International Electronic Journal of Algebra
Volume 6 (2009) 74-94

WEAK HOPF ALGEBRA DUALITY IN WEAK YETTER-DRINFELD CATEGORIES AND APPLICATIONS

Bing-liang Shen and Shuan-hong Wang

Received: 8 June 2008; Revised: 3 March 2009
Communicated by Nanqing Ding

Abstract. Let L be a weak Hopf algebra with a bijective antipode S_L in the sense of [3]. In this paper we show that if H is a finite-dimensional weak Hopf algebra in the weak Yetter-Drinfeld category ${}_L^L\mathcal{WYD}$ in the sense of [1], then its dual H^* is also a weak Hopf algebra in ${}_L^L\mathcal{WYD}$. Also we will apply above result to the representations category $\mathrm{Rep}(L) = {}_L\mathcal{M}$ of a quasitriangular weak Hopf algebra L.

Mathematics Subject Classification (2000): 16W30
Keywords: Weak Hopf algebra, weak Yetter-Drinfeld categories, weak (co)module (co)algebras, duality

1. Introduction

Weak Hopf algebras which are generalizations of ordinary Hopf algebras, were defined by Böhm and Szlachányi in their paper [3]. A general theory for these objects was subsequently developed in [2]. The axioms are the same as the ones for a Hopf algebra, except that the coproduct of the unit, the product of the counit and the antipode condition are replaced by weaker properties. The main motivation for studying weak Hopf algebras comes from quantum field theories, operator algebras and representation theory (cf. [7, 8, 10, 11]). It has turned out that many results of classical Hopf algebra theory can be generalized to weak Hopf algebras. Despite it, the structure of a weak Hopf algebra is much more complicated than that of a Hopf algebra.

In the classical Hopf algebra theory, it is well-known that the dual of a finite-dimensional Hopf algebra is still a Hopf algebra. In 1998, Doi [6] had showed that if H is a finite-dimensional Hopf algebra in the Yetter-Drinfeld category ${}_L^L\mathcal{YD}$ over a Hopf algebra L, then its dual H^* is also a Hopf algebra in ${}_L^L\mathcal{YD}$.

Just like finite-dimensional Hopf algebras, finite-dimensional weak Hopf algebras also obey the mathematical beauty of giving rise to a self-dual notion: the dual of

This work was partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education (20060286006) and the FNS of CHINA (10571026).

it can be canonically endowed with a weak Hopf algebra structure. The notion of a weak Yetter-Drinfeld category ${}_L^L\mathcal{WYD}$ over a weak Hopf algebra L has been introduced by Böhm in [1], and further studied by Caenepeel et al. in [5].

It is very natural to ask whether or not a finite-dimensional weak Hopf algebra in ${}_L^L\mathcal{WYD}$ is self-dual?

In this paper, we discuss this problem, proving that if H is a finite-dimensional weak Hopf algebra in the category ${}_L^L\mathcal{WYD}$ over a weak Hopf algebra L, then its linear dual H^* is also a weak Hopf algebra in ${}_L^L\mathcal{WYD}$.

2. Basic Definitions and Results

In this section, we recall some basic definitions and results related to weak Hopf algebras introduced in [2,3] and also about weak Yetter-Drinfeld categories ${}_L^L\mathcal{WYD}$ given in [1] and [5] that we will need later.

Throughout this paper, k denotes a fixed field. We will work over k. L denotes a weak Hopf algebra with a bijective antipode S_L, and H denotes a weak Hopf algebra in the weak Yetter-Drinfeld category ${}_L^L\mathcal{WYD}$. For an algebra A and a coalgebra C, we have the convolution algebra $Conv(C,A) = Hom(C,A)$ as spaces, but with the multiplication given by

$$(f * g)(c) = m_A(f \otimes g)\Delta_C(c) = f(c_1)g(c_2),$$

for all $f, g \in Hom(C,A), c \in C$.

2.1. Weak bialgebras.

Recall from [2,3] that *a weak k-bialgebra* L is both a k-algebra (m,μ) and a k-coalgebra (Δ,ε) such that $\Delta(hk) = \Delta(h)\Delta(k)$, for all $h, k \in L$, and

$$\Delta^2(1) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2, \tag{1}$$

$$\varepsilon(hkl) = \varepsilon(hk_1)\varepsilon(k_2 l) = \varepsilon(hk_2)\varepsilon(k_1 l), \tag{2}$$

for all $h, k, l \in L$, where $1'$ stands for another copy of 1. We use the Sweedler's notation (see [12]) for the comultiplication. Namely,

$$\Delta(h) = h_1 \otimes h_2.$$

We summarize the elementary properties of weak bialgebras. The maps $\varepsilon_t, \varepsilon_s$: $L \longrightarrow L$ defined by

$$\varepsilon_t(h) = \varepsilon(1_1 h)1_2; \quad \varepsilon_s(h) = 1_1\varepsilon(h1_2).$$

ε_t and ε_s are called the target map and source map, and their imagines L_t and L_s are called the target and source space. For all $g, h \in L$, we have,

$$h_1 \otimes \varepsilon_t(h_2) = 1_1 h \otimes 1_2;\ \ \varepsilon_s(h_1) \otimes h_2 = 1_1 \otimes h1_2.$$

The source and target space can be described as follows:

$$L_t = \{h \in L \mid \varepsilon_t(h) = h\} = \{h \in L \mid \Delta(h) = 1_1 h \otimes 1_2 = h1_1 \otimes 1_2\},$$
$$L_s = \{h \in L \mid \varepsilon_s(h) = h\} = \{h \in L \mid \Delta(h) = 1_1 \otimes h1_2 = 1_1 \otimes 1_2 h\}.$$

We also have

$$\varepsilon_t(h)\varepsilon_s(k) = \varepsilon_s(k)\varepsilon_t(h),$$

and its dual property

$$\varepsilon_s(h_1) \otimes \varepsilon_t(h_2) = \varepsilon_s(h_2) \otimes \varepsilon_t(h_1).$$

Finally $\varepsilon_t(1) = \varepsilon_s(1) = 1$ and

$$\varepsilon_t(h)\varepsilon_t(g) = \varepsilon_t(\varepsilon_t(h)g);\ \ \ \varepsilon_s(h)\varepsilon_s(g) = \varepsilon_s(h\varepsilon_s(g)).$$

This implies that L_t and L_s are subalgebras of L.

2.2. Weak Hopf algebras.

A weak Hopf algebra L is a weak bialgebra together with a k-linear map $S : L \longrightarrow L$ (called the antipode) satisfying

$$S * id = \varepsilon_s,\ id * S = \varepsilon_t,\ S * id * S = S,$$

where $*$ is the convolution product. It follows immediately that

$$S = \varepsilon_s * S = S * \varepsilon_t.$$

If the antipode exists, then it is unique. The antipode S is both an anti-algebra and an anti-coalgebra morphism. We will always assume that S is bijective. If L is a finite-dimensional weak Hopf algebra over k, then S is automatically bijective. Now we recall some properties about S.

By [2], let L be a weak Hopf algebra, then we have the following conclusions:

(1) $\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s, \quad \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t,$ (3)

(2) $x_1 \otimes x_2 S(x_3) = x_1 \otimes \varepsilon_t(x_2) = 1_1 x \otimes 1_2,$ (4)

(3) $S(x_1)x_2 \otimes x_3 = \varepsilon_s(x_1) \otimes x_2 = 1_1 \otimes x 1_2,$ (5)

(4) $x_1 \otimes S(x_2)x_3 = x_1 \otimes \varepsilon_s(x_2) = x 1_1 \otimes S(1_2),$ (6)

(5) $x_1 S(x_2) \otimes x_3 = \varepsilon_t(x_1) \otimes x_2 = S(1_1) \otimes 1_2 x,$ (7)

(6) $x_1 y \otimes x_2 = x_1 \otimes x_2 S(y), \quad \text{for all} \quad y \in L_s,$ (8)

(7) $x_1 \otimes z x_2 = S(z) x_1 \otimes x_2, \quad \text{for all} \quad z \in L_t.$ (9)

Let L be a weak Hopf algebra with a bijective antipode S_L, then L^{cop} is also a weak Hopf algebra with antipode $\bar{S}$ (here $\bar{S}$ is the composite-inverse of the antipode S_L). At this time

$$\bar{S}(h_2)h_1 = \bar{S}\varepsilon_s(h) = \varepsilon(h 1_1)1_2 \triangleq \tilde{\varepsilon}_t(h), \quad h_2\bar{S}(h_1) = \bar{S}\varepsilon_t(h) = 1_1\varepsilon(1_2 h) \triangleq \tilde{\varepsilon}_s(h).$$

We can easily get the following equations from [4, Proposition 4.8]:

(*i*) $\tilde{\varepsilon}_t \circ \varepsilon_t = \varepsilon_t, \qquad \varepsilon_t \circ \tilde{\varepsilon}_t = \tilde{\varepsilon}_t,$

(*ii*) $\tilde{\varepsilon}_s \circ \varepsilon_s = \varepsilon_s, \qquad \varepsilon_s \circ \tilde{\varepsilon}_s = \tilde{\varepsilon}_s,$

(*iii*) $\tilde{\varepsilon}_t \circ \varepsilon_s = \tilde{\varepsilon}_t, \qquad \varepsilon_t \circ \tilde{\varepsilon}_s = \varepsilon_t,$ (10)

(*iv*) $\tilde{\varepsilon}_s \circ \varepsilon_t = \tilde{\varepsilon}_s, \qquad \varepsilon_s \circ \tilde{\varepsilon}_t = \varepsilon_s.$ (11)

2.3. Weak (co)module (co)algebras.

Let L be a weak Hopf algebra.

(i) Recall from [9], an algebra H is *a left weak L-module algebra* if H is left L-module via $l \otimes x \mapsto l \rightarrow x$ such that

(1) $l \rightarrow (xy) = (l_1 \rightarrow x)(l_2 \rightarrow y)$ for all $l \in L,\ x, y \in H$,

(2) $l \rightarrow 1_H = \varepsilon_t(l) \rightarrow 1_H$.

The second equation is equivalent to

$$\tilde{\varepsilon}_s(l) \rightarrow x = x(l \rightarrow 1_H) \qquad \text{or} \qquad \varepsilon_t(l) \rightarrow x = (l \rightarrow 1_H)x. \tag{12}$$

(ii) Recall from [1], an algebra H is *a left weak L-comodule algebra* if H is a left L-comodule via $x \mapsto \sigma_H(x) = x^{-1} \otimes x^0$ such that

(1) $\sigma_H(xy) = \sigma_H(x)\sigma_H(y)$ *i.e.*, $(xy)^{-1} \otimes (xy)^0 = x^{-1}y^{-1} \otimes x^0 y^0$,

(2) $1^{-1} \otimes x1^0 = \varepsilon_s(x^{-1}) \otimes x^0$ for all $x \in H$.

(iii) Recall from [1], a coalgebra H is *a left weak L-module coalgebra* if H is a left L-module via $l\otimes x\mapsto l\rightarrow x$ such that

(1) $\Delta(l\rightarrow x)=(l\rightarrow x)_1\otimes(l\rightarrow x)_2=(l_1\rightarrow x_1)\otimes(l_2\rightarrow x_2)$,

(2) $\varepsilon_s(l)\rightarrow x=x_1\varepsilon(l\rightarrow x_2)$ for all $l\in L,\ x\in H$.

By [4, Proposition 4.13], the second equation is equivalent to

$$\varepsilon(lk\rightarrow h)=\varepsilon(lk_2)\varepsilon(k_1\rightarrow h)\qquad \text{or}\qquad \varepsilon(\varepsilon_s(l)\rightarrow h)=\varepsilon(l\rightarrow h). \tag{13}$$

(iv) A coalgebra H is *a left weak L-comodule coalgebra* if H is a left L-comodule via $x\mapsto\sigma_H(x)=x^{-1}\otimes x^0$ such that

(1) $x^{-1}\otimes(x^0)_1\otimes(x^0)_2={x_1}^{-1}{x_2}^{-1}\otimes{x_1}^0\otimes{x_2}^0$,

(2) $\varepsilon(x^0)x^{-1}=\varepsilon(x^0)\varepsilon_t(x^{-1})$ for all $x\in H$.

2.4. Weak Yetter-Drinfeld category ${}_L^L\mathcal{WYD}$.

Let L be a weak Hopf algebra with a bijective antipode S_L. We recall from [1] and [5] that *the weak Yetter-Drinfeld category* ${}_L^L\mathcal{WYD}$ is the braided monoidal categories whose objects V are both left L-modules and left L-comodules and satisfy the following compatibility conditions:

$$\begin{aligned}
&(1)\quad \sigma_V(v)=v^{-1}\otimes v^0\in L\otimes_t V=\{1_1l\otimes 1_2\rightarrow v\mid \forall\ \ l\in L, v\in V\};\\
&(2)\quad l_1v^{-1}\otimes l_2\rightarrow v^0=(l_1\rightarrow v)^{-1}l_2\otimes(l_1\rightarrow v)^0,\\
&\qquad\ \text{or equivalently,}\\
&(3)\quad \sigma_V(l\rightarrow v)=(l\rightarrow v)^{-1}\otimes(l\rightarrow v)^0=l_1v^{-1}S(l_3)\otimes l_2\rightarrow v^0,\\
&\qquad\ \text{for all } v\in V, l\in L,
\end{aligned}$$

where the L-module action is denoted $l\rightarrow v$ for $l\in L, v\in V$ and the L-comodule structure map by $\sigma_V: V\rightarrow L\otimes V$. We use the following notation:

$$\sigma_V(v)=v^{-1}\otimes v^0,\ (\Delta\otimes id)\sigma_V(v)=(id\otimes\sigma_V)\sigma_V(v)=v^{-2}\otimes v^{-1}\otimes v^0,\quad ...$$

The braiding $\tau=\tau_{V,W}: V\otimes_t W\rightarrow W\otimes_t V$ in this category is given by:

$$\begin{aligned}
&\tau(1_1\rightarrow v\otimes 1_2\rightarrow w)=v^{-1}\rightarrow w\otimes v^0,\\
&\tau^{-1}(1_1\rightarrow w\otimes 1_2\rightarrow v)=v^0\otimes\bar{S}(v^{-1})\rightarrow w.
\end{aligned}$$

Proposition 2.1. *Let $V \in^L_L \mathcal{WYD}$. Then for all $v \in V$, we have*

(1) $\varepsilon_t(v^{-1}) \to v^0 = \varepsilon(v^{-1})v^0 = v,$

(2) $\varepsilon_s(\bar{S}^2(v^{-1})) \to v^0 = v,$

(3) $1_1 \to v^0 \otimes 1_2\bar{S}(v^{-1}) = v^0 \otimes \bar{S}(v^{-1}),$ (14)

(4) $\sigma_v(v) = v^{-1} \otimes v^0 = 1_1 v^{-1} \otimes 1_2 \to v^0 = v^{-1}S(1_2) \otimes 1_1 \to v^0,$ (15)

(5) $\tilde{\varepsilon}_t(v^{-1}) \otimes v^0 = 1_2 \otimes 1_1 \to v,$ (16)

(6) $\tilde{\varepsilon}_s(v^{-1}) \otimes v^0 = 1_1 \otimes 1_2 \to v,$ (17)

(7) $\varepsilon_s(v^{-1}) \otimes v^0 = S(1_2) \otimes 1_1 \to v,\ \varepsilon_t(v^{-1}) \otimes v^0 = S(1_1) \otimes 1_2 \to v.$

Proof. The results of (1)-(4) can be found in [5].

(5) For any $v \in V$, we compute

$$\begin{aligned} & \tilde{\varepsilon}_t(v^{-1}) \otimes v^0 = \bar{S}(\varepsilon_s(v^{-1})) \otimes v^0 \\ = \ & \bar{S}(v^{-1})v^{-2} \otimes v^0 \overset{(3)}{=} 1_2\bar{S}(v^{-1})v^{-2} \otimes 1_1 \to v^0 \\ = \ & \bar{S}(\varepsilon_s(v^{-1})1_1) \otimes \bar{S}(1_2) \to v^0 \\ \overset{(9)}{=} \ & \bar{S}(1_1) \otimes \bar{S}(\bar{S}(\varepsilon_s(v^{-1}))1_2) \to v^0 \\ = \ & 1_2 \otimes 1_1\bar{S}^2(\varepsilon_s(v^{-1})) \to v^0 \\ \overset{(3)}{=} \ & 1_2 \otimes 1_1\varepsilon_s(\bar{S}^2(v^{-1})) \to v^0 \overset{(2)}{=} 1_2 \otimes 1_1 \to v. \end{aligned}$$

(6) For any $v \in V$, we get

$$\begin{aligned} & \tilde{\varepsilon}_s(v^{-1}) \otimes v^0 = \bar{S}(\varepsilon_t(v^{-1})) \otimes v^0 \\ = \ & v^{-1}\bar{S}(v^{-2}) \otimes v^0 = 1_1 v^{-1}\bar{S}(v^{-2}) \otimes 1_2 \to v^0 \\ = \ & 1_1\bar{S}(\varepsilon_t(v^{-1})) \otimes 1_2 \to v^0 \\ \overset{(8)}{=} \ & 1_1 \otimes 1_2\varepsilon_t(v^{-1}) \to v^0 \overset{(1)}{=} 1_1 \otimes 1_2 \to v. \end{aligned}$$

(7) Applying $\varepsilon_s \otimes id$ to the both sides of (5) and $\varepsilon_t \otimes id$ to both sides of (6), we can immediately get (7). □

3. The Dual of Weak Hopf Algebras in Weak Yetter-Drinfeld Categories

In this section, we first give the definition of a weak Hopf algebra in the weak Yetter-Drinfeld category ${}^L_L\mathcal{WYD}$. Then we will show that if H is a finite-dimensional weak Hopf algebra in ${}^L_L\mathcal{WYD}$, then its dual H^* is a weak Hopf algebra in ${}^L_L\mathcal{WYD}$, which generalizes the Hopf case in [6].

Definition 3.1. Let L be a weak Hopf algebra with a bijective antipode S_L. An object $H \in^L_L \mathcal{WYD}$ is called *a weak bialgebra in this category* if it is both a k-algebra and a k-coalgebra satisfying the following conditions:

$$\begin{aligned}(1)\qquad & \Delta(xy) = x_1({x_2}^{-1} \rightarrow y_1) \otimes {x_2}^0 y_2,\\ & \varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2 z),\\ & \varepsilon(xyz) = \varepsilon(x({y_1}^{-1} \rightarrow y_2))\varepsilon({y_1}^0 z),\\ & \Delta^2(1) = 1_1 \otimes 1_2 1'_1 \otimes 1'_2,\\ & \Delta^2(1) = 1_1 \otimes ({1_2}^{-1} \rightarrow 1'_1){1_2}^0 \otimes 1'_2.\end{aligned}$$

(2) H is both a left weak L-module algebra, L-comodule algebra, L-module coalgebra and L-comodule coalgebra.

Furthermore, H is called *a weak Hopf algebra in* ${}^L_L\mathcal{WYD}$ if there exists an antipode $S: H \rightarrow H$ (here S is both left L-linear and L-colinear i.e., S is a morphism in the category of ${}^L_L\mathcal{WYD}$) satisfying

$$\begin{aligned} & x_1 S(x_2) = \varepsilon((x^{-1} \rightarrow 1_1)x^0)1_2,\\ & S(x_1)x_2 = 1_1\varepsilon(({1_2}^{-1} \rightarrow x){1_2}^0),\\ & S(x_1)x_2 S(x_3) = S(x), \text{ for all } x \in H.\end{aligned}$$

An object $H \in^L_L \mathcal{WYD}$ is called an algebra in this category if it is both a k-algebra, a left weak L-module algebra and L-comodule algebra. Similarly an object $H \in^L_L \mathcal{WYD}$ is called a coalgebra in this category if it is both a k-coalgebra, a left weak L-module coalgebra and L-comodule coalgebra.

Similar to the notation of weak Hopf algebras, we denote $\varepsilon_t(x) = \varepsilon((x^{-1} \rightarrow 1_1)x^0)1_2$, $\varepsilon_s(x) = 1_1\varepsilon(({1_2}^{-1} \rightarrow x){1_2}^0)$. As S is both left L-linear and L-colinear, we can easily check that ε_t and ε_s are also both left L-linear and L-colinear.

Assume that H is a weak Hopf algebra in ${}^L_L\mathcal{WYD}$ and finite-dimensional over k. We will make $H^* = Hom(H, k)$ into a weak Hopf algebra in ${}^L_L\mathcal{WYD}$. We first define a left action of L on H^* as:

$$(l \rightarrow f)(h) = f(S_L(l) \rightarrow h), \;\; l \in L, f \in H^*, h \in H$$

and a left coaction of L on H^* as:

$$\sigma_{H^*}: H^* \rightarrow L \otimes H^*, \;\; \sigma_{H^*}(f) = f^{-1} \otimes f^0,$$

where

$$f^0(h)f^{-1} := f(h^0)\bar{S}_L(h^{-1}), \;\; \text{for all } h \in H.$$

Now we have the following proposition.

Proposition 3.2. *With the notation as above. Then* $H^* \in^L_L \mathcal{WYD}$.

Proof. (1) $\sigma_{H^*}(f) = f^{-1} \otimes f^0 \in L \otimes_t H^*$, i.e., $f^{-1}f^0(h) = 1_1 f^{-1}(1_2 \rightarrow f^0)(h)$.

$$\begin{aligned}
1_1 f^{-1}(1_2 \rightarrow f^0)(h) &= 1_1 f^{-1} f^0(S(1_2) \rightarrow h) \\
&= 1_1 f[(S(1_2) \rightarrow h)^0]\bar{S}[(S(1_2) \rightarrow h)^{-1}] \\
&= 1_1 f(S(1_3) \rightarrow h^0)\bar{S}(S(1_4)h^{-1}S^2(1_2)) \\
&= \varepsilon_t(1_1) f(S(1_2) \rightarrow h^0)\bar{S}(h^{-1})1_3 \\
&\overset{(3)}{=} f(S(1_2 1'_1) \rightarrow h^0)S(1_1)\bar{S}(h^{-1})1'_2 \\
&= f(S(1'_1)1_1 \rightarrow h^0)1_2\bar{S}(h^{-1})1'_2 \\
&\overset{(14)}{=} f(S(1_1) \rightarrow h^0)\bar{S}(h^{-1})1_2 \\
&= f(1_2 \rightarrow h^0)\bar{S}(1_1 h^{-1}) \\
&= f(h^0)\bar{S}(h^{-1}) = f^{-1}f^0(h).
\end{aligned}$$

(2) We prove $l_1 f^{-1} \otimes l_2 \rightarrow f^0 = (l_1 \rightarrow f)^{-1} l_2 \otimes (l_1 \rightarrow f)^0$. For all $h \in H$,

$$\begin{aligned}
l_1 f^{-1}(l_2 \rightarrow f^0)(h) &= l_1 f^{-1} f^0(S(l_2) \rightarrow h) \\
&= l_1 f((S(l_2) \rightarrow h)^0)\bar{S}((S(l_2) \rightarrow h)^{-1}) \\
&= l_1 f(S(l_3) \rightarrow h^0)\bar{S}(S(l_4)h^{-1}S^2(l_2)) \\
&= f(S(l_2) \rightarrow h^0)\varepsilon_t(l_1)\bar{S}(h^{-1})l_3 \\
&\overset{(7)}{=} f(S(1_2 l_1) \rightarrow h^0)S(1_1)\bar{S}(h^{-1})l_2 \\
&= f(S(l_1)1_1 \rightarrow h^0)1_2\bar{S}(h^{-1})l_2 \\
&\overset{(14)}{=} f(S(l_1) \rightarrow h^0)\bar{S}(h^{-1})l_2 \\
&= (l_1 \rightarrow f)(h^0)\bar{S}(h^{-1})l_2 \\
&= (l_1 \rightarrow f)^{-1} l_2 (l_1 \rightarrow f)^0(h).
\end{aligned}$$

By (1) and (2), we obtain that $H^* \in {}^L_L\mathcal{WYD}$. □

Lemma 3.3. ([6]) *For any left L-comodule V, define* $\theta_V : H^* \otimes V \rightarrow Hom(H, V)$ *by*

$$\theta_V(f \otimes v)(h) = f(v^{-1} \rightarrow h)v^0, \quad f \in H^*, v \in V, h \in H.$$

Also, define $\theta^{(2)} : H^* \otimes H^* \rightarrow (H \otimes H)^*$ *and* $\theta^{(3)} : H^* \otimes H^* \otimes H^* \rightarrow (H \otimes H \otimes H)^*$ *by*

$$\theta^{(2)}(f \otimes g)(x \otimes y) = f(\bar{S}(y^{-1}) \rightarrow x)g(y^0), \ \ f, g, k \in H^*, x, y, z \in H,$$
$$\theta^{(3)}(f \otimes g \otimes k)(x \otimes y \otimes z) = f(\bar{S}(y^{-1}z^{-2}) \rightarrow x)g(\bar{S}(z^{-1}) \rightarrow y^0)k(z^0).$$

Then $\theta_V, \theta^{(2)}$ *and* $\theta^{(3)}$ *are bijective.*

Lemma 3.4. *If H is a finite-dimensional weak bialgebra in ${}^L_L\mathcal{WYD}$, then H^* is an algebra in ${}^L_L\mathcal{WYD}$, with multiplication $m_{H^*} = (\Delta_H)^* \circ \theta^{(2)}$, unit $\mu_{H^*} = \varepsilon_H$. Explicitly, multiplication is given by*

$$(fg)(x) = f(g^{-1} \rightarrow x_1)g^0(x_2) = f(\bar{S}_L({x_2}^{-1}) \rightarrow x_1)g({x_2}^0), \ f, g \in H^*, x \in H.$$

Proof. We will take three steps as follows:

Step 1): We first prove that H^* is an algebra.

For the associativity, we have

$$\begin{aligned}
& (fg)k(x) \\
= \ & fg(\bar{S}({x_2}^{-1}) \rightarrow x_1)k({x_2}^0) \\
= \ & f(\bar{S}[(\bar{S}({x_2}^{-1}) \rightarrow x_1)_2{}^{-1}] \rightarrow (\bar{S}({x_2}^{-1}) \rightarrow x_1)_1))g((\bar{S}({x_2}^{-1}) \rightarrow x_1)_2{}^0)k({x_2}^0) \\
= \ & f(\bar{S}[(\bar{S}({x_3}^{-2}) \rightarrow x_2)^{-1}]\bar{S}({x_3}^{-1}) \rightarrow x_1)g((\bar{S}({x_3}^{-2}) \rightarrow x_2)^0)k({x_3}^0) \\
= \ & f(\bar{S}[\bar{S}({x_3}^{-2}){x_2}^{-1}{x_3}^{-4}]\bar{S}({x_3}^{-1}) \rightarrow x_1)g(\bar{S}({x_3}^{-3}) \rightarrow {x_2}^0)k({x_3}^0) \\
= \ & f(\bar{S}[{x_3}^{-1}\bar{S}({x_3}^{-2}){x_2}^{-1}{x_3}^{-4}] \rightarrow x_1)g(\bar{S}({x_3}^{-3}) \rightarrow {x_2}^0)k({x_3}^0) \\
= \ & f(\bar{S}[\tilde{\varepsilon}_s({x_3}^{-1}){x_2}^{-1}{x_3}^{-3}] \rightarrow x_1)g(\bar{S}({x_3}^{-2}) \rightarrow {x_2}^0)k({x_3}^0) \\
\overset{(4)}{=} \ & f(\bar{S}[\bar{S}(1_2){x_2}^{-1}{x_3}^{-2}] \rightarrow x_1)g(\bar{S}(1_1{x_3}^{-1}) \rightarrow {x_2}^0)k({x_3}^0) \\
= \ & f(\bar{S}({x_2}^{-1}{x_3}^{-2}) \rightarrow x_1)g(\bar{S}({x_3}^{-1}) \rightarrow {x_2}^0)k({x_3}^0),
\end{aligned}$$

and

$$\begin{aligned}
f(gk)(x) &= f(\bar{S}({x_2}^{-1}) \rightarrow x_1)(gk)({x_2}^0) \\
&= f(\bar{S}({x_2}^{-1}) \rightarrow x_1)g(\bar{S}({x_2}^0{}_2{}^{-1}) \rightarrow {x_2}^0{}_1)k({x_2}^0{}_2{}^0) \\
&= f(\bar{S}({x_2}^{-1}{x_3}^{-1}) \rightarrow x_1)g(\bar{S}({x_3}^{0-1}) \rightarrow {x_2}^0)k({x_3}^{00}) \\
&= f(\bar{S}({x_2}^{-1}{x_3}^{-2}) \rightarrow x_1)g(\bar{S}({x_3}^{-1}) \rightarrow {x_2}^0)k({x_3}^0),
\end{aligned}$$

for all $f, g, k \in H^*, x \in H$.

And for the unit, we compute

$$\begin{aligned}
(\varepsilon f)(x) &= \varepsilon(\bar{S}({x_2}^{-1}) \rightarrow x_1)f({x_2}^0) \\
&\overset{(13)}{=} \varepsilon(\varepsilon_s S^{-1}({x_2}^{-1}) \rightarrow x_1)f({x_2}^0) \\
&= \varepsilon(\tilde{\varepsilon}_s({x_2}^{-1}) \rightarrow x_1)f({x_2}^0) \\
&\overset{(17)}{=} \varepsilon(1_1 \rightarrow x_1)f(1_2 \rightarrow x_2) \\
&= \varepsilon(x_1)f(x_2) = f(x),
\end{aligned}$$

and

$$\begin{aligned}(f\varepsilon)(x) &= f(\bar{S}(x_2{}^{-1}) \rightarrow x_1)\varepsilon(x_2{}^0)\\ &= f(\bar{S}\varepsilon_t(x_2{}^{-1}) \rightarrow x_1)\varepsilon(x_2{}^0)\\ &= f(\tilde{\varepsilon}_s(x_2{}^{-1}) \rightarrow x_1)\varepsilon(x_2{}^0) = f(x),\end{aligned}$$

for all $f \in H^*, x \in H$.

Step 2): We check that H^* is a left weak L-module algebra.

$$\begin{aligned} & (l \rightarrow (fg))(x) = (fg)(S(l) \rightarrow x)\\ =\ & f(\bar{S}((S(l_1) \rightarrow x_2)^{-1}) \rightarrow (S(l_2) \rightarrow x_1))g((S(l_2) \rightarrow x_2)^0)\\ =\ & f(\bar{S}(S(l_3)x_2{}^{-1}S(S(l_1)))S(l_4) \rightarrow x_1)g(S(l_2) \rightarrow x_2{}^0)\\ =\ & f(S(l_1)\bar{S}(x_2{}^{-1})l_3S(l_4) \rightarrow x_1)g(S(l_2) \rightarrow x_2{}^0)\\ \overset{(4)}{=}\ & f(S(l_1)\bar{S}(x_2{}^{-1})1_2 \rightarrow x_1)g(S(l_2)S(1_1) \rightarrow x_2{}^0)\\ =\ & f(S(l_1)\bar{S}(x_2{}^{-1}) \rightarrow x_1)g(S(l_2) \rightarrow x_2{}^0)\\ =\ & (l_1 \rightarrow f)(\bar{S}(x_2{}^{-1}) \rightarrow x_1)(l_2 \rightarrow g)(x_2{}^0)\\ =\ & ((l_1 \rightarrow f)(l_2 \rightarrow g))(x),\end{aligned}$$

and

$$(\varepsilon_t(l) \rightarrow \varepsilon)(x) = \varepsilon(S\varepsilon_t(l) \rightarrow x) \overset{(2.3)}{=} \varepsilon(\varepsilon_s S(l) \rightarrow x) \overset{(2.13)}{=} \varepsilon(S(l) \rightarrow x) = (l \rightarrow \varepsilon)(x).$$

So, we obtain $l \rightarrow 1_{H^*} = \varepsilon_t(l) \rightarrow 1_{H^*}$.

Step 3): We show that H^* is a left weak L-comodule algebra. To prove $(fg)^{-1} \otimes (fg)^0 = f^{-1}g^{-1} \otimes f^0g^0$ in $L \otimes H^*$, for any $x \in H$, we do a calculation:

$$\begin{aligned} & f^0g^0(x)f^{-1}g^{-1} = f^0(\bar{S}(x_2{}^{-1}) \rightarrow x_1)g^0(x_2{}^0)f^{-1}g^{-1}\\ =\ & f^0(\bar{S}(x_2{}^{-2}) \rightarrow x_1)f^{-1}g(x_2{}^0)\bar{S}(x_2{}^{-1})\\ =\ & f((\bar{S}(x_2{}^{-2}) \rightarrow x_1)^0)\bar{S}((\bar{S}(x_2{}^{-2}) \rightarrow x_1)^{-1})g(x_2{}^0)\bar{S}(x_2{}^{-1})\\ =\ & f(\bar{S}(x_2{}^{-3}) \rightarrow x_1{}^0)\bar{S}(x_2{}^{-1}\bar{S}(x_2{}^{-2})x_1{}^{-1}x_2{}^{-4})g(x_2{}^0)\\ \overset{(4)}{=}\ & f(\bar{S}(1_1x_2{}^{-1}) \rightarrow x_1{}^0)\bar{S}(\bar{S}(1_2)x_1{}^{-1}x_2{}^{-2})g(x_2{}^0)\\ =\ & f(\bar{S}(x_2{}^{-1}) \rightarrow x_1{}^0)g(x_2{}^0)\bar{S}(x_1{}^{-1}x_2{}^{-2}) = (fg)(x^0)\bar{S}(x^{-1})\\ =\ & (fg)^0(x)(fg)^{-1},\end{aligned}$$

and

$$\varepsilon_s(\varepsilon^{-1})\varepsilon^0(h) = \varepsilon_s\bar{S}(h^{-1})\varepsilon(h^0) = \bar{S}\varepsilon_t(h^{-1})\varepsilon(h^0) = \bar{S}(h^{-1})\varepsilon(h^0) = \varepsilon^{-1}\varepsilon^0(h).$$

Thus, we have $1_{H^*}{}^{-1} \otimes 1_{H^*}{}^0 = \varepsilon_s(1_{H^*}{}^{-1}) \otimes 1_{H^*}{}^0$. □

Lemma 3.5. *If H is a finite-dimensional weak bialgebra in ${}^L_L\mathcal{WYD}$, then H^* is a coalgebra in ${}^L_L\mathcal{WYD}$, with comultiplication $\Delta_{H^*} = (\theta^{(2)})^{-1} \circ (m_H)^*$, counit ε_{H^*} : $f \to f(1_H)$. Explicitly, comultiplication $\Delta_{H^*}(f) = f_1 \otimes f_2$ is given by*

$$f(xy) = f_1({f_2}^{-1} \to x){f_2}^0(y) = f_1(\bar{S}_L(y^{-1}) \to x_1)f_2(y^0),\ x, y \in H,$$

or equivalently

$$f_1(x)f_2(y) = f((y^{-1} \to x)y^0),\ x, y \in H.$$

Proof. (1) We first check that H^* is a coalgebra. To show the coassociativity, we use the isomorphism $\theta^{(3)}$. For all $f \in H^*$ and $x, y, z \in H$, we do a calculation:

$$\begin{aligned}
f((xy)z) &= f_1(\bar{S}(z^{-1}) \to (xy))f_2(z^0) \\
&= f_1((\bar{S}(z^{-1}) \to x)(\bar{S}(z^{-2}) \to y))f_2(z^0) \\
&= f_{11}(\bar{S}(\bar{S}(z^{-2})y^{-1}z^{-4})\bar{S}(z^{-1}) \to x)f_{12}(\bar{S}(z^{-3}) \to y^0)f_2(z^0) \\
&= f_{11}(\bar{S}(\bar{S}\varepsilon_t(z^{-1})y^{-1}z^{-3}) \to x)f_{12}(\bar{S}(z^{-2}) \to y^0)f_2(z^0) \\
&\overset{(4)}{=} f_{11}(\bar{S}(1_1y^{-1}z^{-2}) \to x)f_{12}(\bar{S}(S(1_2)z^{-1}) \to y^0)f_2(z^0) \\
&= f_{11}(\bar{S}(y^{-1}z^{-2}) \to x)f_{12}(\bar{S}(z^{-1}) \to y^0)f_2(z^0) \\
&= \theta^{(3)}(f_{11} \otimes f_{12} \otimes f_2)(x \otimes y \otimes z), \\
f(x(yz)) &= f_1(\bar{S}(y^{-1}z^{-1}) \to x)f_2(y^0z^0) \\
&= f_1(\bar{S}(y^{-1}z^{-2}) \to x)f_{21}(\bar{S}(z^{-1}) \to y^0)f_{22}(z^0) \\
&= \theta^{(3)}(f_1 \otimes f_{21} \otimes f_{22})(x \otimes y \otimes z).
\end{aligned}$$

So we have $f_{11} \otimes f_{12} \otimes f_{13} = f_1 \otimes f_{21} \otimes f_{22}$ (we denote it by $f_1 \otimes f_2 \otimes f_3$).
And the counit:

$$\begin{aligned}
\varepsilon_{H^*}(f_1)f_2(x) &= f((\varepsilon_t(x^{-1}) \to 1_H)x^0) \\
&= f((S(1_1) \to 1_H)(1_2 \to x)) \\
&= f((1_1 \to 1_H)(1_2 \to x)) = f(x).
\end{aligned}$$

Similarly, we have $f_1\varepsilon_{H^*}(f_2) = f$.
This shows that H^* is a coalgebra.

(2) Next, we prove that H^* is a left weak L-module coalgebra. $\Delta_{H^*}(l \rightarrow f) = (l_1 \rightarrow f_1) \otimes (l_2 \rightarrow f_2)$, for any $x, y \in H$,

$$\begin{aligned}
& \theta^{(2)}((l_1 \rightarrow f_1) \otimes (l_2 \rightarrow f_2))(x \otimes y) \\
= \; & (l_1 \rightarrow f_1)(\bar{S}(y^{-1}) \rightarrow x)(l_2 \rightarrow f_2)(y^0) \\
= \; & f_1(S(l)_2 \bar{S}(y^{-1}) \rightarrow x) f_2(S(l)_1 \rightarrow y^0) \\
= \; & f(((S(l)_1 \rightarrow y^0)^{-1} S(l)_2 \bar{S}(y^{-1}) \rightarrow x)((S(l)_1 \rightarrow y^0)^0)) \\
= \; & f((S(l)_1 y^{-1} \bar{S}(y^{-2}) \rightarrow x)(S(l)_2 \rightarrow y^0)) \\
= \; & f(S(l) \rightarrow ((\tilde{\varepsilon}_s(y^{-1}) \rightarrow x) y^0)) \\
\overset{(17)}{=} \; & f(S(l) \rightarrow ((1_1 \rightarrow x)(1_2 \rightarrow y))) = f(S(l) \rightarrow xy) \\
= \; & (l \rightarrow f)(xy) = \theta^{(2)} \Delta_{H^*}(l \rightarrow f)(x \otimes y),
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon(lk_2) \varepsilon_{H^*}(k_1 \rightarrow f) = \varepsilon(lk_2) f(S(k_1) \rightarrow 1_H) \\
= \; & \varepsilon(lk_2) f(\varepsilon_t S(k_1) \rightarrow 1_H) \overset{(3)}{=} \varepsilon(lk_2) f(S\varepsilon_s(k_1) \rightarrow 1_H) \\
\overset{(5)}{=} \; & \varepsilon(lk1_2) f(S(1_1) \rightarrow 1_H) = \varepsilon(\varepsilon_s(lk) 1_2) f(S(1_1) \rightarrow 1_H) \\
\overset{(5)}{=} \; & \varepsilon((\varepsilon_s(lk))_2) f(S((\varepsilon_s(lk))_1) \rightarrow 1_H) = f(S\varepsilon_s(lk) \rightarrow 1_H) \\
\overset{(3)}{=} \; & f(\varepsilon_t S(lk) \rightarrow 1_H) = f(S(lk) \rightarrow 1_H) = (lk \rightarrow f)(1_H).
\end{aligned}$$

So we have $\varepsilon(lk_2)\varepsilon_{H^*}(k_1 \rightarrow f) = \varepsilon_{H^*}(lk \rightarrow f)$.

(3) Finally we show that H^* is a left weak L-comodule coalgebra. $f^{-1} \otimes (f^0)_1 \otimes (f^0)_2 = {f_1}^{-1} {f_2}^{-1} \otimes {f_1}^0 \otimes {f_2}^0$ in $L \otimes H^* \otimes H^*$, for any $x, y \in H$,

$$\begin{aligned}
& {f_1}^{-1} {f_2}^{-1} \theta^{(2)}({f_1}^0 \otimes {f_2}^0)(x \otimes y) \\
= \; & {f_1}^{-1} {f_2}^{-1} {f_1}^0(\bar{S}(y^{-1}) \rightarrow x) {f_2}^0(y^0) \\
= \; & {f_1}^{-1} f_2(y^0) \bar{S}(y^{-1}) {f_1}^0(\bar{S}(y^{-2}) \rightarrow x) \\
= \; & f_1(\bar{S}(y^{-3}) \rightarrow x^0) \bar{S}(\bar{S}(y^{-2}) x^{-1} y^{-4}) \bar{S}(y^{-1}) f_2(y^0) \\
= \; & f_1(\bar{S}(y^{-2}) \rightarrow x^0) \bar{S}(\bar{S}\varepsilon_t(y^{-1}) x^{-1} y^{-3}) f_2(y^0) \\
\overset{(4)}{=} \; & f_1(\bar{S}(1_1 y^{-1}) \rightarrow x^0) \bar{S}(\bar{S}(1_2) x^{-1} y^{-2}) f_2(y^0) \\
= \; & f_1(\bar{S}(y^{-1}) \rightarrow x^0) f_2(y^0) \bar{S}(x^{-1} y^{-2}) \\
= \; & f(x^0 y^0) \bar{S}(x^{-1} y^{-1}) = f^{-1} f^0(xy) \\
= \; & f^{-1} \theta^{(2)}((f^0)_1 \otimes (f^0)_2)(x \otimes y),
\end{aligned}$$

and

$$f^0(1_H)\varepsilon_t(f^{-1}) = f(1^0)\varepsilon_t\bar{S}(1^{-1}) = f(1^0)\bar{S}\varepsilon_s(1^{-1}) = f(1^0)\bar{S}(1^{-1}) = f^0(1_H)f^{-1}.$$

So, we get $\varepsilon_{H^*}(f^0)f^{-1} = \varepsilon_{H^*}(f^0)\varepsilon_t(f^{-1})$.
By the proof of (1) to (3), we conclude that H^* is a coalgebra in ${}^L_L\mathcal{WYD}$. □

Proposition 3.6. *If H is a finite-dimensional weak bialgebra in ${}^L_L\mathcal{WYD}$, then H^* is a weak bialgebra in ${}^L_L\mathcal{WYD}$ with the structure in Lemma 3.4 and Lemma 3.5.*

Proof. By Lemma 3.4 and Lemma 3.5, we have that H^* is both an algebra and a coalgebra in ${}^L_L\mathcal{WYD}$. So we only need to check the following conditions:

(1) We first show that $\Delta_{H^*}(fg) = f_1({f_2}^{-1} \to g_1) \otimes {f_2}^0 g_2 \in H^* \otimes H^*$ by using $\theta^{(2)}$. For all $x, y \in H$,

$$\begin{aligned}
&\theta^{(2)}(f_1({f_2}^{-1} \to g_1) \otimes {f_2}^0 g_2)(x \otimes y) = (f_1({f_2}^{-1} \to g_1))(\bar{S}(y^{-1}) \to x)({f_2}^0 g_2)(y^0)\\
=\ & (f_1({f_2}^{-1} \to g_1))(\bar{S}(x^{-1}y^{-1}) \to x){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}(\bar{S}({y_1}^{-2}{y_2}^{-3}){x_2}^{-1}{y_1}^{-4}{y_2}^{-5})\bar{S}({y_1}^{-1}{y_2}^{-2}) \to x_1)\\
& \times({f_2}^{-1} \to g_1)(\bar{S}({y_1}^{-3}{y_2}^{-4}) \to {x_2}^0){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}({y_1}^{-1}{y_2}^{-2}\bar{S}({y_1}^{-2}{y_2}^{-3}){x_2}^{-1}{y_1}^{-4}{y_2}^{-5}) \to x_1)\\
& \times({f_2}^{-1} \to g_1)(\bar{S}({y_1}^{-3}{y_2}^{-4}) \to {x_2}^0){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}(\bar{S}\varepsilon_t({y_1}^{-1}{y_2}^{-2}){x_2}^{-1}{y_1}^{-3}{y_2}^{-4}) \to x_1)\\
& \times({f_2}^{-1} \to g_1)(\bar{S}({y_1}^{-2}{y_2}^{-3}) \to {x_2}^0){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
\overset{(4)}{=}\ & f_1(\bar{S}(\bar{S}(1_2){x_2}^{-1}{y_1}^{-2}{y_2}^{-3}) \to x_1)\\
& \times({f_2}^{-1} \to g_1)(\bar{S}(1_1{y_1}^{-1}{y_2}^{-2}) \to {x_2}^0){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}({x_2}^{-1}{y_1}^{-2}{y_2}^{-3}) \to x_1)({f_2}^{-1} \to g_1)(\bar{S}({y_1}^{-1}{y_2}^{-2}) \to {x_2}^0){f_2}^0(\bar{S}({y_2}^{-1}) \to {y_1}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}({x_2}^{-1}{y_1}^{-3}{y_2}^{-5}) \to x_1)f_2(\bar{S}({y_2}^{-2}) \to {y_1}^0)\\
& \times(\bar{S}(\bar{S}({y_2}^{-1}){y_1}^{-1}{y_2}^{-3}) \to g_1)(\bar{S}({y_1}^{-2}{y_2}^{-4}) \to {x_2}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}({x_2}^{-1}{y_1}^{-3}{y_2}^{-5}) \to x_1)f_2(\bar{S}({y_2}^{-2}) \to {y_1}^0)\\
& \times g_1(\bar{S}({y_2}^{-1}){y_1}^{-1}{y_2}^{-3}\bar{S}({y_1}^{-2}{y_2}^{-4}) \to {x_2}^0)g_2({y_2}^0)\\
=\ & f_1(\bar{S}({x_2}^{-1}{y_1}^{-2}{y_2}^{-4}) \to x_1)f_2(\bar{S}({y_2}^{-2}) \to {y_1}^0)g_1(\bar{S}({y_2}^{-1})\bar{S}\varepsilon_t({y_1}^{-1}{y_2}^{-3}) \to {x_2}^0)g_2({y_2}^0)\\
\overset{(4)}{=}\ & f_1(\bar{S}({x_2}^{-1}1_1{y_1}^{-1}{y_2}^{-3}) \to x_1)f_2(\bar{S}({y_2}^{-2}) \to {y_1}^0)g_1(\bar{S}({y_2}^{-1})\bar{S}(1_2) \to {x_2}^0)g_2({y_2}^0)\\
\overset{(15)}{=}\ & f_1(\bar{S}({x_2}^{-1}{y_1}^{-1}{y_2}^{-3}) \to x_1)f_2(\bar{S}({y_2}^{-2}) \to {y_1}^0)g_1(\bar{S}({y_2}^{-1}) \to {x_2}^0)g_2({y_2}^0),
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& (\theta^{(2)}\Delta_{H^*}(fg))(x\otimes y) = (fg)(xy) = f(\bar{S}((xy)_2{}^{-1}) \rightarrow (xy)_1)g((xy)_2{}^{0}) \\
= \;& f(\bar{S}(x_2{}^{-1}y_2{}^{-1}) \rightarrow (x_1(x_2{}^{-2}) \rightarrow y_1))g(x_2{}^{0}y_2{}^{0}) \\
= \;& f((\bar{S}(x_2{}^{-1}y_2{}^{-1}) \rightarrow x_1)(\bar{S}(x_2{}^{-2}y_2{}^{-2})x_2{}^{-3} \rightarrow y_1))g(x_2{}^{0}y_2{}^{0}) \\
= \;& f((\bar{S}(x_2{}^{-1}y_2{}^{-2}) \rightarrow x_1)(\bar{S}(y_2{}^{-3})\bar{S}(x_2{}^{-2})x_2{}^{-3} \rightarrow y_1))g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
= \;& f((\bar{S}(x_2{}^{-1}y_2{}^{-2}) \rightarrow x_1)(\bar{S}(y_2{}^{-3})\bar{S}\varepsilon_s(x_2{}^{-2}) \rightarrow y_1))g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
\overset{(5)}{=} \;& f((\bar{S}(x_2{}^{-1}1_2y_2{}^{-2}) \rightarrow x_1)(\bar{S}(y_2{}^{-3})\bar{S}(1_1) \rightarrow y_1))g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
= \;& f((\bar{S}(x_2{}^{-1}y_2{}^{-2}) \rightarrow x_1)(\bar{S}(y_2{}^{-3}) \rightarrow y_1))g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
= \;& f_1(\bar{S}(\bar{S}(y_2{}^{-3})y_1{}^{-1}y_2{}^{-5})\bar{S}(x_2{}^{-1}y_2{}^{-2}) \rightarrow x_1)f_2(\bar{S}(y_2{}^{-4}) \rightarrow y_1{}^{0})g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
= \;& f_1(\bar{S}(x_2{}^{-1}y_2{}^{-2}\bar{S}(y_2{}^{-3})y_1{}^{-1}y_2{}^{-5}) \rightarrow x_1)f_2(\bar{S}(y_2{}^{-4}) \rightarrow y_1{}^{0})g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
\overset{(4)}{=} \;& f_1(\bar{S}(x_2{}^{-1}\bar{S}(1_2)y_1{}^{-1}y_2{}^{-3}) \rightarrow x_1)f_2(\bar{S}(1_1y_2{}^{-2}) \rightarrow y_1{}^{0})g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}) \\
= \;& f_1(\bar{S}(x_2{}^{-1}y_1{}^{-1}y_2{}^{-3}) \rightarrow x_1)f_2(\bar{S}(y_2{}^{-2}) \rightarrow y_1{}^{0})g_1(\bar{S}(y_2{}^{-1}) \rightarrow x_2{}^{0})g_2(y_2{}^{0}).
\end{aligned}
$$

(2) Next we want to check $\varepsilon_{H^*}(fgk) = \varepsilon_{H^*}(fg_1)\varepsilon_{H^*}(g_2k)$. For all $f, g, k \in H^*$,

$$
\begin{aligned}
\varepsilon_{H^*}(fgk) \;=\;& fgk(1_H) = f(\bar{S}(1_2{}^{-1}1_3{}^{-2}) \rightarrow 1_1)g(\bar{S}(1_3{}^{-1}) \rightarrow 1_2{}^{0})k(1_3{}^{0}) \\
=\;& f(\bar{S}((1_21'_1)^{-1}(1'_2)^{-2}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1}) \rightarrow (1_21'_1)^{0})k(1'_2{}^{0}) \\
& \text{using } \ \Delta^2(1) = 1_1 \otimes 1_21'_1 \otimes 1'_2 \\
=\;& f(\bar{S}(1_2{}^{-1}1'_1{}^{-1}1'_2{}^{-2}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1}) \rightarrow (1_2{}^{0}1'_1{}^{0}))k(1'_2{}^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1'^{-1}) \rightarrow 1_1)g(\bar{S}(1'^{0}{}_2{}^{-1}) \rightarrow (1_2{}^{0}1'^{0}{}_1))k(1'^{0}{}_2{}^{0}) \\
\overset{(17)}{=}\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(\bar{S}((1_2 \rightarrow 1_H)_2{}^{-1}) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1_H)_1])k((1_2 \rightarrow 1_H)_2{}^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(\bar{S}((1_3 \rightarrow 1'_2)^{-1}) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k((1_3 \rightarrow 1'_2)^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(1_5\bar{S}(1'_2{}^{-1})\bar{S}(1_3) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1_4 \rightarrow 1'_2{}^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(1''_2\bar{S}(1'_2{}^{-1})\bar{S}(1_3) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1_41''_1 \rightarrow 1'_2{}^{0}) \\
\overset{(14)}{=}\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1})\bar{S}(1_3) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1_4 \rightarrow 1'_2{}^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(\bar{S}(1_31''_11'_2{}^{-1}) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1''_2 \rightarrow 1'_2{}^{0}) \\
=\;& f(\bar{S}(1_2{}^{-1}1_1) \rightarrow 1_1)g(\bar{S}(1_31'_2{}^{-1}) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1'_2{}^{0}) \\
=\;& f(\bar{S}(1_1)\bar{S}(1_2{}^{-1}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1})\bar{S}(1''_2) \rightarrow [1_2{}^{0}(\tilde{\varepsilon}_s(1''_1)1_2 \rightarrow 1'_1)])k(1'_2{}^{0}) \\
\overset{(12)}{=}\;& f(\bar{S}(1_1)\bar{S}(1_2{}^{-1}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1})\bar{S}(1''_2) \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)(1''_1 \rightarrow 1_H)])k(1'_2{}^{0}) \\
\overset{(12)}{=}\;& f(\bar{S}(1_1)\bar{S}(1_2{}^{-1}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1})\bar{S}(1''_2)1''_1 \rightarrow [1_2{}^{0}(1_2 \rightarrow 1'_1)])k(1'_2{}^{0}) \\
=\;& f(1_2\bar{S}(1_2{}^{-1}) \rightarrow 1_1)g(\bar{S}(1'_2{}^{-1}) \rightarrow [1_2{}^{0}(S(1_1) \rightarrow 1'_1)])k(1'_2{}^{0})
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(12)}{=} f(1_2\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to [1_2{}^0(1_1 \to 1_H)1'_1])k(1'_2{}^0)\\
&\overset{(12)}{=} f(1_2\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to [(\tilde{\varepsilon}_s(1_1) \to 1_2{}^0)1'_1])k(1'_2{}^0)\\
&\overset{(14)}{=} f(1_2\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to [(1_1 \to 1_2{}^0)1'_1])k(1'_2{}^0)\\
&= f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to (1_2{}^0 1'_1))k(1'_2{}^0),
\end{aligned}
$$

and

$$
\begin{aligned}
& \varepsilon_{H^*}(fg_1)\varepsilon_{H^*}(g_2k) = (fg_1, 1_H)(g_2k, 1'_H)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g_1(1_2{}^0)g_2(\bar{S}(1'_2{}^{-1}) \to 1'_1)k(1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(([\bar{S}(1'_2{}^{-1}) \to 1'_1]^{-1} \to 1_2{}^0)[\bar{S}(1'_2{}^{-1}) \to 1'_1]^0)k(1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g[(\bar{S}(1'_2{}^{-1})1'_1{}^{-1}1'_2{}^{-3} \to 1_2{}^0)(\bar{S}(1'_2{}^{-2}) \to 1'_1{}^0)]k(1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to [(1'_1{}^{-1}1'_2{}^{-2} \to 1_2{}^0)1'_1{}^0])k(1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'^0{}_2{}^{-1}) \to [(1'^{-1} \to 1_2{}^0)1'^0{}_1])k(1'^0{}_2{}^0)\\
\overset{(17)}{=} \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}[(1_2 \to 1_H)_2{}^{-1}] \to [(1_1 \to 1_2{}^0)(1_2 \to 1_H)_1])k((1_2 \to 1_H)_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}((1_3 \to 1'_2)^{-1}) \to [(1_1 \to 1_2{}^0)(1_2 \to 1'_1)])k((1_3 \to 1'_2)^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}((1_2 \to 1'_2)^{-1})1_1 \to (1_2{}^0 1'_1))k((1_2 \to 1'_2)^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(1_4\bar{S}(1'_2{}^{-1})\bar{S}(1_2)1_1 \to (1_2{}^0 1'_1))k(1_3 \to 1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(1_3\bar{S}(1'_2{}^{-1})\tilde{\varepsilon}_t(1_1) \to (1_2{}^0 1'_1))k(1_2 \to 1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(1_2\bar{S}(1'_2{}^{-1})\bar{S}(1''_1) \to (1_2{}^0 1'_1))k(1_1 1''_2 \to 1'_2{}^0)\\
= \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(1_2\bar{S}(1'_2{}^{-1}) \to (1_2{}^0 1'_1))k(1_1 \to 1'_2{}^0)\\
\overset{(14)}{=} \;& f(\bar{S}(1_2{}^{-1}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to (1_2{}^0 1'_1))k(1'_2{}^0).
\end{aligned}
$$

Similarly, using $\Delta^2(1) = 1_1 \otimes (1_2{}^{-1} \to 1'_1)1_2{}^0 \otimes 1'_2$, one can also get that

$$
\begin{aligned}
& \varepsilon_{H^*}(fgk) = \varepsilon_{H^*}(f(g_1{}^{-1} \to g_2))\varepsilon_{H^*}(g_1{}^0 k)\\
= \;& f(\bar{S}(1_2{}^{-2}) \to 1_1)g(\bar{S}(1'_2{}^{-1}) \to ((1_2{}^{-1} \to 1'_1)1_2{}^0))k(1'_2{}^0).
\end{aligned}
$$

(3) To verify that $\Delta^2(\varepsilon) = \varepsilon_1 \otimes \varepsilon_2\tilde{\varepsilon}_1 \otimes \tilde{\varepsilon}_2$ using the isomorphism $\theta^{(3)}$. For all $x, y, z \in H$, we compute

$$
\begin{aligned}
& \theta^{(3)}(\varepsilon_1 \otimes \varepsilon_2\tilde{\varepsilon}_1 \otimes \tilde{\varepsilon}_2)(x \otimes y \otimes z)\\
= \;& \varepsilon_1(\bar{S}(y^{-1}z^{-2}) \to x)\varepsilon_2\tilde{\varepsilon}_1(\bar{S}(z^{-1}) \to y^0)\tilde{\varepsilon}_2(z^0)\\
= \;& \varepsilon_1(\bar{S}(y^{-1}z^{-3}) \to x)\varepsilon_2(\bar{S}[(\bar{S}(z^{-2}) \to y^0{}_2)^{-1}] \to (\bar{S}(z^{-1}) \to y^0{}_1))\\
& \times\tilde{\varepsilon}_1((\bar{S}(z^{-2}) \to y^0{}_2)^0)\tilde{\varepsilon}_2(z^0)\\
= \;& \varepsilon_1(\bar{S}(y^{-1}z^{-5}) \to x)\varepsilon_2(\bar{S}(z^{-1}\bar{S}(z^{-2})y^0{}_2{}^{-1}z^{-4}) \to y^0{}_1)\tilde{\varepsilon}_1(\bar{S}(z^{-3}) \to y^0{}_2{}^0)\tilde{\varepsilon}_2(z^0)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(4)}{=} && \varepsilon_1(\bar S(y^{-1}z^{-3}) \rightarrow x)\varepsilon_2(\bar S(\bar S(1_2){y^0}_2{}^{-1}z^{-2}) \rightarrow {y^0}_1)\tilde\varepsilon_1(\bar S(1_1z^{-1}) \rightarrow {{y^0}_2}^0)\tilde\varepsilon_2(z^0)\\
&= && \varepsilon_1(\bar S(y^{-1}z^{-3}) \rightarrow x)\varepsilon_2(\bar S({y^0}_2{}^{-1}z^{-2}) \rightarrow {y^0}_1)\tilde\varepsilon_1(\bar S(z^{-1}) \rightarrow {{y^0}_2}^0)\tilde\varepsilon_2(z^0)\\
&= && \varepsilon(([\bar S({y^0}_2{}^{-1}z^{-3}) \rightarrow {y^0}_1]^{-1}\bar S(y^{-1}z^{-4}) \rightarrow x)[\bar S({y^0}_2{}^{-1}z^{-3}) \rightarrow {y^0}_1]^0)\\
& && \times\varepsilon((z^{-1}\bar S(z^{-2}) \rightarrow {{y^0}_2}^0)z^0)\\
&= && \varepsilon((\bar S({y^0}_2{}^{-1}z^{-3}){y^0}_1{}^{-1}{y^0}_2{}^{-3}z^{-5}\bar S(y^{-1}z^{-6}) \rightarrow x)(\bar S({y^0}_2{}^{-2}z^{-4}) \rightarrow {{y^0}_1}^0))\\
& && \times\varepsilon((z^{-1}\bar S(z^{-2}) \rightarrow {{y^0}_2}^0)z^0)\\
&\overset{(7)}{=} && \varepsilon((\bar S({y^0}_2{}^{-1}z^{-3}){y^0}_1{}^{-1}{y^0}_2{}^{-3}\bar S(y^{-1}) \rightarrow x)(\bar S({y^0}_2{}^{-2}z^{-4}) \rightarrow {{y^0}_1}^0))\\
& && \times\varepsilon((z^{-1}\bar S(z^{-2}) \rightarrow {{y^0}_2}^0)z^0)\\
&\overset{(4)}{=} && \varepsilon((\bar S({y^0}_2{}^{-1}1_1z^{-1}){y^0}_1{}^{-1}{y^0}_2{}^{-3}\bar S(y^{-1}) \rightarrow x)(\bar S({y^0}_2{}^{-2}z^{-2}) \rightarrow {{y^0}_1}^0))\\
& && \times\varepsilon((\bar S(1_2) \rightarrow {{y^0}_2}^0)z^0)\\
&\overset{(15)}{=} && \varepsilon((\bar S({y^0}_2{}^{-1}z^{-1}){y^0}_1{}^{-1}{y^0}_2{}^{-3}\bar S(y^{-1}) \rightarrow x)(\bar S({y^0}_2{}^{-2}z^{-2}) \rightarrow {{y^0}_1}^0))\varepsilon({{y^0}_2}^0z^0)\\
&= && \varepsilon(\bar S({y^0}_2{}^{-1}z^{-1}) \rightarrow [({y^0}_1{}^{-1}{y^0}_2{}^{-2}\bar S(y^{-1}) \rightarrow x){{y^0}_1}^0])\varepsilon({{y^0}_2}^0z^0)\\
&\overset{(13)}{=} && \varepsilon(\tilde\varepsilon_s({y^0}_2{}^{-1}z^{-1}) \rightarrow [({y^0}_1{}^{-1}{y^0}_2{}^{-2}\bar S(y^{-1}) \rightarrow x){{y^0}_1}^0])\varepsilon({{y^0}_2}^0z^0)\\
&\overset{(17)}{=} && \varepsilon(1_1 \rightarrow [({y^0}_1{}^{-1}{y^0}_2{}^{-1}\bar S(y^{-1}) \rightarrow x){{y^0}_1}^0])\varepsilon(1_2 \rightarrow {{y^0}_2}^0z)\\
&= && \varepsilon(1_1 \rightarrow [(y^{-1}\bar S(y^{-2}) \rightarrow x){y^0}_1])\varepsilon(1_2 \rightarrow {y^0}_2z)\\
&= && \varepsilon((1_1\tilde\varepsilon_s(y^{-1}) \rightarrow x)(1_21'_1 \rightarrow {y^0}_1))\varepsilon(1'_2 \rightarrow {y^0}_2z)\\
&\overset{(12)}{=} && \varepsilon((1_1\tilde\varepsilon_s(y^{-1}) \rightarrow x)(1_21'_1 \rightarrow {y^0}_1))\varepsilon((1'_2 \rightarrow {y^0}_2)z)\\
&= && \varepsilon((1_1\tilde\varepsilon_s(y^{-1}) \rightarrow x)(1_2 \rightarrow {y^0}_1))\varepsilon({y^0}_2z)\\
&= && \varepsilon((\tilde\varepsilon_s(y^{-1}) \rightarrow x){y^0}_1)\varepsilon({y^0}_2z)\\
&= && \varepsilon((\tilde\varepsilon_s(y^{-1}) \rightarrow x)y^0z) \ \text{ by } \ \varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2z)\\
&\overset{(17)}{=} && \varepsilon((1_1 \rightarrow x)(1_2 \rightarrow y)z)\\
&= && \varepsilon(xyz) = \theta^{(3)}\Delta^2(\varepsilon)(x\otimes y\otimes z).
\end{aligned}
$$

The last equality is given in the proof (1) of Lemma 3.5. So $\Delta^2(\varepsilon) = \varepsilon_1\otimes\varepsilon_2\tilde\varepsilon_1\otimes\tilde\varepsilon_2$.

Similarly, by $\varepsilon(xyz) = \varepsilon(x({y_1}^{-1} \rightarrow y_2))\varepsilon({y_1}^0z)$, one can prove that $\Delta^2(\varepsilon) = \varepsilon_1 \otimes ({\varepsilon_2}^{-1} \rightarrow \tilde\varepsilon_1){\varepsilon_2}^0 \otimes \tilde\varepsilon_2$.

From all above, H^* is a weak bialgebra in ${}^L_L\mathcal{WYD}$. □

We now can arrive at the main result of this paper.

Theorem 3.7. *If H is a finite-dimensional weak Hopf algebra in ${}^L_L\mathcal{WYD}$, then H^* is a weak Hopf algebra in ${}^L_L\mathcal{WYD}$ with the antipode $(S_H)^*$. In particular H^{**} is a weak Hopf algebra in ${}^L_L\mathcal{WYD}$. If $(S_L)^2 = id_L$, then the canonical map $\tau : H \rightarrow H^{**}$ given by $\tau(x)(f) = f(x)$ is a weak Hopf algebra isomorphism in ${}^L_L\mathcal{WYD}$.*

Proof. We have proved H^* is a weak bialgebra in ${}^L_L\mathcal{WYD}$. Now we need to check the antipode $(S_H)^*$. It is easy to check that $(S_H)^*$ is a morphism in the category of ${}^L_L\mathcal{WYD}$ as S is both left L-linear and L-colinear. Firstly, we show that $f_1S_{H^*}(f_2) = \varepsilon_{H^*}((f^{-1} \rightarrow \varepsilon_1)f^0)\varepsilon_2$, for any $x \in H$,

$$\begin{aligned}
(f_1S_{H^*}(f_2))(x) &= f_1(\bar{S}(x_2{}^{-1}) \rightarrow x_1)S(f_2)(x_2{}^0) \\
&= f_1(\bar{S}(x_2{}^{-1}) \rightarrow x_1)f_2(S(x_2{}^0)) \\
&= f(([(S(x_2{}^0)]^{-1}\bar{S}(x_2{}^{-1}) \rightarrow x_1)([S(x_2{}^0)]^0)) \\
&= f((x_2{}^{-1}\bar{S}(x_2{}^{-2}) \rightarrow x_1)S(x_2{}^0)) \quad (S \text{ is L-colinear}) \\
&= f((\tilde{\varepsilon}_s(x_2{}^{-1}) \rightarrow x_1)S(x_2{}^0)) \\
&\overset{(16)}{=} f((1_1 \rightarrow x_1)S(1_2 \rightarrow x_2)) \\
&= f(x_1S(x_2)) = f(\varepsilon_t(x)) \\
&= \varepsilon((x^{-1} \rightarrow 1_1)x^0)f(1_2),
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon_{H^*}((f^{-1} \rightarrow \varepsilon_1)f^0)\varepsilon_2(x) \\
=\ & (f^{-1} \rightarrow \varepsilon_1)(\bar{S}(1_2{}^{-1}) \rightarrow 1_1)f^0(1_2{}^0)\varepsilon_2(x) \\
=\ & \varepsilon_1(1_2{}^{-1}\bar{S}(1_2{}^{-2}) \rightarrow 1_1)f(1_2{}^0)\varepsilon_2(x) \\
=\ & \varepsilon_1(\tilde{\varepsilon}_s(1_2{}^{-1}) \rightarrow 1_1)f(1_2{}^0)\varepsilon_2(x) \\
\overset{(16)}{=}\ & \varepsilon_1(1_1)f(1_2)\varepsilon_2(x) \\
=\ & \varepsilon((x^{-1} \rightarrow 1_1)x^0)f(1_2).
\end{aligned}$$

Similarly, we can check that $S_{H^*}(f_1)f_2 = \varepsilon_1\varepsilon_{H^*}((\varepsilon_2{}^{-1} \rightarrow f)\varepsilon_2{}^0)$.
Next, we compute $S_{H^*}(f_1)f_2S_{H^*}(f_3) = S_{H^*}(f)$, for any $x \in H$,

$$\begin{aligned}
&(S_{H^*}(f_1)f_2S_{H^*}(f_3))(x) = S_{H^*}(f_1)(\bar{S}(x_2{}^{-1}) \rightarrow x_1)f_2(\varepsilon_t(x_2{}^0)) \\
=\ & f_1(\bar{S}(x_2{}^{-1}) \rightarrow S(x_1))f_2(\varepsilon_t(x_2{}^0)) \quad (S \text{ is L-linear}) \\
=\ & f((x_2{}^{-1}\bar{S}(x_2{}^{-2}) \rightarrow S(x_1))\varepsilon_t(x_2{}^0)) \quad (\varepsilon_t \text{ is L-colinear}) \\
=\ & f((\tilde{\varepsilon}_s(x_2{}^{-1}) \rightarrow S(x_1))\varepsilon_t(x_2{}^0)) \\
\overset{(16)}{=}\ & f((1_1 \rightarrow S(x_1))\varepsilon_t(1_2 \rightarrow x_2)) = f(S(1_1 \rightarrow x_1)\varepsilon_t(1_2 \rightarrow x_2)) \\
=\ & f(S(x_1)\varepsilon_t(x_2)) = f(S(x)) = S_{H^*}(f)(x).
\end{aligned}$$

Therefore, H is a weak Hopf algebra in ${}^L_L\mathcal{WYD}$.

Finally, we verify that if $(S_L)^2 = id_L$, then τ is a weak Hopf algebra isomorphism in ${}^L_L\mathcal{WYD}$. Firstly, we check that τ is both L-linear and L-colinear, for any $f \in H^*$,

$$\begin{aligned} & (l \rightarrow \tau(x))(f) = \tau(x)(S(l) \rightarrow f) = (S(l) \rightarrow f)(x) \\ = \quad & f(S^2(l) \rightarrow x) = f(l \rightarrow x) = \tau(l \rightarrow x)(f), \end{aligned}$$

and

$$\begin{aligned} & \tau(x)^{-1}\tau(x)^0(f) = \tau(x)(f^0)\bar{S}(f^{-1}) = \bar{S}(f^0(x)f^{-1}) \\ = \quad & \bar{S}(f(x^0)\bar{S}(x^{-1})) = f(x^0)x^{-1} = x^{-1}\tau(x)^0(f). \end{aligned}$$

Next we show that τ is an algebra map.

$$\begin{aligned} (\tau(x) * \tau(y))(f) &= \tau(x)(\bar{S}({f_2}^{-1}) \rightarrow f_1)\tau(y)({f_2}^0) \\ &= (\bar{S}({f_2}^{-1}) \rightarrow f_1)(x){f_2}^0(y) = f_1({f_2}^{-1} \rightarrow x){f_2}^0(y) \\ &= f_1(\bar{S}(y^{-1}) \rightarrow x)f_2(y^0) = f(xy) = \tau(xy)(f), \end{aligned}$$

and

$$\tau(1_H)(f) = f(1_H) = \varepsilon_{H^*}(f) = 1_{H^{**}}(f).$$

Similarly, τ is a coalgebra map.
The whole proof is completed. $\square$

4. Applications

In this section, we will apply our results to the representations category $\mathrm{Rep}(L) = {}_L\mathcal{M}$ of a quasitriangular weak Hopf algebra L.

Definition 4.1. ([10]) *A quasitriangular weak Hopf algebra* is a pair $(L, \mathcal{R})$ where L is a weak Hopf algebra and $\mathcal{R} \in \Delta^{op}(1)(L \otimes_k L)\Delta(1)$ (called the $\mathcal{R}$-matrix) satisfying the following conditions:

$$\Delta^{op}(h)\mathcal{R} = \mathcal{R}\Delta(h), \tag{18}$$

for all $h \in L$, where Δ^{op} denotes the comultiplication opposite to Δ,

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{19}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, etc., as usual, and such that there exists $\overline{\mathcal{R}} \in \Delta(1)(L \otimes_k L)\Delta^{op}(1)$ with

$$\mathcal{R}\overline{\mathcal{R}} = \Delta^{op}(1), \qquad \overline{\mathcal{R}}\mathcal{R} = \Delta(1). \tag{20}$$

Furthermore, $(L, \mathcal{R})$ is called *triangular* if $\overline{\mathcal{R}} = \mathcal{R}_{21}$, where we write $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$, then $\mathcal{R}_{21} = \mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)}$.

Note that $\overline{\mathcal{R}}$ is uniquely determined by $\mathcal{R}$ and $(S\otimes id)(\mathcal{R}) = (id\otimes \bar{S})(\mathcal{R}) = \overline{\mathcal{R}}$. $\mathcal{R}$ satisfies the quantum Yang-Baxter equation. By [10, Lemma 5.3], we can obtain that

$$(\varepsilon\otimes id)\mathcal{R} = 1 = (id\otimes\varepsilon)\mathcal{R}. \tag{21}$$

Proposition 4.2. *([10]) The category Rep(L)* $= {}_L\mathcal{M}$ *is a braided monoidal category. The braiding* $\tau_{V,W} : V\otimes W \to W\otimes V$ *is defined by*

$$\tau_{V,W}(x) = \mathcal{R}^{(2)} \rightharpoonup x^{(2)} \otimes \mathcal{R}^{(1)} \rightharpoonup x^{(1)}, \tag{22}$$

where $x = x^{(1)}\otimes x^{(2)} \in V\otimes W$ *and* $\mathcal{R} = \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)} \in \Delta^{op}(1)(L\otimes_k L)\Delta(1)$, *and the inverse of* $\tau_{V,W}$ *is given by*

$$\tau^{-1}{}_{V,W}(y) = \overline{\mathcal{R}}^{(1)} \rightharpoonup y^{(2)} \otimes \overline{\mathcal{R}}^{(2)} \rightharpoonup y^{(1)},$$

where $y = y^{(1)}\otimes y^{(2)} \in V\otimes W$ *and* $\overline{\mathcal{R}} = \overline{\mathcal{R}}^{(1)}\otimes\overline{\mathcal{R}}^{(2)} \in \Delta(1)(L\otimes_k L)\Delta^{op}(1)$.

Lemma 4.3. *Let* $V\in {}_L\mathcal{M}$, *then* $V \in {}^L_L\mathcal{WYD}$.

Proof. We first construct a left L-coaction over V via

$$\sigma_V : V \longrightarrow L\otimes V, \quad v\mapsto \mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)} \rightharpoonup v.$$

$(id\otimes\sigma_V)\circ\sigma_V = (\Delta_L\otimes id)\circ\sigma_V$ follows Eq. (19), and $(\varepsilon_L\otimes id)\circ\sigma_V = id$ follows Eq. (21). So V is a left L-comodule with σ_V.

Next let us check the compatibility conditions for V. Since $\mathcal{R} = \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)} \in \Delta^{op}(1)(L\otimes_k L)\Delta(1)$, we immediately get $\sigma_V(v) \in L\otimes_t V$. Using Eq. (18), one can obtain that $l_1 v^{-1}\otimes l_2 \rightharpoonup v^0 = (l_1\rightharpoonup v)^{-1}l_2\otimes(l_1\rightharpoonup v)^0$. Therefore, $V\in {}^L_L\mathcal{WYD}$. □

Note that the matrix R give rise to a natural braiding for ${}_L\mathcal{M}$ and ${}^L_L\mathcal{WYD}$.

Definition 4.4. Let $(L,\mathcal{R})$ be a quasitriangular weak Hopf algebra. An object $H\in {}_L\mathcal{M}$ is called *a weak bialgebra in this category* if it is both a k-algebra and a k-coalgebra satisfying the following conditions:

(1)
$$\Delta(xy) = x_1(\mathcal{R}^{(2)}\rightharpoonup y_1)\otimes(\mathcal{R}^{(1)}\rightharpoonup x_2)y_2,$$
$$\varepsilon(xyz) = \varepsilon(xy_1)\varepsilon(y_2z),$$
$$\varepsilon(xyz) = \varepsilon(x(\mathcal{R}^{(2)}\rightharpoonup y_2))\varepsilon((\mathcal{R}^{(1)}\rightharpoonup y_1)z),$$
$$\Delta^2(1) = 1_1\otimes 1_2 1'_1\otimes 1'_2,$$
$$\Delta^2(1) = 1_1\otimes(\mathcal{R}^{(2)}\rightharpoonup 1'_1)(\mathcal{R}^{(1)}\rightharpoonup 1_2)\otimes 1'_2.$$

(2) H is both left weak L-module algebra and L-module coalgebra.

Furthermore, H is called *a weak Hopf algebra in* ${}_L\mathcal{M}$ if there exists an antipode $S : H \to H$ (S is left L-linear) satisfying

$$\varepsilon_t(x) = x_1 S(x_2) = \varepsilon((\mathcal{R}^{(2)} \to 1_1)(\mathcal{R}^{(1)} \to x))1_2,$$
$$\varepsilon_s(x) = S(x_1)x_2 = 1_1\varepsilon((\mathcal{R}^{(2)} \to x)(\mathcal{R}^{(1)} \to 1_2)),$$
$$S(x_1)x_2S(x_3) = S(x), \text{ for all } x \in H.$$

Assume that H is a weak Hopf algebra in ${}_L\mathcal{M}$ and finite dimensional over k. Now we will make its dual space $H^* = Hom(H, k)$ into a weak Hopf algebra in ${}_L\mathcal{M}$.

Proposition 4.5. *If H is a finite-dimensional weak Hopf algebra in ${}_L\mathcal{M}$, then H^* is also a weak Hopf algebra in ${}_L\mathcal{M}$ with the following structures:*
left L-module action $(l \to f)(h) = f(S_L(l) \to h)$, for all $l \in L, f \in H^*, h \in H$; multiplication is given by

$$(fg)(x) = f(\bar{S}_L(\mathcal{R}^{(2)}) \to x_1)g(\mathcal{R}^{(1)} \to x_2) = f(\overline{\mathcal{R}}^{(2)} \to x_1)g(\overline{\mathcal{R}}^{(1)} \to x_2),$$

unit $u_{H^*} = \varepsilon_H$, comultiplication $\Delta_{H^*}(f) = f_1 \otimes f_1$ is defined as

$$f_1(x)f_2(y) = f((\mathcal{R}^{(2)} \to x)(\mathcal{R}^{(1)} \to y)),$$

or equivalently

$$f(xy) = f_1(\bar{S}_L(\mathcal{R}^{(2)}) \to x)f_2(\mathcal{R}^{(1)} \to y) = f_1(\overline{R}^{(2)} \to x)f_2(\overline{\mathcal{R}}^{(1)} \to y),$$

counit $\varepsilon_{H^*} : f \mapsto f(1_H)$ and antipode $S_{H^*} = (S_H)^*$.

In particular, H^{**} is also a weak Hopf algebra in ${}_L\mathcal{M}$.

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] G. Böhm, *Doi-Hopf modules over weak Hopf algebras*, Comm. Algebra, 28 (2000), 4687-4698.

[2] G. Böhm, F. Nill, K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra, 221 (1999), 385-438.

[3] G. Böhm, K. Szlachányi, *A coassociative C^*-quantum group with nonintegral dimensions*, Lett. in Math. Phys., 35 (1996), 437-456.

[4] S. Caenepeel, E. De Groot, *Modules over weak entwining structures*, Contemporary Mathematics, 267 (2000), 31-54.

[5] S. Caenepeel, D.G. Wang, Y.M. Yin, *Yetter-Drinfeld modules over weak bialgebras*, Ann. Univ. Ferrara Sez. VII - Sc. Mat., 51 (2005), 69-98.

[6] Y. Doi, *Hopf modules in Yetter-Drifeld categories*, Comm. Algebra, 26(9) (1998), 3057-3070.

[7] P. Etingof, D. Nikshych, *Dynamical quantum groups at roots of 1*, Duke Math. J., 108 (2001), 135-168.

[8] L. Kadison, D. Nikshych, *Frobenius extensions and weak Hopf algebras*, J. Algebra, 244 (2001), 312-342.

[9] D. Nikshych, *A duality theorem for quantum groupoids*, Contemporary Mathematics, 267 (2000), 237-243.

[10] D. Nikshych, V. Turaev, L. Vainerman, *Invariants of knots and 3-manifolds from quantum groupoids*, Topology and its application, 127 (2003), 91-123.

[11] D. Nikshych, L. Vainerman, *A characterization of depth 2 subfactors of* Π_1 *factors*, J. Funct. Anal., 171 (2000), 278-307.

[12] M. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

Bing-liang Shen and **Shuan-hong Wang**
Department of Mathematics
Southeast University
Nanjing, Jiangsu 210096, P. R. of China
e-mail: bingliangshen@yahoo.com.cn (B.L. Shen),
shuanhwang2002@yahoo.com (S.H. Wang)