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A FORMULA FOR REDUCTION NUMBER OF AN IDEAL RELATIVE TO A NOETHERIAN MODULE

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Abstract. Let $(A, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and E be a finitely generated d dimensional Cohen-Macaulay A-module. Let $\mathfrak{b}$ be an ideal of A such that $\mathrm{ht}_E \mathfrak{b} = 0$ and $\lambda(\mathfrak{b}, E) = 1$. Assume that $\mathfrak{b}_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Min}(E/\mathfrak{b}E)$. Let $r(\mathfrak{b}, E) > 0$. We show that if $G_{\mathfrak{b}}(E)$ is Cohen-Macaulay, then $r(\mathfrak{b}, E) = a(G_{\mathfrak{b}}(E)) + 1$.

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1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with infinite residue field $k = A/\mathfrak{m}$ and let E be a d-dimensional finitely generated A-module. Let $\mathfrak{b}$ be an ideal of A. An ideal $\mathfrak{a} \subseteq \mathfrak{b}$ is called a *reduction* of $\mathfrak{b}$ relative to E if $\mathfrak{a}\mathfrak{b}^n E = \mathfrak{b}^{n+1} E$ for some nonnegative integer n, (see [2, Definition 4.6.4]). We denote by $r_{\mathfrak{a}}(\mathfrak{b}, E)$ the least integer with this property. A reduction $\mathfrak{a}$ of $\mathfrak{b}$ relative to E is called a *minimal reduction* if it does not properly contain any other reduction of $\mathfrak{b}$ relative to E. Since k is infinite it is well known that minimal reductions relative to E always exist; see [15, section 4] and [2, Proposition 4.5.8]. In this case we define the reduction number of $\mathfrak{b}$ relative to E by

$$r(\mathfrak{b}, E) = \min\{r_{\mathfrak{a}}(\mathfrak{b}, E) : \mathfrak{a} \text{ is a minimal reduction of } \mathfrak{b} \text{ relative to } E\}.$$

With $E = A$ the correspondence definitions for ideals almost immediately yields; (see [11]). In this case we set $r(\mathfrak{b}) := r(\mathfrak{b}, A)$ and call it the reduction number of $\mathfrak{b}$. In order to state and prove our results we set up a few more notation. We denote by $R_{\mathfrak{b}}(E)$ (resp. by $G_{\mathfrak{b}}(E)$) the Rees module of E associated to $\mathfrak{b}$ (resp. the associated graded module of E with respect to $\mathfrak{b}$), namely:

$$R_{\mathfrak{b}}(E) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n E \quad \text{and} \quad G_{\mathfrak{b}}(E) := \bigoplus_{n=0}^{\infty} \mathfrak{b}^n E/\mathfrak{b}^{n+1} E = R_{\mathfrak{b}}(E)/\mathfrak{b}R_{\mathfrak{b}}(E).$$

In the case $E = A$ we denote it by $R(\mathfrak{b})$ (resp. by $G(\mathfrak{b})$) and call it the Rees algebra (resp. the associated graded ring) of $\mathfrak{b}$ simply. Then both $R_{\mathfrak{b}}(E)$ and $G_{\mathfrak{b}}(E)$ are finitely generated graded $R(\mathfrak{b})$-module. We denote by $\mathbf{m}$ the unique homogeneous maximal ideal of $R(\mathfrak{b})$, i.e., $\mathbf{m} := \mathfrak{m}R(\mathfrak{b}) + R(\mathfrak{b})_+$. Then following [2, Definition 4.5.7], the *analytic spread* of $\mathfrak{b}$ relative to E is defined to be $\lambda(\mathfrak{b}, E) = \dim(R_{\mathfrak{b}}(E)/\mathfrak{m}R_{\mathfrak{b}}(E)) = \dim(G_{\mathfrak{b}}(E)/\mathfrak{m}G_{\mathfrak{b}}(E))$, where $\dim(-)$ denotes Krull dimension. Set also $\lambda(\mathfrak{b}) = \dim(R(\mathfrak{b})/\mathfrak{m}R(\mathfrak{b}))$. We note that in general $\mathrm{ht}_E\mathfrak{b} \leq \lambda(\mathfrak{b}, E) \leq d = \dim E$ and that by [7, (9.7) Theorem] $\dim(G_{\mathfrak{b}}(E)) = \dim E$. The reduction number of an ideal was introduced by Sally [12], where he used explicitly the presence of small reduction number of the maximal ideal $\mathfrak{m}$ in a Cohen-Macaulay local ring in order to study Cohen-Macaulay property of associated graded ring $G(\mathfrak{m})$. For further results and usefulness of this notion see [3,8,15]. A question due to Sally [13], which attained much attention is; when the reduction number of $\mathfrak{b}$ is independent of the choice of minimal reduction? Some partial solutions of this problem were given in [8,9,10]. Most of results are based on the "a-invariant" and the end of some local cohomology modules. So it is suitable to describe them briefly. A nice reference for this material is [5], and the textbook by Brodmann and Sharp [1, Chapters 15, 18]. Let $S = \oplus_{n \geq 0} S_i$ be a Noetherian graded ring with $(S_0, \mathfrak{n}_0)$ a local ring. Let S_+ be the irrelevant ideal of S and $\mathfrak{N} = \mathfrak{n}_0 S + S_+$ denote the maximal homogeneous ideal of S. Let L be a Noetherain graded S-module of dimension s. If $H^i_{\mathfrak{u}}(L)$ denotes the i-th graded local cohomology of L with support in graded ideal $\mathfrak{u}$ of S, then it is well known that the n-th homogeneous component of $H^i_{S_+}(L)$ i.e., $[H^i_{S_+}(L)_n]$ is finitely generated for all $i \geq 0$ and all $n \in \mathbb{Z}$, and it is zero for large values of n. We set

$$a_i(L) = \mathrm{Max}\{n \in \mathbb{Z} : [H^i_{\mathfrak{N}}(L)]_n \neq 0\},$$

and

$$\bar{a}_i(L) = \mathrm{Max}\{n \in \mathbb{Z} : [H^i_{S_+}(L)]_n \neq 0\}.$$

(Convention: If $H^i_{\mathfrak{N}}(L) = 0$ (resp. $H^i_{S_+}(L) = 0$) we set $a_i(L) = -\infty$ (resp. $\bar{a}_i(L) = -\infty$)). Then for convenience $a_s(L)$ is denoted simply as $a(L)$ and called a-invariant of L.

In [8], Hoa by combining Trungs's approaches in [14] and an idea of [6] proved that for large values of n the reduction number of $\mathfrak{b}^n$ is independent of n and any minimal reduction of $\mathfrak{b}^n$ and he computed the asymptotic value of $r(\mathfrak{b}^n)$. More exactly he proved that for $n > \mathrm{Max}\{|\bar{a}_i(G(\mathfrak{b}))| : \bar{a}_i(G(\mathfrak{b})) \neq 0\}$, $r(\mathfrak{b}^n) = \lambda(\mathfrak{b})$ if

$\bar{a}_{\lambda(\mathfrak{b})}(G(\mathfrak{b})) \geq 0$ and otherwise $r(\mathfrak{b}^n) = \lambda(\mathfrak{b}) - 1$. On the other hand in [10], Marley proved that if A is Cohen-Macaulay, $\mathfrak{b}$ is $\mathfrak{m}$-primary ideal and $\operatorname{grade}(G(\mathfrak{b})_+, G(\mathfrak{b})) > 0$, then $r(\mathfrak{b}) = a(G(\mathfrak{b})) + \dim A$ (see also [9, proposition 5.6]).

In this paper using some ideas of [4], under some assumptions on $\mathfrak{b}$ and E, we find a formula for the invariant $r(\mathfrak{b}, E)$. More precisely we prove:

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a local ring and let E be a finitely generated d dimensional Cohen-Macaulay A-module. Let $\mathfrak{b}$ be an ideal of A such that $ht_E\mathfrak{b} = 0$, $\lambda(\mathfrak{b}, E) = 1$, $r(\mathfrak{b}, E) > 0$ and $\mathfrak{b}_\mathfrak{p} = 0$ for all $\mathfrak{p} \in Min(E/\mathfrak{b}E)$. If $G_\mathfrak{b}(E)$ is Cohen-Macaulay, then $r(\mathfrak{b}, E) = a(G_\mathfrak{b}(E)) + 1$.*

2. Proof of Theorem 1.1

We first prove some auxiliary lemmas.

Lemma 2.1. *Suppose that E is Cohen-Macaulay and that $\mathfrak{b}_\mathfrak{p} = 0$ for each $\mathfrak{p} \in Min(E/\mathfrak{b}E)$. Let $b \in \mathfrak{b}$ such that $\sqrt{0 :_A E + (b)} = \sqrt{0 :_A E + \mathfrak{b}}$. Then $(0 :_E b) \cap \mathfrak{b}E = 0$.*

Proof. Let $0_E = Q_1 \cap ... \cap Q_n$ be a minimal primary decomposition of 0_E, with associated primes $\mathfrak{p}_\mathfrak{i} = \sqrt{Q_i :_A E}$ for each $i = 1, ..., n$, enumerated in such a way that $\mathfrak{b} \subseteq \mathfrak{p}_\mathfrak{i}$ for $i = 1, ..., t$ and $\mathfrak{b} \nsubseteq \mathfrak{p}_\mathfrak{i}$ for $i = t+1, ..., n$. Since E is Cohen-Macaulay, we have $\operatorname{ht}_E\mathfrak{p}_\mathfrak{i} = 0$ for $i = 1, ..., n$. Since $\mathfrak{b}_\mathfrak{p} = 0$ for each $\mathfrak{p} \in \operatorname{Min}(E/\mathfrak{b}E)$, we have $\mathfrak{b}E \subseteq Q_1 \cap ... \cap Q_t$. Now suppose $x \in \mathfrak{b}E$ such that $bx = 0_E$. This in particular gives that $bx \in Q_i$ for $i = t+1, ..., n$. Since $\sqrt{0 :_A E + (b)} = \sqrt{0 :_A E + \mathfrak{b}}$ and $\mathfrak{b} \nsubseteq \mathfrak{p}_\mathfrak{i}$, so b is not an element of $\mathfrak{p}_\mathfrak{i}$ for $i = t+1, ..., n$. Therefore $x \in Q_i$ for $i = t+1, ..., n$. So $x \in \mathfrak{b}E \cap Q_{t+1} \cap ... \cap Q_n \subseteq Q_1 \cap ... \cap Q_n = 0_E$ and the claim follows. □

We remind the terminology we are using with respect to $G(\mathfrak{b})$. If $x \in A$ then x^* denotes the initial form of x in $G(\mathfrak{b})$, (i.e., the image of x in $\mathfrak{b}^n/\mathfrak{b}^{n+1}$, where $x \in \mathfrak{b}^n \setminus \mathfrak{b}^{n+1}$) and for each ideal $\mathfrak{u}$ of A, the notation $\mathfrak{u}^*$ denotes the ideal $\mathfrak{u}G(\mathfrak{b})$.

Lemma 2.2. *Let $x \in \mathfrak{m} \setminus \mathfrak{b}$ be such that x^* be a $G_\mathfrak{b}(E)$-regular element in $G(\mathfrak{b})$. Then $r(\mathfrak{b} + (x)/(x), E/xE) = r(\mathfrak{b}, E)$.*

Proof. Let $\mathfrak{a}$ be a minimal reduction of $\mathfrak{b}$ relative to E such that $r = r(\mathfrak{b}, E) = r_\mathfrak{a}(\mathfrak{b}, E)$. Then an easy calculation gives that $(\mathfrak{a} + (x)/(x))(\mathfrak{b} + (x)/(x))^r E/xE = (\mathfrak{b} + (x)/(x))^{r+1} E/xE$ and thus $r(\mathfrak{b} + (x)/(x), E/xE) \leq r(\mathfrak{b}, E)$.

To prove the opposite inequality, suppose that $\mathfrak{c}/(x)$ be a minimal reduction of $\mathfrak{b} + (x)/(x)$ relative to E/xE satisfying $r' = r_{\mathfrak{c}/(x)}(\mathfrak{b} + (x)/(x), E/xE) = r(\mathfrak{b} +$

$(x)/(x), E/xE)$. Then $\mathfrak{c}/(x)(\mathfrak{b}+(x)/(x))^{r'}E/xE = (\mathfrak{b}+(x)/(x))^{r'+1}E/xE$ which gives that $\mathfrak{c}\mathfrak{b}^{r'}E + xE = \mathfrak{b}^{r'+1}E + xE$ so that $\mathfrak{b}^{r'+1}E \subseteq \mathfrak{c}\mathfrak{b}^{r'}E + xE$. Therefore $\mathfrak{b}^{r'+1}E = \mathfrak{b}^{r'+1}E \cap (\mathfrak{c}\mathfrak{b}^{r'}E + xE) = \mathfrak{c}\mathfrak{b}^{r'}E + xE \cap \mathfrak{b}^{r'+1}E$. Since x^* is $G_{\mathfrak{b}}(E)$-regular element, we have $xE \cap \mathfrak{b}^{r'+1}E = x\mathfrak{b}^{r'+1}E$. Hence $\mathfrak{b}^{r'+1}E = \mathfrak{c}\mathfrak{b}^{r'}E + x\mathfrak{b}^{r'+1}E$. Now using Nakayama's lemma we deduce that $\mathfrak{b}^{r'+1}E = \mathfrak{c}\mathfrak{b}^{r'}E$. Consequently $r(\mathfrak{b}, E) \leq r'$ and the proof of the claim is complete. □

Lemma 2.3. *Let $x \in \mathfrak{m} \setminus \mathfrak{b}$ and assume that x^* is $G_{\mathfrak{b}}(E)$-regular. Let $\mathfrak{b}' = \mathfrak{b} + (x)/(x)$. Then $R_{\mathfrak{b}'}(E/xE) = R_{\mathfrak{b}}(E)/xR_{\mathfrak{b}}(E)$.*

Proof. We can write

$$R_{\mathfrak{b}'}(E/xE) = \bigoplus_{n\geq 0} \mathfrak{b}'^n E/xE = \bigoplus_{n\geq 0} (\mathfrak{b}'^n + (x))E/xE \cong \bigoplus_{n\geq 0} \mathfrak{b}'^n E/xE \cap \mathfrak{b}'^n E.$$

Now since x^* is $G_{\mathfrak{b}}(E)$-regular element, the last module is equal to $\oplus_{n\geq 0}\mathfrak{b}'^n E/x\mathfrak{b}'^n E$ which is isomorphic to $R_{\mathfrak{b}}(E)/xR_{\mathfrak{b}}(E)$. □

Lemma 2.4. *Let S, L, $\mathfrak{N}$ and $\mathfrak{n}_0$ be as in section 1 such that L is annihilated by some power of S_+. Then for any $i \geq 0$ and $n \in \mathbb{Z}$ we have an isomorphism $[H^i_{\mathfrak{N}}(L)]_n \cong H^i_{\mathfrak{n}_0}(L_n)$ of S_0-modules.*

Proof. There exists $t \in \mathbb{N}$ such that $S^t_+ L = 0$. Thus L is an S/S^t_+-module and so by [1, 4.2.1], we may assume that $S_k = 0$ for all large values of k. But in this case we have $\mathfrak{N} = \sqrt{\mathfrak{n}_0 S}$, which gives that $H^i_{\mathfrak{N}}(L) \cong H^i_{\mathfrak{n}_0}(L)$ and the result follows by [1, 13.1.10]. □

Here we note that if $\mathfrak{a}$ is a reduction of $\mathfrak{b}$ relative to E, then $\lambda(\mathfrak{b}, E) \leq \mu(\mathfrak{a})$ the number of elements of any minimal generating set for $\mathfrak{a}$ and equality holds if and only if $\mathfrak{a}$ is a minimal reduction of $\mathfrak{b}$ relative to E (see [2, Proposition 4.5.8] and also [15, section 4]). Keeping this in mind we state the following lemma which proves Theorem 1.1 in some special case.

Lemma 2.5. *Let E be a one dimensional Cohen-Macaulay A-module and $\mathfrak{b}$ be an ideal of A such that $ht_E\mathfrak{b} = 0$ and $\lambda(\mathfrak{b}, E) = 1$. Assume that $\mathfrak{b}_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in Min(E/\mathfrak{b}E)$. Let (b) be a minimal reduction of $\mathfrak{b}$ relative to E and $r = r_{(b)}(\mathfrak{b}, E) > 0$. Then $r = a(G_{\mathfrak{b}}(E)) + 1$.*

Proof. We first show that $a = a(G_{\mathfrak{b}}(E)) \leq r - 1$.
Let $N = (0 :_{G_{\mathfrak{b}}(E)} b^*)$ and $\bar{G} = G_{\mathfrak{b}}(E)/b^*G_{\mathfrak{b}}(E)$. The short exact sequences

$$0 \longrightarrow N(-1) \longrightarrow G_{\mathfrak{b}}(E)(-1) \xrightarrow{b^*} b^*G_{\mathfrak{b}}(E) \longrightarrow 0,$$

and

$$0 \longrightarrow b^*G_{\mathfrak{b}}(E) \longrightarrow G_{\mathfrak{b}}(E) \longrightarrow \bar{G} \longrightarrow 0,$$

of graded $G(\mathfrak{b})$-modules induce the exact sequences of graded local cohomology modules, from which we deduce the exact sequences

$$0 \to H^0_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1} \to H^1_{\mathbf{m}}(N)_a \xrightarrow{f} H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_a \xrightarrow{g} H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1} \to 0,$$

and

$$0 \to H^0_{\mathbf{m}}(\bar{G})_{a+1} \to H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1} \xrightarrow{h} H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_{a+1} \xrightarrow{k} H^1_{\mathbf{m}}(\bar{G})_{a+1} \to 0,$$

of local cohomology modules (Note that by [7, (9.7) Theorem] we have $\dim N \leq 1$ and $\dim(b^*G_{\mathfrak{b}}(E)) \leq 1$).

We consider the following two cases:

(i) If $g : H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_a \to H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1}$ is the zero map, then $H^1_{\mathbf{m}}(N)_a \neq 0$ and in particular $N_a \neq 0$ by Lemma 2.4. We claim that $a = 0$ or else $a < r - 1$. Suppose the contrary $a > 0$ and $a \geq r - 1$ (note that $r > 0$). Let $0 \neq x^* \in N_a$. Then $x \in \mathfrak{b}^a E \setminus \mathfrak{b}^{a+1}E$ and $b^*x^* = 0$. This means that $bx \in \mathfrak{b}^{a+2}E = (b)\mathfrak{b}^{a+1}E$ (note that $a + 1 \geq r$). Thus there exists $y \in \mathfrak{b}^{a+1}E$ such that $bx = by$. This gives that $x - y \in (0 :_E b) \cap \mathfrak{b}E$ and so in view of Lemma 2.1, we have $x = y \in \mathfrak{b}^{a+1}E$, which is a contradiction. So the claim is true and we have $a \leq r - 1$ in this case.

(ii) If $g : H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_a \to H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1}$ is not the zero map. Then there exists $x \in H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_a$ such that $0 \neq g(x) \in H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_{a+1}$ and $h(g(x)) \in H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_{a+1} = 0$ by the definition of a. Therefore by the second exact sequence $0 \neq g(x) \in H^0_{\mathbf{m}}(\bar{G})_{a+1}$. This means that $H^0_{\mathbf{m}}(\bar{G})_{a+1} \neq 0$ and so $\bar{G}_{a+1} \neq 0$. From this it follows that $(b)\mathfrak{b}^a E \neq \mathfrak{b}^{a+1}E$ and so $a < r$ by the definition of r. Thus $a \leq r - 1$ and the claim is also true in this case.

Now we show that $a \geq r - 1$. It follows from the first exact sequence that $H^1_{\mathbf{m}}(b^*G_{\mathfrak{b}}(E))_n = 0$ for all $n \geq a + 2$. Hence by the second exact sequence we have $H^0_{\mathbf{m}}(\bar{G})_n = 0$ for all $n \geq a + 1$. Also by the second exact sequence we deduce that $H^1_{\mathbf{m}}(\bar{G})_n = 0$ for all $n \geq a + 1$. Therefore by Lemma 2.4 we have $H^0_{\mathfrak{m}}(\bar{G}_{a+2}) = 0$ and $H^1_{\mathfrak{m}}(\bar{G}_{a+2}) = 0$. From this it follows that If $\bar{G}_{a+2} \neq 0$ then $\operatorname{grade}(\mathfrak{m}, \bar{G}_{a+2}) > 1 = \dim E$, which is impossible. So $\bar{G}_{a+2} = 0$. Thus we must have $(b)\mathfrak{b}^{a+1}E + \mathfrak{b}^{a+3}E = \mathfrak{b}^{a+2}E$. It follows by the Nakayama's lemma that $(b)\mathfrak{b}^{a+1}E = \mathfrak{b}^{a+2}E$. Therefore $a + 1 \geq r$ if $a+1 > 0$. But $H^1_{\mathbf{m}}(\bar{G})_0 = H^1_{\mathfrak{m}}(\bar{G}_0) = H^1_{\mathfrak{m}}(E/\mathfrak{b}E) \neq 0$ (note that since E is Cohen-Macaulay and $\mathrm{ht}_E \mathfrak{b} = 0$, we have $\dim E/\mathfrak{b}E = 1$). Thus by the second exact sequence we have $H^1_{\mathbf{m}}(G_{\mathfrak{b}}(E))_0 \neq 0$. Hence $a \geq 0$ and $a+1 > 0$. The proof now is completed. □

Remark 2.6. *Let* $\mathfrak{b}_{\mathfrak{p}} = 0$ *for all* $\mathfrak{p} \in Min(E/\mathfrak{b}E)$, *then the set*

$$\mathcal{P} = \{\mathfrak{p} \in Supp(E) : \mathfrak{b} \subseteq \mathfrak{p}, \mathfrak{b}_{\mathfrak{p}} = 0 \text{ and } ht_E\mathfrak{p} = 1\},$$

as a minimal elements of a Zariski-closed set is a finite set.

Proof of Theorem 1.1. We proceed by induction on $d = \dim E \geq 1$. The case $d = 1$ was settled in previous lemma. So let $d \geq 2$. Since $G_{\mathfrak{b}}(E)$ is Cohen-Macaulay and $\lambda(\mathfrak{b}, E) = 1$, so we have $\text{grade}(\mathfrak{m}^*, G_{\mathfrak{b}}(E)) = \text{ht}_{G_{\mathfrak{b}}(E)}\mathfrak{m}^* - \dim G_{\mathfrak{b}}(E)/\mathfrak{m}G_{\mathfrak{b}}(E) = d - 1 \geq 1$. Since k is infinite, it follows from this that there exists a $G_{\mathfrak{b}}(E)$-regular element, of degree zero, say x^*, in $G(\mathfrak{b})$ (that is in fact in $\mathfrak{m}/\mathfrak{b}$). With the same assumption as in Remark 2.6 we have $\mathfrak{m} \nsubseteq \cup_{\mathfrak{p}\in\mathcal{P}}\mathfrak{p}$. Hence we may select x^* in such a way that $x \in \mathfrak{m} \setminus \cup_{\mathfrak{p}\in\mathcal{P}}\mathfrak{p}$. Then it follows that $\text{ht}_E\mathfrak{b} = \text{ht}_{E/xE}(\mathfrak{b} + (x)/(x))$, $\mathfrak{b}_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \text{Min}(E/(\mathfrak{b} + (x))E)$ and $\dim(E/(\mathfrak{b} + (x))E) < \dim E = d$. We note that E/xE and $G_{\mathfrak{b}+(x)/(x)}(E/xE) \cong G_{\mathfrak{b}}(E)/x^*G_{\mathfrak{b}}(E)$ are Cohen-Macaulay and that by applying the local cohomology functors $H^i_{\mathbf{m}}(-)$ to the exact sequence

$$0 \longrightarrow G_{\mathfrak{b}}(E) \xrightarrow{x^*} G_{\mathfrak{b}}(E) \longrightarrow G_{\mathfrak{b}}(E)/x^*G_{\mathfrak{b}}(E) \longrightarrow 0,$$

and using the fact that x^* is of degree zero, it is easy to see that $a(G_{\mathfrak{b}}(E)/x^*G_{\mathfrak{b}}(E)) = a(G_{\mathfrak{b}}(E))$. Now using Lemma 2.2, we have $r(\mathfrak{b} + (x)/(x), E/xE) = r(\mathfrak{b}, E)$. Also by Lemma 2.3 we have $R_{\mathfrak{b}+(x)/(x)}(E/xE) \cong R_{\mathfrak{b}}(E)/xR_{\mathfrak{b}}(E)$. Therefore

$$\lambda(\mathfrak{b} + (x)/(x), E/xE) = \dim(R_{\mathfrak{b}+(x)/(x)}(E/xE)/\mathfrak{m}/(x)R_{\mathfrak{b}+(x)/(x)}(E/xE))$$

$$= \dim(R_{\mathfrak{b}}(E)/\mathfrak{m}R_{\mathfrak{b}}(E)) = \lambda(\mathfrak{b}, E) = 1.$$

So we can reduce to the case $d = 1$ and the proof of the Theorem follows by Lemma 2.5.

We proved Theorem 1.1, with the assumption that $\text{ht}_E(\mathfrak{b}) = 0$. For ideals $\mathfrak{b}$ of arbitrary $\text{ht}_E(\mathfrak{b})$ we could not prove the same result. Although in the ring version it has been proved in [4] for ideals of arbitrary hight. So the following question arises.

Question. Does Theorem 1.1 hold true for ideals $\mathfrak{b}$ of arbitrary $\text{ht}_E(\mathfrak{b})$?

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