International Electronic Journal of Algebra
Volume 6 (2009) 95-106

FACTORING CERTAIN INFINITE ABELIAN GROUPS BY DISTORTED CYCLIC SUBSETS

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Received: 11 November 2008; Revised: 13 March 2009
Communicated by Abdullah Harmancı

Abstract. We will prove that two results on factoring finite abelian groups into a product of subsets, related to Hajós's and Rédei's theorems, can be extended for certain infinite torsion abelian groups.

Mathematics Subject Classification (2000): Primary 20K01; Secondary 05B45, 52C22, 68R05
Keywords: factorization of finite abelian groups, periodic factorization, full-rank factorization, Hajós-Rédei theory.

1. Introduction

Let G be a finite abelian group and let $A_1, \dots, A_n$ be subsets of G. If the product $A_1 \cdots A_n$ is direct and is equal to G, then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G. A subset A of G in the form

$$A = \{e, a, a^2, \dots, a^{r-1}\}$$

is called a cyclic subset of G. In order to avoid trivial cases we assume that $r \geq 2$ and that $|a| \geq r$. Clearly A is a subgroup of G if and only if $a^r = e$. It is a famous result of G. Hajós [3] that if a finite abelian group is factored as a direct product of its cyclic subsets, then at least one of the factors must be a subgroup.

A subset A of G in the form

$$A = \{e, a, a^2, \dots, a^{i-1}, a^i d, a^{i+1}, \dots, a^{r-1}\}$$

is called a distorted cyclic subset. Here we assume that $a^i d \neq a^j$ for each j, $0 \leq j \leq r-1$. If $d = e$ then clearly a distorted cyclic subset coincides with a cyclic subset. A. D. Sands [5] showed that if a finite abelian group is a direct product of distorted cyclic subsets, then at least one of the factors must be a subgroup. This is a generalization of Hajós's theorem. We will show that Sands's result holds for certain infinite groups too.

A subset A of G is called normalized if $e \in A$. If each A_i is normalized then we say that the factorization $G = A_1 \cdots A_n$ is normalized. The next theorem of L.

Rédei is one of the most striking results of the factorization theory of abelian group. If a finite abelian group is factored into normalized subsets of prime cardinalities, then at least one the factors must be a subgroup.

We say that a subset A of G is periodic if there is an element $g \in G \setminus \{e\}$ such that $Ag = A$. The element g is called a period of A. Rédei's theorem can be reformulated in terms of periodic subsets in the following way. If $G = A_1 \cdots A_n$ is a factorization of the finite abelian group G and each $|A_i|$ is a prime, then at least one of the factors must be periodic. Examples show that the condition that each $|A_i|$ is a prime cannot be dropped from the theorem. However, for 2-groups K. Amin, K. Corrádi and A. D. Sands ([1] Theorem 15) proved a slightly more general version. Namely, if $G = BA_1 \cdots A_n$ is a factorization of the finite abelian 2-group G such that $|B| = 4$ and $|A_1| = \cdots = |A_n| = 2$, then at least one of the factors is periodic. S. Szabó [7] extend the above result proving that if $G = BA_1 \cdots A_n$ is a factorization of the finite abelian group G such that $|B| = 4$ and each $|A_i|$ is a prime, then at least one of the factors must be periodic. In this paper we will show that the result holds for a class of infinite torsion abelian groups.

We define factorizations for infinite groups. We assume that each factor in a factorization contains the identity element. In short we consider only normalized factorizations of an infinite group. Let A_i, $i \in I$ be a collection of finite subsets of an abelian group G such that $e \in A_i$ for each i, $i \in I$. If each element $g \in G$ can be written in the form

$$g = \prod_{i \in I} a_i, \;\; a_i \in A_i$$

uniquely, where only finitely many of the a_i's are not equal to the identity element e, then we say that G is factored into its subsets A_i, $i \in I$. We also will say that the equation

$$G = \prod_{i \in I} A_i$$

is a factorization of G. Let p be a prime. The multiplicative group of all (p^α)th roots of unity will be denoted by $C(p^\infty)$. The group $C(p^\infty)$ is the so-called Prüfer group.

We would like to point out a difference between factoring finite and infinite groups. For a finite abelian group G if the product of its subsets A and B is direct and if $|A||B|$ is equal to $|G|$, then the product AB is a factorization of G, that is, each element $g \in G$ is uniquely represented in the form $g = ab$, $a \in A$, $b \in B$. This does not hold for infinite groups. To see why let $G = AB$ be a normalized factorization of the infinite abelian group G, where A is an infinite subset of G. Set $A' = A \setminus \{a\}$, where $a \in A \setminus \{e\}$. Plainly, $|A'||B|$ is equal to $|G|$. As the product

AB is direct it follows that the product $A'B$ is also direct. On the other hand, the product $A'B$ cannot be equal to G. However, for finite A this phenomenon cannot occur. E. J. Eigen and V. S. Prasad [2] have proved the following. Let $G = AB$ be a factorization of an abelian group, where A is finite. If A' is a subset of G such that $|A| = |A'|$ and the product $A'B$ is direct, then $G = A'B$ is a factorization of G.

We will use a corollary of this result. Note that the product AB is direct if and only if $A^{-1}A \cap BB^{-1} = \{e\}$. Therefore if the product AB is direct then so is the product $A^{-1}B$. Thus if A is finite, then A can be replaced by A^{-1} in each factorization $G = AB$ to get the factorization $G = A^{-1}B$. Here $A^{-1} = \{a^{-1} :\ a \in A\}$ and in general $A^t = \{a^t :\ a \in A\}$ for each integer t.

2. Distorted cyclic factors

A. D. Sands [5] proved that in a factorization of a finite abelian group a distorted cyclic factor always can be replaced by an associated cyclic subset. A weaker version of this result holds for infinite abelian groups too.

Lemma 2.1. *Let G be an abelian group. Let*

$$A = \{e, a, a^2, \ldots, a^{i-1}, a^i d, a^{i+1}, \ldots, a^{r-1}\}$$

be a distorted cyclic subset of G, where

$$C = \{e, a, a^2, \ldots, a^{r-1}\}$$

is a cyclic subset associated with A. If $r \geq 4$, then in the normalized factorization $G = AB$ the factor A can be replaced by C to get the normalized factorization $G = CB$.

Proof. We distinguish the following two cases

Case 1: $i = r - 1$.

Case 2: $1 \leq i \leq r - 2$.

Let us settle case 1 first. As $G = AB$ is a factorization of G the sets

$$B,\ aB, \ldots, a^{r-2}B,\ a^{r-1}dB \tag{1}$$

form a partition of G. Multiplying the factorization $G = AB$ by a we get the factorization $G = aG = (aA)B$ of G. The sets

$$aB, \ldots, a^{r-2}B,\ a^{r-1}B,\ a^r dB \tag{2}$$

form a partition of G. Comparing the two partitions we get

$$B \cup a^{r-1}dB = a^{r-1}B \cup a^r dB.$$

If $a^{r-1}dB \cap a^r dB \neq \emptyset$, then $B \cap aB \neq \emptyset$ which contradicts (1). Thus $a^{r-1}dB \cap a^r dB = \emptyset$ and so $dB \subset B$.

In the factorization $G = AB$ replace the finite factor A by A^{-1} to get the factorization $G = A^{-1}B$. One can draw the conclusion that $d^{-1}B \subset B$. Multiplying by d we get $B \subset dB$. From $dB \subset B$ and $B \subset dB$ it follows that $B = dB$. Plugging this to (1) we get that the sets

$$B, aB, \ldots, a^{r-2}B, a^{r-1}B$$

form a partition of G and so $G = CB$ is a factorization of G.

Let us turn to case 2. We would like to prove that $G = AB$ is a factorization of G, that is, the sets

$$B,\ aB,\ a^2B, \ldots, a^{r-1}B \tag{3}$$

form a partition of G. Since $G = AB$ is a factorization of G the sets

$$B,\ aB, \ldots, a^{i-1}B,\ a^i dB,\ a^{i+1}B, \ldots, a^{r-1}B \tag{4}$$

form a partition of G. In particular $a^u C \cap a^v C = \emptyset$ for each u, v, $u \neq v$, $0 \leq u, v \leq r-1$ whenever $u \neq i$ and $v \neq i$.

In order to verify that (3) is a partition of G we establish that $B = dB$. We do this by showing that $dB \subset B$ and $B \subset dB$. The containment $dB \subset B$ follows from the fact that $a^u B \cap a^i B = \emptyset$ holds for each u, $u \neq i$, $0 \leq u \leq r-1$. To prove this assume on the contrary that $a^u C \cap a^i C \neq \emptyset$ for some u, $u \neq i$, $0 \leq u \leq r-1$. We distinguish two cases depending on $u < i$ or $i < u$.

Let us consider the $u < i$ case first. Multiplying $a^u B \cap a^i B \neq \emptyset$ by a^k gives that

$$a^{u+k}B \cap a^{i+k}B \neq \emptyset. \tag{5}$$

If

$$0 \leq u+k,\ i+k \leq r-1, \tag{6}$$

$$u+k \neq i,\ i+k \neq i, \tag{7}$$

then (5) violates partition (4). Plainly (6) is equivalent to $-u \leq k \leq r-1-i$ and so there are $r-1-i+u+1$ choices for k. Using $i \leq r-2$, $0 \leq u$ we get

$$2 = r-1-(r-2)+(0)+1 \leq r-1-i+u+1$$

and so there are at least two choices for k. If $i \neq r-2$ or $0 \neq u$, then there are at least three choices for k and we can get a contradiction. Thus we may assume that $i = r-2$ and $0 = u$. Now with the $k = 1$ choice (6) and (7) are satisfied and we get a contradiction.

Finally consider the $i < u$ case. Multiplying $a^i B \cap a^u B \neq \emptyset$ by a^k gives that

$$a^{i+k}B \cap a^{u+k}B \neq \emptyset. \tag{8}$$

If

$$0 \leq i + k, \ u + k \leq r - 1, \tag{9}$$

$$i + k \neq i, \ u + k \neq i, \tag{10}$$

then (8) contradicts to partition (4). Clearly (9) is equivalent to $-i \leq k \leq r-1-u$ and so there are $r - 1 - u + i + 1$ choices for k. Using $u \leq r - 1$, $1 \leq i$ we get

$$2 = r - 1 - (r - 1) + (1) + 1 \leq r - 1 - u + i + 1$$

and so there are at least two choices for k. If $u \neq r - 1$ or $1 \neq i$, then there are at least three choices for k and we can get a contradiction. Thus we may assume that $u = r - 1$ and $1 = i$. Now with the $k = -1$ choice (9) and (10) are satisfied and we get a contradiction. Therefore $dB \subset B$ as we claimed.

In the factorization $G = AB$ replace the finite factor A by A^{-1} to get the factorization $G = A^{-1}B$. From this factorization we can conclude that $d^{-1}B \subset B$ or equivalently $B \subset dB$.

This completes the proof. □

Lemma 2.2. *Let G be an abelian group. Let*

$$A = \{e, a, a^2, \ldots, a^{i-1}, a^i d, a^{i+1}, \ldots, a^{r-1}\}$$

be a distorted cyclic subset of G, where $r \geq 4$. A is a subgroup of G if and only if $d = e$ and $a^r = e$.

Proof. If $d = e$, then A is a cyclic subset of G. If $a^r = e$, then this cyclic subset is a subgroup of G.

Next assume that A is a subgroup of G and try to show that $d = e$ and $a^r = e$. We distinguish the following three cases

Case 1: $i = 1$.
Case 2: $i = r - 1$.
Case 3: $2 \leq i \leq r - 2$.

Let us settle case 1 first. Now

$$A = \{e, ad, a^2, \ldots, a^{r-1}\}.$$

As $a^{r-2}, a^{r-1} \in A$, it follows that $a \in A$. If $a = ad$, then $e = d$. Therefore A is a cyclic subgroup of G and so $a^r = e$, as required. Thus we may assume $a \neq ad$ and consequently $a \in \{e, a^2, \ldots, a^{r-1}\}$. If $a = e$, then $a^2 = \cdots = a^{r-1} = e$ which is a contradiction.

If $a = a^2$, then $a = e$. This reduces the problem to the earlier case. If $a = a^j$, $3 \leq j \leq r - 1$, then we get the $e = a^{j-1}$ contradiction.

Let us turn to case 2. Now

$$A = \{e, a, a^2, \dots, a^{r-2}, a^{r-1}d\}.$$

As $a \in A$, it follows that $a^{r-1} \in A$. If $a^{r-1} = a^{r-1}d$, then $e = d$ and so A is a cyclic subgroup of G. It follows that $a^r = e$, as required.

If $a^{r-1} = e$, then the cyclic subset $\{e, a, a^2, \dots, a^{r-2}\}$ is a subgroup of G. Consequently it is a subgroup of A. It follows the contradiction that $r-1$ divides r. If $a^{r-1} = a^j$, $1 \leq j \leq r-2$, then we get the $a^{r-2} = a^{j-1}$ contradiction.

We may turn to case 3. Now $a \in A$ implies $a^i \in A$. If $a^i = a^i d$, then $e = d$ and A is a cyclic subgroup of G. It follows that $a^r = e$, as required. We may assume that $a^i = a^j$, where $i \neq j$. We distinguish two cases depending on $j < i$ or $i < j$.

Assume first that $j < i$. Multiply $a^j = a^i$ by a^{-u} to get $a^{j-u} = a^{i-u}$. If

$$0 \leq j - u, \ i - u \leq i - 1, \tag{11}$$

then we get a contradiction. (11) is equivalent to

$$-j \leq -u, \ -u \leq -1.$$

There are $(-1) - (-j) + 1 = j$ choices for u. Therefore in the $j \geq 1$ case we are done. So we may assume that $j = 0$.

Multiply $a^0 = a^i$ by a^u to get $a^u = a^{i+u}$. If

$$0 \leq u \leq i - 1, \ i + 1 \leq i + u \leq r - 1,$$

then we get a contradiction. Note that the $u = 1$ choice is suitable, as $2 \leq i \leq r-2$.

Next assume that $i < j$. Multiply $a^i = a^j$ by a^u to get $a^{i+u} = a^{j+u}$. If

$$i + 1 \leq i + u, \ j + u \leq r - 1, \tag{12}$$

then we get a contradiction. (12) is equivalent to

$$1 \leq u, \ u \leq r - 1 - j.$$

There are $(r-1-j) - (-1) + 1 = r - j + 1$ choices for u. If $r - j + 1 \geq 1$, then we get a contradiction. Equivalently if $j \leq r-2$, then we get a contradiction. Thus we may assume that $j = r - 1$. Multiplying $a^i = a^{r-1}$ by a^{-u} we get $a^{i-u} = a^{r-1-u}$. If

$$0 \leq i - u \leq i - 1, \ i + 1 \leq r - 1 - u \leq r - 1,$$

then we get a contradiction. If $i \leq r - 3$, then the $u = 1$ choice is suitable. Thus we may assume that $i = r - 2$. Now $a^{r-2} = a^{r-1}$. This provides the $e = a$ contradiction. $\square$

The reader may notice that in the proof above in case 2 it is enough to assume that $r \geq 3$. In this particular case a stronger result holds. We will need this result and for easier reference we spell it out as a lemma.

Lemma 2.3. *Let G be an abelian group. Let*

$$A = \{e, a, a^2, \ldots, a^{r-2}, a^{r-1}d\}$$

be a distorted cyclic subset of G, where $r \geq 3$. A is a subgroup of G if and only if $d = e$ and $a^r = e$.

The distorted cyclic subset A of an abelian group is called a reducible subset if the following hold.

(1) A is not a subgroup.
(2) There is a subset B of G such that $G = AB$ is a factorization of G.
(3) There are non-subgroup distorted cyclic subsets A_1, A_2 for which $G = A_1A_2B$ is a factorization of G.

Lemma 2.4. *Let G be an abelian group and let A be a non-subgroup distorted cyclic subset of G. If $|A| = st$, $s \geq 3$, $t \geq 3$, then A is reducible.*

Proof. Suppose there is a subset B of G such that $G = AB$ is a factorization of G. Let

$$A = \{e, a, a^2, \ldots, a^{i-1}, a^i d, a^{i+1}, \ldots, a^{r-1}\}$$

and let

$$C = \{e, a, a^2, \ldots, a^{r-1}\}$$

be a cyclic subset associated with A, where $r = st$. Set

$$\begin{aligned}
C_1 &= \{e, a, a^2, \ldots, a^{s-1}\}, \\
C_2 &= \{e, a^s, a^{2s}, \ldots, a^{(t-1)s}\}, \\
A_1 &= C_1, \\
A_2 &= \{e, a^s, a^{2s}, \ldots, a^{(t-1)s}d\}.
\end{aligned}$$

Note that the product C_1C_2 is direct and is equal to C.

By Lemma 2.1, in the factorization $G = AB$ the factor A can be replaced by C to get the factorization $G = CB$. We can read off from the proof that $B = dB$ also holds. The sets

$$B,\ aB,\ a^2B, \ldots, a^{r-1}B$$

form a partition of G. Using $C = C_1C_2$ we get that the sets

$$\begin{array}{cccccc} B, & a^sB, & a^{2s}B, & \dots, & a^{(t-2)s}B, & a^{(t-1)s}B, \\ aB, & a^{1+s}B, & a^{1+2s}B, & \dots, & a^{1+(t-2)s}B, & a^{1+(t-1)s}B, \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{s-1}B, & a^{s-1+s}B, & a^{s-1+2s}B, & \dots, & a^{s-1+(t-2)s}B, & a^{s-1+(t-1)s}B \end{array}$$

form a partition of G. Using $B = dB$ we get that the sets

$$\begin{array}{cccccc} B, & a^sB, & a^{2s}B, & \dots, & a^{(t-2)s}B, & a^{(t-1)s}dB, \\ aB, & a^{1+s}B, & a^{1+2s}B, & \dots, & a^{1+(t-2)s}B, & a^{1+(t-1)s}dB, \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{s-1}B, & a^{s-1+s}B, & a^{s-1+2s}B, & \dots, & a^{s-1+(t-2)s}B, & a^{s-1+(t-1)s}dB \end{array}$$

form a partition of G and therefore $G = A_1A_2B$ is a factorization of G.

If A_1 is a subgroup of G, then $a^s = e$. This violates the factorization $G = CB$. Thus A_1 cannot be a subgroup of G. If A_2 is a subgroup of G, then, by Lemma 2.3, $a^{st} = a^r = e$ and $d = e$. Consequently, by Lemma 2.2, A is a subgroup of G. This is not the case. Thus A_2 is not a subgroup of G.

This completes the proof. □

Theorem 2.5. *Let $p_1, \dots, p_s$ be distinct odd primes and let*

$$G = H \times \Big[\prod_{i=1}^{s} C(p_i^\infty)\Big],$$

where H is a finite abelian group of odd order and p_i does not divide $|H|$ for each i, $1 \le i \le s$. If

$$G = \prod_{i=1}^{\infty} A_i \tag{13}$$

is a factorization of G and each A_i is a finite distorted cyclic subset of G, then A_i is a subgroup of G for some i, $1 \le i < \infty$.

Proof. We claim that q divides $|H|p_1 \cdots p_s$ for each prime divisor q of $|A_k|$ for each factor A_k of the factorization (13).

To prove the claim assume on the contrary that there is a factor A_k of the factorization (13) and a prime divisor q of $|A_k|$ such that q does not divide $|H|p_1 \cdots p_s$. We may assume that $k = 1$ since this is only a matter of indexing the factors. Setting

$$B = \prod_{i=2}^{\infty} A_i$$

the factorization (13) can be written in the form $G = A_1B$. We distinguish the next two cases.

Case 1: $|A_1| \neq q$.

Case 2: $|A_1| = q$.

In case 1 let

$$A_1 = \{e, a, a^2, \dots, a^{i-1}, a^i d, a^{i+1}, \dots, a^{r-1}\}$$

and let

$$C = \{e, a, a^2, \dots, a^{r-1}\}$$

be a cyclic subset associated with A_1. Now plainly $|A_1| \geq 4$ and so, by Lemma 2.1, in the factorization $G = A_1 B$, the factor A_1 can be replaced by C to get the factorization $G = CB$. There are cyclic subsets of prime cardinality $C_1, \dots, C_u$ such that the product $C_1 \cdots C_s$ is direct and is equal to C. Some of $|C_1|, \dots, |C_u|$ is equal to q. We may assume that $|C_1| = q$ since this is only a matter of rearranging the factors. Because of the structure of G, each element of G has finite order. Let m be the order of c. Clearly each prime divisor of m divides $|H|p_1 \cdots p_s$. Therefore q and m are relatively prime. By the Chinese remainder theorem, there is an integer t such that

$$\begin{aligned} t &\equiv 0 \pmod{m}, \\ t &\equiv 1 \pmod{q}. \end{aligned}$$

Plainly t is relatively prime to q. By Proposition 3 of [4], in the factorization $G = C_1 \cdots C_u B$ the factor C_1 can be replaced by C_1^t to get the factorization $G = C_1^t C_2 \cdots C_u B$. Note that $(c^i)^t = (c^t)^i = e$ and so the element e appears in C_1^t with multiplicity q. This is a contradiction.

In case 2, let m be the least common multiple of the orders of the elements of A_1. Define the integer t in the same way as in case 1. Replace A_1 by A_1^t in the factorization $G = A_1 B$ to get $G = A_1^t B$. Now we get the contradiction that the element e appears in A_1^t with multiplicity q.

By the assumption of the theorem $|H|p_1 \cdots p_s$ is odd. The claim we have just verified gives in particular that each $|A_i|$ is odd.

In order to prove the theorem assume on the contrary that in factorization (13) none of the factors is a subgroup of G. By Lemma 2.4, in the factorization (13) each A_i can be replaced by a product of non-subgroup distorted cyclic subsets whose cardinalities are primes. (It may happen that $|A_i|$ is a prime. In this case of course we do not replace A_i.) We end up with a factorization in which each factors is normalized, has prime cardinality, and is not a subgroup of G. For this factorization Theorem 1 of [6] is applicable and implies that one of the factors is a subgroup of G.

This contradiction completes the proof. □

3. Factors of prime cardinality

In this section we consider factorizations in which each factor with one possible exception has prime cardinality and one factor may have order four.

Theorem 3.1. *Let*

$$G = H \times \Big[\prod_{i=1}^{r} C(p_i^{\infty})\Big],$$

where H is a finite abelian group, $p_1, \ldots, p_r$ are distinct primes, p_i does not divide $|H|$ for each i, $1 \le i \le r$. If

$$G = B \prod_{i=1}^{\infty} A_i \tag{14}$$

is a normalized factorization of G such that $|B| = 4$ and each $|A_i|$ is a prime, then one of the factors $B, A_1, A_2, \ldots$ is periodic.

Proof. Let $|H|$ be the product of the (not necessarily distinct) primes $p_{r+1}, \ldots, p_s$. By the assumptions of the theorem

$$\{p_1, \ldots, p_r\} \cap \{p_{r+1}, \ldots, p_s\} = \emptyset.$$

For a factor A_j of the factorization (14) with $|A_j| = p$, where p is a prime let A_j' be the set of the p-components of the elements of A_j. We claim that in the factorization (14) A_j can be replaced by A_j'. In order to prove the claim let m be a common multiple of the orders of the p'-components of the elements of A_j and let n be a common multiple of the orders of the p-components of the elements A_j. Such m, n do exist since each element of G has a finite order. As m and n are relatively primes by the Chinese remainder theorem, the system of congruences

$$\begin{aligned} t &\equiv 0 \pmod{m}, \\ t &\equiv 1 \pmod{n}, \end{aligned}$$

is solvable. By Proposition 3 of [4], A_j can be replaced by A_j^t. As $A_j^t = A_j'$, the claim is proved.

The fact that A_j can be replaced by A_j' implies that the elements of A_j' are distinct. Further we can conclude that if $|A_j| = p$, where p is a prime, then p must be one of the primes $p_1, \ldots, p_s$. It follows in the same manner that the 2-components of the elements of B are distinct. These elements form a set B' and $|B'| = 4$.

From the factorization

$$G = B' \prod_{i=1}^{\infty} A_i'$$

we draw further conclusions. Let p be one of the primes $p_1, \ldots, p_r$. If p is odd, then the product of all the A_j''s with $|A_j'| = p$ forms a factorization of the subgroup

$C(p^\infty)$ of G. If $p=2$, then the product of B' and all the A'_j's with $|A'_j| = p$ form a factorization of the subgroup $C(p^\infty)$ of G. If $|H|$ is odd, then the product of all the A'_j's with $|A'_j| \bigm| |H|$ forms a factorization of H. If $|H|$ is even, then the product of B' and all the A'_j with $|A'_j| \bigm| |H|$ forms a factorization of H.

The subgroups of $C(p_i^\infty)$ form a chain. For each integer $j \geq 0$ there is a unique subgroup of order p^j. Let $H_{i,0}, H_{i,1}, \ldots$ be all the subgroups of $C(p_i^\infty)$. We assume that $|H_{i,j}| = p_i^j$. There are factors $A''_{i,1}, A''_{i,2}, \ldots$ among $A'_1, A'_2, \ldots$ such that

$$A''_{i,1} = H_{i,1},\ A''_{i,1}A''_{i,2} = H_{i,2}, \ldots$$

Note that each nonidentity element of $A''_{i,j}$ must have order p_i^j.

Now let us go back to factorization (14). To prove the theorem we assume the contrary that none of the factors $B, A_1, A_2, \ldots$ is periodic. Choose a factor A_j and assume that $|A_j| = p$, where p is a prime. (We know that p is one of the primes $p_1, \ldots, p_s$.) The p-components of the elements of A_j are distinct and form a set A'_j with $|A'_j| = p$. If A'_j is not a subgroup of G, then replace A_j by $C_j = A'_j$. If A'_j is a subgroup of G, then there is an element in A_j whose q-component is not the identity element since A_j is not a subgroup of G. Here q is a prime $p \neq q$. Now A_j can be replaced by C_j such that C_j is not a subgroup of G and the orders of the elements of C_j divide pq. In other words C_j is constructed from the subgroup A'_j by multiplying some elements of A'_j by some elements of order q. Let us consider the factorization

$$G = B \prod_{i=1}^{\infty} C_i.$$

Here none of the factors $B, C_1, C_2, \ldots$ is periodic. For each i, $1 \leq i \leq r$ there is an integer $\alpha(i)$ such that the orders of the p_i-components of the elements of B are less than or equal to $p_i^{\alpha(i)}$ and $\alpha(i) \geq 1$. The elements of $C(p_i^\infty)$ whose order is less than or equal to $p_i^{\alpha(i)}$ form the unique subgroup $H_{i,\alpha(i)}$ of $C(p_i^\infty)$. Set

$$K = HH_{1,\alpha(1)} \cdots H_{r,\alpha(r)}.$$

Clearly, $B \subset K$. Let $D_1, \ldots, D_n$ be all the C_i factors for which $C_i \subset K$. We claim that $K = BD_1 \cdots D_n$ is a factorization of K. As $B, D_1, \ldots, D_n \subset K$, it is enough to verify that $|B||D_1| \cdots |D_n| = |K|$. In order to verify this equation let $D_1, \ldots, D_m$ be the factors among $D_1, \ldots, D_n$ whose cardinality is one of $p_{r+1}, \ldots, p_s$.

Assume first that $4 \bigm| |H|$. Note that $|B||D_{m+1}| \cdots |D_n| = |H|$. Further

$$\begin{aligned} |D_1| \cdots |D_m| &= \big(|A''_{1,1}| \cdots |A''_{1,\alpha(1)}|\big) \cdots \big(|A''_{r,1}| \cdots |A''_{r,\alpha(r)}|\big) \\ &= p_1^{\alpha(1)} \cdots p_r^{\alpha(r)} \\ &= |H_{1,\alpha(1)}| \cdots |H_{r,\alpha(r)}|. \end{aligned}$$

Assume next that $4 \mid |H_{1,\alpha(1)}| \cdots |H_{r,\alpha(r)}|$. Note that $|D_{m+1}| \cdots |D_n| = |H|$ and

$$|B||D_1| \cdots |D_m| = |H_{1,\alpha(1)}| \cdots |H_{r,\alpha(r)}|.$$

Thus $K = BD_1 \cdots D_n$ is a factorization of the finite abelian group K. By Theorem 1 of [7], one of the factors is periodic. This contradiction completes the proof. □

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