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HOMOMORPHIC IMAGES OF SEMIHEREDITARY RINGS

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Abstract. An example is provided of a left semihereditary ring R with an idempotent ideal I such that R/I is not left semihereditary. Two related positive results are provided: a) a left semiartinian, left p.p. ring has nonsingular Loewy factors; b) if R is a commutative von Neumann regular ring and D is an idempotent ideal of $R[X]$ then $R[X]/D$ is semihereditary.

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It is well-known that a factor of a left hereditary ring by an idempotent ideal yields a left hereditary ring ([1] or [8]). A similar result has been claimed to be true for left semihereditary rings [2, Lemma 1.5]. Unfortunately, the proof given there has a flaw. In this note, a counterexample to the claim is exhibited. Two additional results are provided, including a correct proof of a result in [2] which relied on the claim.

The example is essentially due to S.U. Chase [4], constructed to exhibit a ring which is right hereditary but not left p.p.

Example. Let $S = \mathbb{Z}/2\mathbb{Z}$, let T be the ring consisting of all eventually constant sequences whose entries come from S and let I be the ideal consisting of those sequences in T that are eventually zero. Then T is a hereditary Boolean ring and $T/I \approx S$. The triangular matrix ring $A = \left[\begin{smallmatrix} S & 0 \\ S & T \end{smallmatrix}\right]$ is right hereditary but not left p.p.; complete details for these assertions can be found in [5, p.110–111]. Now consider the triangular matrix ring $R = \left[\begin{smallmatrix} T & 0 \\ T & T \end{smallmatrix}\right]$. This ring is left and right semihereditary; see, e.g., [6, p.114, Ex.14]. Observe that $D = \left[\begin{smallmatrix} I & 0 \\ I & 0 \end{smallmatrix}\right]$ is an ideal of R, $D^2 = D$ because $I^2 = I$, and $A \approx R/D$. Since A is not left p.p., R/D is not left semihereditary.

The proof of [2, Corollary 1.6] relies on the incorrect assertion regarding factors of semihereditary rings by idempotent ideals. However [2, Corollary 1.6] is true as will be shown. Before proceeding to showing this, needed terminology used in [2] will be stated.

The (left) *Loewy series* for the ring R is the ascending chain $\{S_\alpha(R)\}_{\alpha\geq 0}$ of ideals defined recursively for ordinals $\alpha \geq 0$ as follows: $S_0(R) = 0$, $S_{\alpha+1}(R)/S_\alpha(R) =$ left socle $(R/S_\alpha(R))$ and $S_\alpha(R) = \bigcup_{\beta<\alpha} S_\beta(R)$ when α is a limit ordinal. The ring R is (left) *semiartinian* if $R = S_\alpha(R)$ for some ordinal α. The least such ordinal, necessarily a non-limit ordinal under the assumption that R has an identity element, is called the *Loewy length* of R.

A semiartinian ring R is a (left) *NLF-ring* (Nonsingular Loewy Factor rings) if $R/S_\alpha(R)$ is a left nonsingular ring for each member $S_\alpha(R)$ of the Loewy series. As shown in [2, Theorem 1.3], the semiartinian ring R is an NLF-ring if and only if $S_\alpha^2(R) = S_\alpha(R)$ for each member of the Loewy series. Additionally, [2, Corollary 1.6] asserts that a semihereditary semiartinian ring is an NLF-ring. The next result covers [2, Corollary 1.6].

Theorem 1. *If R is a left semiartinian, left p.p. ring then R is a left NLF-ring.*

Proof. Left p.p. rings are left nonsingular. Thus it suffices to show that if R is a left p.p. ring with left socle S, then R/S is a left p.p. ring. Assuming R is left p.p., observe that $S^2 = S$. Indeed, if U is a minimal left ideal of R, then, because U is projective, $U \approx Re$ for some idempotent e; if u is the image of e under an isomorphism from Re to U, then $U = Reu \subseteq S^2$. Now let $(Rx + S)/S = (R/S)(x+S)$ be a principal left ideal of R/S. Then $(Rx+S)/S \approx Rx/(Rx\cap S)$ as R- or R/S-modules. The claim is that $Rx\cap S = S(Rx\cap S)$. Indeed, $S = (Rx\cap S)\oplus W$ for some left ideal $W \subseteq S$. If $a \in Rx \cap S$, then $a = \sum_i u_i v_i$, where $u_i, v_i \in S$. For each i, $v_i = x_i + w_i$, where $x_i \in Rx \cap S$, $w_i \in W$. Then

$$a = \sum_i u_i x_i + \sum_i u_i w_i \ ,$$

hence

$$\sum u_i w_i = a - \sum_i u_i x_i \in (Rx \cap S) \cap W = 0 \ .$$

Therefore $a \in S(Rx\cap S)$ and hence $Rx\cap S = S(Rx\cap S)$. The "Dual Basis Lemma" (e.g., [3, Proposition 3.1, p.132]) can now be used to show that $(Rx + S)/S$ is a projective R/S-module. For Rx is a projective R-module so there exists an R-homomorphism $f : Rx \to R$ such that $a = f(a)x$ for all $a \in Rx$; i.e., $\{x; f\}$ is a "dual basis" for Rx. Then

$$f(Rx \cap S) = f(S(Rx \cap S)) = Sf(Rx \cap S) \subseteq S$$

and so f induces an R/S-homomorphism

$$f^* : (Rx + S)/S \to R/S$$

such that

$$a + S = f^*(a + S)(x + S) \text{ for all } a \in Rx\ .$$

Accordingly $\{x + S; f^*\}$ is a "dual basis" for $(Rx + S)/S$ and so $(Rx + S)/S$ is a projective R/S-module. Thus R/S is a left p.p.-ring, as needed. □

Note that a similar proof can be used to show that: *if R is a left semihereditary ring with left socle S, then R/S is also left semihereditary*. Thus in some instances factors of semihereditary rings by idempotent ideals yield semihereditary rings. Of course this happens for von Neumann regular rings and one way commutative semihereditary rings arise is as polynomial rings over von Neumann regular rings [7]. The next result implies that the homomorphic property is preserved in this class of semihereditary rings.

Theorem 2. *Let R be a von Neumann regular ring. An ideal D of $R[X]$ is idempotent if and only if $D = (D \cap R)[X]$.*

Proof. If $A = D \cap R$, then A is an ideal of R, $A^2 = A$ and so $D = A[X] = A \cdot A[X] \subseteq D^2$. For the converse, suppose D is idempotent and $(D \cap R)[X] \subsetneq D$. By passing to $(R/(D \cap R))[X]$, it can be assumed that $D \neq 0$ and $D \cap R = 0$. Now let p be a polynomial of minimal degree in D, say $p = a_0 + a_1X + \cdots + a_nX^n$ with $a_n \neq 0$. Then $n \geq 1$ because $D \cap R = 0$. Note first that if $r \in R$ and $ra_n = 0$, then $ra_i = 0$ for all i, since $rp \in D$ and has lower degree if it is nonzero. Similarly, $a_nr = 0$ implies $a_ir = 0$ for all i. Now $a_n = a_nba_n$, hence letting $e = ba_n$, we have $e^2 = e$, $a_n = a_ne$ and hence $a_i = a_ie$ for all i, since $a_n(1-e) = 0$. Thus $p = pe \in De$. We claim that $De = R[x]p$. To show this, let $q = b_kX^k + \cdots + b_1X + b_0 \in De$; then $b_i = b_ie$ for all i. Note that $k \geq n$ if $q \neq 0$. Then $b_k = b_ke = b_k \cdot ba_n$ so $q_1 = q - b_kbX^{k-n}p$ is zero or of lower degree than q. Thus a left division algorithm exists and, by the minimality of the degree of p, we get $q = gp$. Thus $De = R[X]p$, as claimed. Hence $De = D^2e = DR[X]p = Dp$, and because $p \in De$, we have $p = hp$ for some $h \in D$, where h is of degree $k \geq 0$. If $k \geq 1$, then $h = c_kX^k + h_1$ where $h_1 = 0$ or $\deg(h_1) < \deg(h)$. Then $p = hp = c_kX^k \cdot p + h_1p$. If $c_ka_n \neq 0$ then the right hand product has degree greater than p. It follows that $c_ka_n = 0$, hence $c_kp = 0$, thus $p = h_1p$. Repeating this argument we conclude that $c_jp = 0$ for $1 \leq j \leq k$ and hence $p = c_0p$. It follows that $a_n = c_0a_n \neq 0$. But then we have $a_n = c_0a_n = h \cdot a_n \in D$, a contradiction. □

Corollary. *If R is a commutative von Neumann regular ring and D is an idempotent ideal of $R[X]$, then $R[X]/D$ is semihereditary.*

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