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LAWS OF METABELIAN PRODUCTS OF ABELIAN GROUPS

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Abstract. In this paper, it is proved that all nontrivial laws in form $\prod_{i<j}[x_i, x_j]^{\lambda_{ij}}$ of metabelian products of abelian groups are products of transforms of Jacobi products if not all the factors are torsion groups. This result generalizes the well-known result of Bachmuth on the laws of free metabelian groups. Using this, n-symmetric words of metabelian products are completely described. Moreover, an example is constructed to show that the above result is not necessarily true if all the factors are torsion groups.

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1. Introduction

We first fix some notation which will be used throughout the paper. For a positive integer n, let F_n denote the *free group* of rank n with basis $\{x_1, \ldots, x_n\}$. Then $A_n := F_n/F_n'$ and $M_n := F_n/F_n''$ will denote the *free abelian group* and the *free metabelian group* of rank n, respectively. Similarly, for a positive integer m, let $F := H_1 * \cdots * H_m$ be the *free product* of some nontrivial abelian groups $H_1, \ldots, H_m$. Then $A := F/F'$ and $M := F/F''$ will denote the corresponding abelian product and metabelian product, respectively. For a group G, let $Z(G)$ be the *integral group ring*, and for each element $\omega \in Z(G)$, we use $\bar{\omega}$ and $\tilde{\omega}$ to denote its natural image in $Z(G/G')$ and $Z(G/G'')$, respectively. The commutator of two elements a, b is denoted by $[a, b] = aba^{-1}b^{-1}$, and the conjugate is denoted by $a^b = bab^{-1}$. Moreover, we use $a = b(mod\ F_n'')$ to denote that there exists an element $c \in F_n''$ such that $a = bc$.

Definition 1.1. Let G be a group. An element $\omega \in F_n$ is called a *law* of G if $\omega(g_1, \ldots, g_n) = 1$ for any $g_1, \ldots, g_n$ in G. The group of all laws of G in F_n is called the *law group* of G and denoted by $L(G)$.

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Let $\omega \in F_n$. Since F_n' is generated by the commutators $[x_i, x_j]^\mu$, where $1 \leq i \leq j \leq n$ and $\mu \in F_n$, and since any two commutators commute modulo F_n'', ω can be expressed as

$$\omega = \omega_1\omega_2(mod\ F_n'') = x_1^{\mu_1}\cdots x_n^{\mu_n}\prod_{i<j}[x_i, x_j]^{\lambda_{ij}}(mod\ F_n''), \tag{1}$$

where $\omega_1 = x_1^{\mu_1}\cdots x_n^{\mu_n}$, $\omega_2 = \prod_{i<j}[x_i, x_j]^{\lambda_{ij}}$, $\mu_i \in Z$ and $\lambda_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_j^{\pm}]$.

For the free metabelian group M_n, since M_n' is abelian, M_n' may be regarded as a left $Z(M_n/M_n')$-module in the natural way, where the module action is induced by the conjugation in M_n. It is known that the *Jacobi products*

$$[x_i, x_j]^{1-\bar{x}_k}[x_j, x_k]^{1-\bar{x}_i}[x_k, x_i]^{1-\bar{x}_j}$$

are laws for each metabelian group. The following well-known theorem obtained by Bachmuth [1], shows that its partial inverse is true.

Theorem 1.2. *Let $\omega \in F_n'$ and write $\omega = \omega_2(mod\ F_n'')$ as in form (1). Then ω is a law of the free metabelian group M_n if and only if ω_2 is a product of transforms of Jacobi products.*

The main purpose of this paper is to prove the following theorem, which generalizes the above Bachmuth's result.

Theorem 1.3. *Let $m, n \geq 2$ be integers. Let M be the metabelian product of some nontrivial abelian groups $H_1, \ldots, H_m$. If not all factors H_i are torsion groups, then $L(G) = F_n''$. More precisely, let $\omega = \omega_2(mod\ F_n'')$ as in form (1), then ω is a law of M in F_n if and only if $\omega_1 = 1$ and ω_2 is a product of transforms of Jacobi products.*

In the case that all factors H_i are torsion groups, the following example shows that Theorem 1.3 is not necessarily true.

Example 1.4. Let $n \geq 2$ be an integer. Let G be the metabelian product of cyclic groups $\langle a_1\rangle$ and $\langle a_2\rangle$ with $o(a_1) = o(a_2) = 2$. Then $\prod_{i<j}[x_i, x_j]^{(1+\bar{x}_i)(1+\bar{x}_j)}$ is a law of G, and cannot be expressed as a product of transforms of Jacobi products.

This example is discussed in details in Section 4.

As an application of Theorem 1.3, a characterization of *symmetric words* of metabelian groups is given in Section 4.

Definition 1.5. Let G be a group. A word $\omega(x_1, \ldots, x_n) \in F_n$ is called a n-symmetric word for the group G, if $\omega(g_1, \ldots, g_n) = \omega(g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ for all $g_1, \ldots, g_n$ in G and σ in the symmetric group S_n.

Symmetric words for a group are closely related to the fixed points of the automorphisms permuting generators in their corresponding relatively free groups [13]. The problem of characterizing the symmetric words for a given group G was initiated by Plonka [10], who gave a complete description for nilpotent groups of class ≤ 3. More results on the symmetric words can be found in [4,5,6,7].

Our second result, stated in the following theorem, completely describes n-symmetric words of metabelian products M.

Theorem 1.6. *Each n-symmetric word of M can be expressed in the form*

$$\omega = \prod_{i<j}[x_i, x_j]^{u(\bar{x}_i,\bar{x}_j)+v(\bar{x}_i,\bar{x}_j)p(\bar{x}_1,\ldots,\wedge\bar{x}_i,\ldots,\wedge\bar{x}_j,\ldots,\bar{x}_n)},$$

where u and v are sums of polynomials of the forms $x^r y^s - x^s y^r$ with integral coefficients, and p is an $(n-2)$-symmetric polynomial.

This paper is organized as follows. After this introduction, we recall, in Section 2, some necessary information on Fox derivatives, Magnus embedding, Shmel'kin embedding and generalized derivatives. Then in Section 3, we first prove several technical lemmas and then Theorem 1.3. Finally in Section 4, we discuss Example 1.4 and prove Theorem 1.6.

2. Preliminaries

We first recall some necessary information of *Fox derivatives* [2] and *Magnus embedding* [8].

Left Fox derivatives ∂_i $(i = 1, 2, \cdots, n)$ are defined to be linear mappings from $Z(M_n)$ to $Z(A_n)$, which satisfy for any elements $\tilde{u}, \tilde{v} \in Z(M_n)$ the following rules:

(1) $\partial_i(\tilde{u} + \tilde{v}) = \partial_i(\tilde{u}) + \partial_i(\tilde{v})$;

(2) $\partial_i(\tilde{x}_j) = \delta_{ij}$;

(3) $\partial_i(\tilde{u}\tilde{v}) = \bar{u}\partial_i(\tilde{v}) + \epsilon(\tilde{v})\partial_i(\tilde{u})$, where δ_{ij} is kronecker symbol, ϵ is the trivialization map of $Z(M_n) \to Z$.

For every $\tilde{\omega} \in M_n$, $\bar{s} \in Z(A_n)$, we have the following equality [1]:

$$\partial_i(\tilde{w}^{\bar{s}}) = \bar{s} \cdot \partial_i(\tilde{w}).$$

Let T_n be a left $Z(A_n)$-module with basis $\{t_1, \ldots, t_n\}$. Consider a matrix group W_n such that

$$W_n = \begin{pmatrix} A_n & T_n \\ 0 & 1 \end{pmatrix}.$$

Then the map

$$\beta(\tilde{\omega}) = \begin{pmatrix} \bar{\omega} & \partial_1(\tilde{\omega})t_1 + \cdots + \partial_n(\tilde{\omega})t_n \\ 0 & 1 \end{pmatrix},$$

where $\tilde{\omega} \in M_n$, is an embedding (called the Magnus embedding) of M_n into W_n.

The Magnus embedding was generalized into *Shmel'kin embedding* in [12], and the reduced *generalized derivatives* [3] play similar role in the investigation of the group F/R' to the role of the Fox derivatives in the investigation of the group M_n, where R is a normal subgroup of F with $R \bigcap H_i = 1$ for $i = 1, \ldots, m$. Write $H := F/R$. Let T be a free left $Z(H)$-module with basis $\{t_1, \ldots, t_m\}$. Consider a matrix group W such that

$$W = \begin{pmatrix} H & T \\ 0 & 1 \end{pmatrix}.$$

The map

$$a_i \to \begin{pmatrix} \bar{a}_i & (\bar{a}_i - 1)t_i \\ 0 & 1 \end{pmatrix},$$

where $a_i \in H_i$ $(i = 1, \ldots, m)$ and $\bar{a}_i$ is the natural image of a_i in H, determines a homomorphism $\sigma : F \to W$ with kernel R'. The resulting embedding (we also write σ to denote this embedding) of the group F/R' into the group W is called the Shmel'kin embedding. It is known that the Shmel'kin embedding can be applied to the case where $R = F'$.

As shown in [3], for any element $\tilde{\omega} \in F/R'$, we have

$$\sigma(\tilde{\omega}) = \begin{pmatrix} \bar{\omega} & D_1(\tilde{\omega})t_1 + \cdots + D_m(\tilde{\omega})t_n \\ 0 & 1 \end{pmatrix},$$

where D_i $(i = 1, \ldots, m)$ are the generalized derivatives.

Similar to the Fox derivatives, the generalized derivatives

$$D_i : Z(F/R') \to Z(F/R)$$

satisfy the following rules:

(1) $D_i(\tilde{u} + \tilde{v}) = D_i(\tilde{u}) + D_i(\tilde{v})$ for any $\tilde{u}$, $\tilde{v} \in Z(F/R')$;

(2) $D_i(\tilde{u}\tilde{v}) = \bar{u}D_i(\tilde{v}) + \epsilon(\tilde{v})D_i(\tilde{u})$ for any $\tilde{u}$ and $\tilde{v} \in Z(F/R')$;

(3) $D_i(\tilde{u}) = \bar{u} - 1$ for any $u \in A_i$;

(4) $D_i(\tilde{u}) = 0$ for any $u \in A_j, j \neq i$.

where ϵ is the trivialization map of $Z(F/R') \to Z$.

Generalized derivatives and Fox derivatives have the following relation [3]: for any $u(\tilde{x}_1, \ldots, \tilde{x}_n) \in M_n$ and for all $g_1, \cdots, g_n \in M$,

$$D_i(u(g_1, \cdots, g_n)) = \sum_{j=1}^{n} D_i g_i \partial_j u(g_1, \cdots, g_n),$$

where $\partial_j u(g_1, \ldots, g_n)$ denotes the value of $\partial_j u(x_1, \ldots, x_n)$ on $(g_1, \ldots, g_n)$.

3. Proof of Theorem 1.3

The following lemma can be found in [9], it allows to use equations on group rings to study metabelian groups.

Lemma 3.1. *Let a, b be elements of a group. Consider the equation*

$$(1-a)(1-b)x = 0$$

then

(1) if $o(a) = o(b) = +\infty$, the equation has only the zero solution in $Z(\langle a\rangle \times \langle b\rangle)$;

(2) if $o(a) = m_1$, $o(b) = m_2$, all the solutions of the equation in $Z(\langle a\rangle \times \langle b\rangle)$ are as follows:

$$x = g(b)\sum_{i=1}^{m_1-1} a^i + h(a)\sum_{j=1}^{m_2-1} b^j,$$

where $g(b) \in Z(\langle b\rangle)$, $h(a) \in Z(\langle a\rangle)$;

(3) if $o(a) = +\infty$, $o(b) = m$, all the solutions of the equation in $Z(\langle a\rangle \times \langle b\rangle)$ are as follows:

$$x = f(a)\sum_{i=1}^{m-1} b^i,$$

where $f(a) \in Z(\langle a\rangle)$.

We now prove two technical lemmas which play important roles in the proof of Theorem 1.3, and are of independent interest.

Lemma 3.2. *Let $\omega \in F_n'$. Then ω can be expressed as*

$$\omega = \prod_{i<j}[x_i, x_j]^{\lambda_{ij}}\upsilon,$$

where $\lambda_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_j^{\pm}]$, and $\upsilon \in F_n''$.

Proof. By definition, F_n' is generated by the elements $[x_i, x_j]^{\mu}$, where $1 \le i < j \le n$ and $\mu \in F_n$. Since F_n'/F_n'' is abelian, we can write

$$\omega = \prod_{i<j}[x_i, x_j]^{\mu_{ij}}\omega_1,$$

where $\mu_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_n^{\pm}]$, and $\omega_1 \in F_n''$. Then it suffices to prove that all factors $[x_i, x_j]^{\mu_{ij}}$ can be expressed as in the form of lemma. Equivalently, it suffices to prove that if $\nu_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_j^{\pm}]$ then $[x_i, x_j]^{\nu_{ij}\bar{x}_k}$ and $[x_i, x_j]^{\nu_{ij}\bar{x}_k^{-1}}$ can be expressed as in the form of lemma for each $k > j$.

By Jacobi identity

$$[x_i, x_j]^{1-\bar{x}_k}[x_j, x_k]^{1-\bar{x}_i}[x_k, x_i]^{1-\bar{x}_j} = 1 \ mod \ F_n'',$$

we have

$$\begin{aligned}
[x_i, x_j]^{\nu_{ij}\bar{x}_k} &= [x_i, x_j]^{\nu_{ij}}[x_i, x_j]^{(\bar{x}_k-1)\nu_{ij}} \\
&= [x_i, x_j]^{\nu_{ij}}([x_j, x_k]^{\bar{x}_i-1}[x_k, x_i]^{\bar{x}_j} - 1)^{\nu_{ij}} \ mod \ F_n'' \\
&= [x_i, x_j]^{\nu_{ij}}[x_i, x_k]^{\nu_{ij}(\bar{x}_j-1)}[x_j, x_k]^{\nu_{ij}(1-\bar{x}_i)} \ mod \ F_n''
\end{aligned}$$

and similarly, we have

$$[x_i, x_j]^{\nu_{ij}\bar{x}_k^{-1}} = [x_i, x_j]^{\nu_{ij}}[x_i, x_k]^{\nu_{ij}(1-\bar{x}_j)\bar{x}_k^{-1}}[x_j, x_k]^{\nu_{ij}(\bar{x}_i-1)\bar{x}_k^{-1}} \ mod \ F_n''.$$

So the lemma is true. □

Lemma 3.3. *Let G be a metabelian group and ω as in Lemma 3.2. Then ω is a law of G if and only if*

$$\omega_k = \prod_{i<k}[x_i, x_k]^{\lambda_{ik}}$$

is a law of G for each $k \geq 2$.

Proof. Note that $\omega = \prod_{k=2}^{n}\omega_k\upsilon$, where $\upsilon \in F''$. Since υ is naturally a law of G, if all ω_k are laws of G, then ω is a law of G.

Set $u_k = \prod_{i<j\leq k}[x_i, x_j]^{\lambda_{ij}}$, where $\lambda_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_j^{\pm}]$. For any elements $g_1, \ldots, g_k \in G$, we have

$$u_k(g_1, \ldots, g_k) = \omega(g_1, \ldots, g_k, 1, \ldots, 1) = 1.$$

Thus u_k is a law of G for each $k \geq 2$, and so $\omega_k = u_{k-1}^{-1}u_k$ is a law. □

We can now prove our main theorem.

The inclusion $F_n'' \subseteq L(G)$ is obvious.

Without loss of generality, we may assume that H_1 is a torsion free group and $a_1 \in H_1$ is an element of infinite order. Obviously, $A = F/F' = H_1 \times \cdots \times H_m$.

Suppose $\omega \in L(M)$, that is, ω is a law. By Lemma 2.2, we may write

$$\omega = x_1^{\mu_1} \cdots x_n^{\mu_n} \prod_{i<j}[x_i, x_j]^{\lambda_{ij}} \ mod \ F_n'',$$

where $\mu_i \in Z$ and $\lambda_{ij} \in Z[\bar{x}_1^{\pm}, \ldots, \bar{x}_j^{\pm}]$.

Choose $\tilde{g}_i = \tilde{a}_1$, $\tilde{g}_j = 1$ for $j \neq i$. Then

$$\tilde{a}_1^{\mu_i} = \omega(\tilde{g}_1, \ldots, \tilde{g}_n) = 1,$$

hence $\mu_i = 0$ since $o(\tilde{a}_1) = o(a_1) = +\infty$. By Lemma 3.3, for $2 \leq k \leq n$, the elements

$$\omega_k = [x_1, x_k]^{\lambda_{1k}} \cdots [x_{k-1}, x_k]^{\lambda_{k-1,k}} \tag{2}$$

are laws of G. In the following, we distinguish two cases depending on whether or not there exists some H_i with $i \geq 2$ which is not a torsion group.

Case 1. Assume that some H_i with $i \geq 2$ is not a torsion group.

Without loss of generality, assume that H_2 is not a torsion group, and $a_2 \in H_2$ is an element of infinite order. For any positive integers $s_1, s_2, \ldots, s_k$, by (2), we have

$$\omega_k(\tilde{a}_1^{s_1}, \tilde{a}_2^{s_2}, \ldots, \tilde{a}_2^{s_k}) = [\tilde{a}_1^{s_1}, \tilde{a}_2^{s_k}]^{\lambda_{1k}(\bar{a}_1^{s_1}, \bar{a}_2^{s_2}, \ldots, \bar{a}_2^{s_k})} = 1.$$

Applying the generalized derivative D_1, it follows that

$$-\lambda_{1k}(\bar{a}_1^{s_1}, \bar{a}_2^{s_2}, \ldots, \bar{a}_2^{s_k})(1 - \bar{a}_1^{s_1})(1 - \bar{a}_2^{s_2}) = 0.$$

By Lemma 3.1, we have

$$\lambda_{1k}(\bar{a}_1^{s_1}, \bar{a}_2^{s_2}, \ldots, \bar{a}_2^{s_k}) = 0. \tag{3}$$

On the other hand, suppose $\lambda_{1k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_2) \neq 0$. Let $c\bar{x}_1^{t_1}\bar{x}_2^{t_2} \cdots \bar{x}_k^{t_k}$ be the initial monomial of $\lambda_{1k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_2)$ in the lexicographic order, where $c \neq 0$ is an integer. Then we can choose $s_1, s_2, \ldots, s_k$ sufficiently large (if necessary) such that the degree of $(c(\bar{a}_1^{s_1})^{t_1}(\bar{a}_2^{s_2})^{t_2} \cdots (\bar{a}_2^{s_k})^{t_k})$ is bigger than the degrees of other monomials in $\lambda_{1k}(\bar{a}_1^{s_1}, \bar{a}_2^{s_2}, \ldots, \bar{a}_2^{s_k})$, which contradicts equation (3). So $\lambda_{1k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) = 0$.

Similarly, we have $\lambda_{ik}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) = 0$ for $i = 2, \ldots, k-1$. Hence $\omega \in F_n''$.

Case 2. Assume that $H_2, \ldots, H_n$ are torsion groups.

Choose $a_2 \in H_2$ with $o(a_2) = r > 1$. For any positive integers $s_1, s_2, \cdots, s_k$, by (2), we have

$$\begin{aligned}&[\tilde{a}_1^{s_1}\tilde{a}_2, \tilde{a}_1^{s_k}]^{\lambda_{1k}(\bar{a}_1^{s_1}\bar{a}_2, \bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k})}\\ &= \omega_k(\tilde{a}_1^{s_1}\tilde{a}_2, \tilde{a}_1^{s_2}, \ldots, \tilde{a}_1^{s_k}) = 1.\end{aligned}$$

Applying the generalized derivative D_1, it follows that

$$\lambda_{1k}(\bar{a}_1^{s_1}\bar{a}_2, \bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k})\bar{a}_1^{s_1}(1 - \bar{a}_1^{s_k})(1 - \bar{a}_2) = 0$$

Since $\bar{a}_1^{s_1}$ is invertible in $Z[\bar{a}_1^{\pm}, \bar{a}_2^{\pm}]$, by Lemma 3.1, there exists $f(\bar{a}_1) \in Z[\bar{a}_1^{\pm}]$ such that

$$\lambda_{1k}(\bar{a}_1^{s_1}\bar{a}_2, \bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k}) = f(\bar{a}_1)(1 + \bar{a}_2 + \cdots + \bar{a}_2^{r-1}). \tag{4}$$

On the other hand, if $\lambda_{1k}(\bar{x}_1, \ldots, \bar{x}_k) \neq 0$, write

$$\lambda_{1k}(\bar{x}_1, \ldots, \bar{x}_k) = \bar{x}_1^s f_0(\bar{x}_2, \ldots, \bar{x}_k) + \bar{x}_1^{s-1} f_1(\bar{x}_2, \ldots, \bar{x}_k) + \cdots$$

with $f_0(\bar{x}_2, \ldots, \bar{x}_k) \neq 0$. Since $o(\bar{a}_1) = +\infty$, it is not difficult to show that there exist positive integers $s_2, \cdots, s_k$ such that $f_0(\bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k}) \neq 0$. Then we can choose s_1 sufficiently large (if necessary) such that the degree of $f_0(\bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k})\bar{a}_1^{s_1 s}$ is bigger than the degree of $f_i(\bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k})\bar{a}_1^{\nu_1(s-i)}$ for each $i \geq 1$. Since $o(a_2) = r > 1$, we may rewrite

$$\lambda_{1k}(\bar{a}_1^{s_1}\bar{a}_2, \bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k}) = g_0(\bar{a}_1) + g_1(\bar{a}_1)\bar{a}_2 + \cdots + g_{r-1}(\bar{a}_1)\bar{a}_2^{r-1},$$

we then have that degree of $g_t(\bar{a}_1)$ is bigger than degree of $g_i(\bar{a}_1)$ for each $i \neq t$, where t is the smallest nonnegative residue of s modulo r. So $\lambda_{1k}(\bar{a}_1^{s_1}\bar{a}_2, \bar{a}_1^{s_2}, \ldots, \bar{a}_1^{s_k}) \neq 0$, which contradicts the equation (4). Hence $\lambda_{1k}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k) = 0$.

With the similar discussion as in case 1, we have $\omega \in F_n''$.

Summarizing, we have proved that $L(M) = F_n''$. Then the last statement in Theorem 1.3 is a direct consequence of [1, Theorem 1.3]. □

4. Example and symmetric words

The following example shows that in Theorem 1.3 it is necessary to assume that not all factors H_i are torsion groups.

Example 1.4. Let $\omega := \prod_{i<j}[x_i, x_j]^{(1+\bar{x}_i)(1+\bar{x}_j)}$. To prove ω is a law of G, it suffices to prove that $\omega_{i,j} := [x_i, x_j]^{(1+\bar{x}_i)(1+\bar{x}_j)}$ is a law of G for each $i < j$.

Note that each element of G can be expressed in one of the following four forms:

$$\text{(i) } (\tilde{a}_1\tilde{a}_2)^k; \text{ (ii) } (\tilde{a}_2\tilde{a}_1)^k; \text{ (iii) } (\tilde{a}_1\tilde{a}_2)^k\tilde{a}_1; \text{ (iv) } (\tilde{a}_2\tilde{a}_1)^k\tilde{a}_2.$$

where k is a positive integer. Let $\tilde{g}_1$ and $\tilde{g}_2$ be any elements of G.

If $\tilde{g}_1$ and $\tilde{g}_2$ are elements of the form (i) or (ii), then $\tilde{g}_1$ and $\tilde{g}_2$ commute and then $\omega_{i,j}(\tilde{g}_1, \tilde{g}_2) = 1$. Thus assume that there exists at least one of $\tilde{g}_1$ and $\tilde{g}_2$ in form (iii) or (iv), upon the symmetry of a_1 and a_2, we may suppose that

$$(1 + \bar{g}_1)(1 + \bar{g}_2) \in Z(A/A')(1 + \bar{a}_1).$$

Then

$$D_1(\omega(\tilde{g}_1, \tilde{g}_2)) = (1 + \bar{g}_1)(1 + \bar{g}_2)D_1([\tilde{g}_1, \tilde{g}_2]) \in Z(A/A')(1 + a_1)(a_1 - 1) = \{0\},$$

that is, $D_1(\omega(\tilde{g}_1, \tilde{g}_2)) = 0$. Obviously, $\overline{\omega_{i,j}(\tilde{g}_1, \tilde{g}_2)} = 1$, so $\omega(\tilde{g}_1, \tilde{g}_2) = 1$ by Shmel'kin embedding. Hence ω is a law of the group G.

Further, applying Fox derivative, by direct computation, we have

$$\partial_1\omega = (1+\bar{x}_1)(n - \sum_{i<j}\bar{x}_j^2) \neq 0,$$

by the Magnus embedding, $\omega \notin F_n''$. So ω cannot be expressed as a product of transforms of Jacobi products. □

Finally, we prove Theorem 1.6.

Let ω be a n-symmetric word of M. By definition, we have that

$$\omega(g_1,\ldots,g_n) = \omega(g_{\sigma(1)},\ldots,g_{\sigma(n)}) \tag{5}$$

for all $g_1,\cdots,g_n$ in M and σ in the symmetric group S_n. It is easy to show that the equation (5) is equivalent to that

$$\omega(x_1,\ldots,x_n)(\omega(x_{\sigma(1)},\ldots,x_{\sigma(n)}))^{-1}$$

is a law of M in F_n. This discussion means that the n-symmetric words of a group are perfectly determined by its law group. Now, since $L(M) = L(M_n) = F_n''$, we know that M and M_n have the same n-symmetric words set, and the Theorem 1.6 is true by [7, Theorem 1]. □

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