow[d] & L_\zeta \arrow[d] & & & & & \\
0 \arrow[r] & \Omega^n\mathbb{Z} \arrow[r] \arrow[d, "\hat{\zeta}"] & F_{n-1} \arrow[r] \arrow[d] & F_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & F_0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & F_{n-1}/L_\zeta \arrow[r] \arrow[d] & F_{n-2} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0. \\
 & 0 & 0 & & & & &
\end{tikzcd}$$

The obstruction to rechoosing the resolution so that F_{n-1}/L_ζ is free, is the element

$$[F_{n-1}/L_\zeta] = -[L_\zeta] = \sigma(\zeta) \in \tilde{K}_0(\mathbb{Z}G).$$

This accounts for the negative sign in the description of the map. The image under σ of $\hat{H}^0(G,\mathbb{Z})^\times \cong \mathbb{Z}/|G|^\times$ is called the Swan subgroup $T_G \subseteq \tilde{K}_0(\mathbb{Z}G)$. It should be noted that $\{\pm 1\}$ is in the kernel of σ on $\hat{H}^0(G,\mathbb{Z})^\times$. We obtain a well defined map

$$\bar{\sigma}\colon \hat{H}^*(G,\mathbb{Z})^\times/\hat{H}^0(G,\mathbb{Z})^\times \ (\cong \mathbb{Z}) \ \to \tilde{K}_0(\mathbb{Z}G)/T_G.$$

If $\hat{H}^*(G,\mathbb{Z})^\times/\hat{H}^0(G,\mathbb{Z})^\times$ contains a non-zero element of degree n, then that element is unique, and we write σ_n for its image in $\tilde{K}_0(\mathbb{Z}G)/T_G$. The interpretation of T_G here is that it is the indeterminacy caused by the freedom to replace ζ by $m\zeta$, where m is an integer coprime to $|G|$.

If G acts freely on a finite CW complex X with the homotopy type of an $(n-1)$-sphere, then the cellular chains on X form a complex of finitely generated free $\mathbb{Z}G$-modules with homology only in degrees zero and $n-1$:

$$0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to 0.$$

If $m \geq n$, then since projective $\mathbb{Z}G$-modules are weakly injective (Cartan and Eilenberg [2], Section XII.1), we may strip off the terms in this complex in degrees bigger than n and insert the homology of the complex to make an exact sequence

$$0 \to \mathbb{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbb{Z} \to 0$$

with F_i free ($i < n-1$) and F_{n-1} stably free. It follows that the element $\sigma_n \in \tilde{K}_0(\mathbb{Z}G)/T_G$ is zero. The element σ_n is called the Swan obstruction, because it is the obstruction to the existence of such a complex.

If we invert $|G|$ then we have $\hat{H}^*(G, \mathbb{Z}(1/|G|)) = 0$. It follows that the sequence of $\mathbb{Z}(1/|G|)G$-modules

$$0 \to L_\zeta(1/|G|) \to \Omega^n \mathbb{Z}(1/|G|) \to \mathbb{Z}(1/|G|) \to 0$$

necessarily splits, giving

$$0 = [L_\zeta(1/|G|)] \in \tilde{K}_0(\mathbb{Z}(1/|G|)G).$$

If p is a prime dividing $|G|$ and $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers, then $\tilde{K}_0(\mathbb{Z}_p^\wedge G)$ is a finitely generated free abelian group, by the Krull–Schmidt theorem. Since $\tilde{K}_0(\mathbb{Z}G)$ is finite, it follows that the inclusion $\mathbb{Z} \to \mathbb{Z}_p^\wedge$ induces the zero map $\tilde{K}_0(\mathbb{Z}G) \to \tilde{K}_0(\mathbb{Z}_p^\wedge G)$. Now examine the long exact sequence in K-theory of the arithmetic square

$$\begin{aligned} K_1(\mathbb{Z}(1/|G|)G) \oplus \bigoplus_{p|\,|G|} K_1(\mathbb{Z}_p^\wedge G) \to \bigoplus_{p|\,|G|} K_1(\mathbb{Q}_p^\wedge G) \to \\ K_0(\mathbb{Z}G) \to K_0(\mathbb{Z}(1/|G|)G) \oplus \bigoplus_{p|\,|G|} K_0(\mathbb{Z}_p^\wedge G) \to \bigoplus_{p|\,|G|} K_0(\mathbb{Q}_p^\wedge G) \end{aligned}$$

(see Davis and Milgram [3], page 246). Using the fact that $\tilde{K}_0(\mathbb{Z}G)$ is finite and $K_0(\mathbb{Z}_p^\wedge G)$ is $\mathbb{Z}$-free, we see that a representative of σ_n in $\tilde{K}_0(\mathbb{Z}G)$ goes to zero in the next term in this sequence. So σ_n lifts to an element of $\bigoplus_{p|\,|G|} K_1(\mathbb{Q}_p^\wedge G)$ that is well defined modulo the inverse image of T_G.

The calculations in K_1 have been carried out for the groups considered in this paper by Milgram [6]) and later by Madsen [5]) and Bentzen and Madsen [1]. The result of these calculations is as follows. Let A denote the ring $\mathbb{Z}[\eta_p, \eta_q]$ where $\eta_p = \zeta_p + \zeta_p^{-1}$ and $\eta_q = \zeta_q + \zeta_q^{-1}$, and ζ_p and ζ_q are primitive pth and qth roots of unity respectively, with p and q distinct odd primes. Denote by $A^\times$ the units in A and by A^* the totally positive units in A. Consider the reductions modulo p and q

$$\Phi_A \colon A^\times \to (A/pA)^\times_{(2)} \oplus (A/qA)^\times_{(2)}.$$

Here, (2) denotes 2-primary component. Then $\sigma_4 = 0$ if and only if

(i) p and q are not both congruent to 3 modulo 8, and
(ii) $(4,4)$ is in the image of Φ_A restricted to A^*

(Proposition 5.2 of [1]).

Condition (i) is guaranteed since we have assumed that $q \equiv 1 \pmod 4$. Condition (ii) is therefore necessary, in order that the two generators given in Theorem 4.8 can be replaced by a single generator. It seems plausible that this arithmetic condition is also sufficient, but the author has not succeeded in proving this.

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