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# LAX GROUP CORINGS

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Abstract. As a generalization of the notion of a group coring in the sense of Caenepeel et al. [7], we introduce the notion of a lax group coring. Firstly, we provide a large class of examples of such a lax group coring by considering so-called lax group entwining structures and partial group entwining structures. Secondly, over a lax group entwining structure one can consider two different categories of modules $\mathcal{M}(\psi)_A^C$ and $\mathcal{M}(\psi)_A^{\pi-C}$, which are in fact nothing else than categories of (group) comodules over the group coring one can associate to each lax group entwining structure. Finally, we study the category ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}^\pi$ of $\pi$-graded modules over the $\pi$-graded $A$-ring $\underline{A^{op}\sharp C^*}$, and show that it is isomorphic to the category $\mathcal{M}(\psi)_A^{\pi-C}$. Moreover, if $\pi$ is a finite group, then we have an equivalence of categories between ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}$ and $\mathcal{M}(\psi)_A^C$.

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## Introduction

On one hand, recently, the concept of a group coring was introduced by Caenepeel et al. [7] as a powerful tool to study group coalgebras and Hopf group coalgebras. Then it is natural to ask how to construct new examples of such group corings. This is one of the motivations of writing this paper. On the other hand, we can learn a lot of structural properties from [2] and [16] about corings and weak corings. In particular, in [6] the authors introduced the notion of a lax coring, inspired by the theory of partial actions of discrete groups on $C^*$-algebras [10] (a further investigation of these partial actions from a purely algebraic point of view was carried out in [8, 9, 11]). It is also natural to study how to unify these notions above to one, and then to study and apply its structural properties to the theory of Hopf group coalgebras. This is another motivation for our paper.

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In fact, let $\pi$ be a group (written multiplicatively and with unit 1) and $A$ a (non-commutative) associative ring with unit $1_A$. We introduce the notion of a left unital lax $\pi$-$A$-coring, i.e. a triple $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon)$ consisting of a family $\mathcal{C} = \{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ of left unital $A$-bimodules, a family $\Delta = \{\Delta_{\alpha,\beta} : \mathcal{C}_{\alpha\beta} \to \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta\}_{\alpha,\beta\in\pi}$ of $A$-bimodule maps and an $A$-bimodule map $\varepsilon : \mathcal{C}_1 \to A$, such that

$$(\Delta_{\alpha,\beta} \otimes_A \mathcal{C}_\gamma)\Delta_{\alpha\beta,\gamma} = (\mathcal{C}_\alpha \otimes_A \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$$

and

$$c = \varepsilon(c_{(1,1)})c_{(2,\alpha)} = c_{(1,\alpha)}\varepsilon(c_{(2,1)}),$$

for all $\alpha, \beta, \gamma \in \pi$ and $c \in \underline{\mathcal{C}}_\alpha$. Here we used the notation $\underline{\mathcal{C}}_\alpha = \mathcal{C}_\alpha 1_A, \Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)}$. The definition is designed in such a way that $\underline{\mathcal{C}} = \{\mathcal{C}_\alpha 1_A\}_{\alpha\in\pi}$ can be given the structure of an ordinary $\pi$-$A$-coring in the sense of [7].

If we take the group $\pi = \{1\}$ to be trivial, we get that $\mathcal{C}_1 = (\mathcal{C}_1, \Delta_{1,1}, \varepsilon)$ is a left unital lax $A$-coring as introduced in [6]. If all the $A$-bimodules $\mathcal{C}_\alpha$ are also right unital, then we recover the notion of $\pi$-$A$-coring in the sense of [7]. If $A = k$ is a commutative ring or a field, and the left and the right $k$-actions on $\mathcal{C}_\alpha$ coincide, for all $\alpha \in \pi$, then we recover the notion of a $\pi$-coalgebra over $k$, as in [13].

This paper is organized as follows.

In Section 1, we recall some basic definitions and results about group corings and Hopf $\pi$-coalgebras that we will need later. In Section 2, we study some relations between lax group corings and their dual notions, a class of lax group rings (cf. Proposition 2.6). In Section 3, we provide a class of examples by introducing lax group entwining structures and show that for a lax group entwining structure $(A, C, \psi)$, we have two different isomorphisms of categories: $\mathcal{M}^{\pi-\underline{\mathcal{C}}} \cong \mathcal{M}(\psi)_A^{\pi-C}$ and $\mathcal{M}^{\underline{\mathcal{C}}} \cong \mathcal{M}(\psi)_A^C$ (cf. Proposition 3.5). In the final section we investigate the notion of a lax and partial group smash product structure. In particular, we prove that the category ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}^\pi$ of $\pi$-graded modules over $\underline{A^{op}\sharp C^*}$ which is a direct summand of the lax group smash product $A^{op}\sharp C^*$ is isomorphic to the category $\mathcal{M}(\psi)_A^{\pi-C}$. Furthermore, if $\pi$ is a finite group, then we have an equivalence of categories between ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}$ and $\mathcal{M}(\psi)_A^C$ (cf. Proposition 4.7).

## 1. Preliminaries

Throughout this paper we will adopt the following notational conventions. Let $\pi$ be a group with unit 1. For an object $M$ in a category, $M$ will also denote the identity morphism on $M$. $A$ will be an associative ring with unit $1_A$, and $k$ will be a commutative ring. If a tensor product is written without index, then it is assumed to be taken over $k$, that is, $\otimes = \otimes_k$. For a right (not necessarily unital) $A$-module $N$, we will denote $\underline{N} = N1_A$.

Recall from [13] that a $\pi$-coalgebra is a family of $k$-modules $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of $k$-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \longrightarrow C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$ and a $k$-linear map $\varepsilon : C_1 \longrightarrow k$, such that $\Delta$ is coassociative in the sense that

- $(\Delta_{\alpha,\beta} \otimes C_\gamma)\Delta_{\alpha\beta,\gamma} = (C_\alpha \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},$ for any $\alpha, \beta, \gamma \in \pi$, and
- $(C_\alpha \otimes \varepsilon)\Delta_{\alpha,\,1} = C_\alpha = (\varepsilon \otimes C_\alpha)\Delta_{1,\,\alpha},$ for all $\alpha \in \pi$.

We use the Sweedler's notation (see [13]) for a comultiplication in the following way: for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write $\Delta_{\alpha,\beta}(c) = c_{(1,\,\alpha)} \otimes c_{(2,\beta)}$.

Let $C = \{C_\alpha\}_{\alpha\in\pi}$ be a $\pi$-coalgebra. Set $C^*_\alpha = \mathrm{Hom}(C_\alpha, k)$, for every $\alpha \in \pi$. Take $f \in \mathrm{Hom}(C_\alpha, k), g \in \mathrm{Hom}(C_\beta, k)$, we define the convolution product as follows:

$$f * g = m(f \otimes g)\Delta_{\alpha,\,\beta} \in \mathrm{Hom}_k(C_{\alpha\beta}, A),\ (f * g)(c) = f(c_{(1,\,\alpha)})g(c_{(2,\,\beta)})$$

for any $c \in C_{\alpha\beta}$.

Furthermore, if every $C_\alpha$ with $\alpha \in \pi$ is finitely generated projective, then we call $C = \{C_\alpha\}_{\alpha\in\pi}$ a $\pi$-coalgebra of finite type.

One verifies that the $k$-module $C^* = \bigoplus_{\alpha\in\pi} C^*_\alpha$ endowed with the convolution product $*$ and the unit element $\varepsilon$, is a $\pi$-graded algebra, called convolution algebra.

We now recall from [13, 15] that a semi-Hopf $\pi$-coalgebra is a $\pi$-coalgebra $H = (\{H_\alpha\}, \Delta, \varepsilon)$ such that

(1) each $H_\alpha$ is an algebra with multiplication $m_\alpha$ and unit element $1_\alpha \in H_\alpha$,

(2) $\varepsilon : H_1 \to k$ and $\Delta_{\alpha,\,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra homomorphisms, for all $\alpha,\ \beta \in \pi$.

Recall from [7] that a $\pi$-$A$-coring $\underline{\mathcal{C}}$ is a family $\underline{\mathcal{C}} = \{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ of $A$-bimodules together with $A$-bimodule maps

$$\Delta = \{\Delta_{\alpha,\,\beta} : \mathcal{C}_{\alpha\beta} \longrightarrow \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta\}_{\alpha,\,\beta\in\pi} \text{ and } \varepsilon : \mathcal{C}_1 \longrightarrow A$$

satisfying the coassociativity in the sense that, for any $\alpha,\ \beta,\ \gamma \in \pi$,

$$(\Delta_{\alpha,\,\beta} \otimes_A \mathcal{C}_\gamma)\Delta_{\alpha\beta,\,\gamma} = (\mathcal{C}_\alpha \otimes_A \Delta_{\beta,\,\gamma})\Delta_{\alpha,\,\beta\gamma}, \tag{1}$$

and the counit properties in the sense that, for all $\alpha \in \pi$,

$$(\mathcal{C}_\alpha \otimes_A \varepsilon)\Delta_{\alpha,\,1} = \mathcal{C}_\alpha = (\varepsilon \otimes_A \mathcal{C}_\alpha)\Delta_{1,\,\alpha}.$$

Let $\underline{\mathcal{C}} = \{\mathcal{C}_\alpha, \Delta, \varepsilon\}_{\alpha,\,\beta\in\pi}$ be a $\pi$-$A$-coring. Recall from [7] that we can define two different types of comodules over $\underline{\mathcal{C}}$. A right $\underline{\mathcal{C}}$-comodule is a right $A$-module $M$

together with a family of right $A$-linear maps $(\rho_\alpha^M)_{\alpha\in\pi}$, $\rho_\alpha^M : M \longrightarrow M\otimes_A \mathcal{C}_\alpha$, such that

$$(\rho_\alpha^M \otimes_A \mathcal{C}_\beta)\circ\rho_\beta^M = (M\otimes_A \Delta_{\alpha,\,\beta})\circ\rho_{\alpha\beta}^M, \tag{2}$$

and

$$(M\otimes_A \varepsilon)\circ\rho_1^M = M. \tag{3}$$

We will use the following Sweedler-type notation:

$$\rho_\alpha^M(m) = m_{[0]}\otimes_A m_{[1,\,\alpha]}.$$

In this case, Eq.(2) justifies the notation

$$\begin{aligned}(\rho_\alpha^M \otimes_A \mathcal{C}_\beta)\circ\rho_\beta^M(m) &= (M\otimes_A \Delta_{\alpha,\,\beta})\circ\rho_{\alpha\beta}^M(m)\\ &= m_{[0]}\otimes_A m_{[1,\,\alpha]}\otimes_A m_{[2,\,\beta]},\end{aligned}$$

and Eq.(3) is equivalent to $m_{[0]}\varepsilon(m_{[1,1]}) = m$, for all $m\in M$.

A morphism of right $\underline{\mathcal{C}}$-comodules is a right $A$-linear map $f : M \longrightarrow N$ satisfying the condition

$$\rho_\alpha^N\circ f = (f\otimes_A \mathcal{C}_\alpha)\circ\rho_\alpha^M$$

for all $\alpha\in\pi$. $\mathcal{M}^{\underline{\mathcal{C}}}$ will be our notation for the category of right $\underline{\mathcal{C}}$-comodules.

A right $\pi$-$\underline{\mathcal{C}}$-comodule $\underline{M}$ is a family of right $A$-modules $(M_\alpha)_{\alpha\in\pi}$, together with a family of right $A$-linear maps

$$\rho_{\alpha,\,\beta} = \rho_{\alpha,\,\beta}^M : M_{\alpha\beta}\longrightarrow M_\alpha\otimes_A \mathcal{C}_\beta$$

such that

$$(M_\alpha\otimes_A \Delta_{\beta,\,\gamma})\circ\rho_{\alpha,\,\beta\gamma} = (\rho_{\alpha,\,\beta}\otimes_A \mathcal{C}_\gamma)\circ\rho_{\alpha\beta,\,\gamma}, \tag{4}$$

and

$$(M_\alpha\otimes_A \varepsilon)\circ\rho_{\alpha,\,1} = M_\alpha, \tag{5}$$

for all $\alpha,\ \beta,\ \gamma\in\pi$. We now use the following Sweedler-type notation:

$$\rho_{\alpha,\,\beta}(m) = m_{[0,\,\alpha]}\otimes_A m_{[1,\,\beta]},$$

for $m\in M_{\alpha\beta}$. Then Eq.(4) justifies the notation

$$\begin{aligned}(M_\alpha\otimes_A \Delta_{\beta,\,\gamma})\circ\rho_{\alpha,\,\beta\gamma}(m) &= (\rho_{\alpha,\,\beta}\otimes_A \mathcal{C}_\gamma)\circ\rho_{\alpha\beta,\,\gamma}(m)\\ &= m_{[0,\,\alpha]}\otimes_A m_{[1,\,\beta]}\otimes_A m_{[2,\,\gamma]},\end{aligned}$$

for all $m\in M_{\alpha\beta\gamma}$, and Eq.(5) implies that $m_{[0,\,\alpha]}\varepsilon(m_{[1,\,1]}) = m$, for all $m\in M_\alpha$. A morphism between two right $\pi$-$\underline{\mathcal{C}}$-comodules $\underline{M}$ and $\underline{N}$ is a family of right $A$-linear maps $f_\alpha : M_\alpha\longrightarrow N_\alpha$ such that

$$(f_\alpha\otimes_A \mathcal{C}_\beta)\circ\rho_{\alpha,\,\beta}^M = \rho_{\alpha,\,\beta}^N\circ f_{\alpha\beta}.$$

The category of right $\pi$-$\underline{\mathcal{C}}$-comodules will be denoted by $\mathcal{M}^{\pi-\underline{\mathcal{C}}}$.

## 2. Lax Group Corings And Lax Group Rings

In this section we will introduce the notion of a lax group coring and investigate its structural properties.

We take a family of left unital $A$-bimodules $\mathcal{C} = \{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ and $(A, A)$-bimodule maps $\Delta = \{\Delta_{\alpha,\beta} : \mathcal{C}_{\alpha\beta} \longrightarrow \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta\}$ and $\varepsilon : \mathcal{C}_1 \longrightarrow A$ satisfying (1).

We consider a family of projections $p = \{p_\alpha : \mathcal{C}_\alpha \to \underline{\mathcal{C}}_\alpha = \mathcal{C}_\alpha 1_A\}_{\alpha\in\pi}$. Obviously the inclusion map $\iota_\alpha$ is a right inverse of $p_\alpha$. For every $c \in \mathcal{C}_{\alpha\beta}$, we have that

$$\Delta_{\alpha,\beta}(c1_A) = \Delta_{\alpha,\beta}(c)1_A = c_{(1,\,\alpha)} \otimes_A 1_A c_{(2,\,\beta)}1_A = c_{(1,\,\alpha)}1_A \otimes_A c_{(2,\,\beta)}1_A \in \underline{\mathcal{C}}_\alpha \otimes_A \underline{\mathcal{C}}_\beta$$

So $\Delta_{\alpha,\,\beta}$ can be restricted to a map $\underline{\Delta}_{\alpha,\,\beta} : \underline{\mathcal{C}}_{\alpha\beta} \to \underline{\mathcal{C}}_\alpha \otimes_A \underline{\mathcal{C}}_\beta$. $\varepsilon \circ \iota_1$ is then the restriction of $\varepsilon$ to $\underline{\mathcal{C}}_1$.

**Definition 2.1.** We call $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon)$ a left unital lax $\pi$-$A$-coring if $\underline{\mathcal{C}} = (\underline{\mathcal{C}} = \{\underline{\mathcal{C}}_\alpha\}_{\alpha\in\pi}, \underline{\Delta} = \{\underline{\Delta}_{\alpha,\,\beta}\}_{\alpha,\beta\in\pi}, \varepsilon\circ\iota_1)$ is a $\pi$-$A$-coring. This is equivalent to the equality

$$c = \varepsilon(c_{(1,1)})c_{(2,\alpha)} = c_{(1,\alpha)}\varepsilon(c_{(2,1)}), \tag{6}$$

for all $c \in \underline{\mathcal{C}}_\alpha$ and $\alpha \in \pi$.

**Remark 2.2.** (1) If all $A$-bimodules $\mathcal{C}_\alpha$ are also right unital, then $\mathcal{C}$ becomes a $\pi$-$A$-coring in the sense of [7]. In particular, when $\pi$ is trivial, $\mathcal{C} = \mathcal{C}_1$ is an $A$-coring.

(2) If $\pi = \{1\}$ is trivial, then $\mathcal{C}$ is a left unital lax $A$-coring (see Section 1 of [6]).

(3) If $A = k$ is a commutative ring or a field, and the left and right $k$-actions on $\mathcal{C}_\alpha$ coincide, for all $\alpha \in \pi$, then we recover the notion of a $\pi$-coalgebra over $k$, as in [13].

Recall that a $\pi$-$A$-ring $\mathcal{R} = (\mathcal{R}, \mu, \eta)$ is a triple consisting of a family $\mathcal{R} = \{\mathcal{R}_\alpha\}_{\alpha\in\pi}$ of $A$-bimodules, a family $A$-bimodule maps $\mu = \{\mu_{\alpha,\,\beta} : \mathcal{R}_\alpha \otimes \mathcal{R}_\beta \longrightarrow \mathcal{R}_{\alpha\beta}\}_{\alpha,\,\beta\in\pi}$ and an $A$-bimodule map $\eta : A \to \mathcal{R}_1$, such that

$$\mu_{\alpha\beta,\,\gamma} \circ (\mu_{\alpha,\,\beta} \otimes_A \mathcal{R}_\gamma) = \mu_{\alpha,\,\beta\gamma} \circ (\mathcal{R}_\alpha \otimes_A \mu_{\beta,\,\gamma}), \tag{7}$$

and

$$\mu_{1,\,\alpha} \circ (\eta \otimes_A \mathcal{R}_\alpha) = \mu_{\alpha,\,1} \circ (\mathcal{R}_\alpha \otimes_A \eta) = \mathcal{R}_\alpha, \tag{8}$$

for all $\alpha$, $\beta$, $\gamma \in \pi$. A morphism between two $\pi$-$A$-rings $\mathcal{R} = (\mathcal{R}, \mu, \eta)$ and $\mathcal{R}' = (\mathcal{R}', \mu', \eta')$ is a family $\varphi = \{\varphi_\alpha : \mathcal{R}_\alpha \to \mathcal{R}'_\alpha\}_{\alpha\in\pi}$ of $A$-bimodule maps such that $\varphi_{\alpha\beta} \circ \mu_{\alpha,\,\beta} = \mu'_{\alpha,\,\beta} \circ (\varphi_\alpha \otimes_A \varphi_\beta)$, and $\varphi_1 \circ \eta = \eta'$. The category of $\pi$-$A$-rings and morphisms between them form a category, which we will denote by $\pi$-$A$-**Ring**.

Over a $\pi$-$A$-ring $\mathcal{R} = (\mathcal{R}, \mu, \eta)$, we can define two different types of modules and we denote $1_{\mathcal{R}} = \eta(1_A)$. A left $\mathcal{R}$-module is a left $A$-module $M$ together with a family of left $A$-linear maps $\rightharpoonup = \{\rightharpoonup_\alpha : \mathcal{R}_\alpha \otimes_A M \longrightarrow M\}_{\alpha\in\pi}$ satisfying

$$s \rightharpoonup_\beta (r \rightharpoonup_\alpha m) = sr \rightharpoonup_{\beta\alpha} m, \text{ for all } m \in M, r \in \mathcal{R}_\alpha \text{ and } s \in \mathcal{R}_\beta;$$

and

$$1_{\mathcal{R}} \rightharpoonup_1 m = m.$$

A morphism of two left $\mathcal{R}$-modules $M$ and $N$ is a left $A$-linear map $f : M \longrightarrow N$ satisfying the condition

$$f(r \rightharpoonup_\alpha m) = r \rightharpoonup_\alpha f(m),$$

for all $m \in M$, and $r \in \mathcal{R}_\alpha$. We will use ${}_{\mathcal{R}}\mathcal{M}$ to denote the category of left $\mathcal{R}$-modules.

A left $\pi$-$\mathcal{R}$-module $\underline{M}$ is a family of left $A$-modules $(M_\alpha)_{\alpha\in G}$ together with a family of left $A$-linear maps $\rightharpoonup = \{\rightharpoonup_{\alpha,\beta} : \mathcal{R}_\alpha \otimes_A M_\beta \longrightarrow M_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ such that

$$s \rightharpoonup_{\gamma,\alpha\beta} (r \rightharpoonup_{\alpha,\beta} m) = sr \rightharpoonup_{\gamma\alpha,\beta} m, \text{ for all } m \in M_\beta, r \in \mathcal{R}_\alpha \text{ and } s \in \mathcal{R}_\gamma;$$

and

$$1_{\mathcal{R}} \rightharpoonup_{1,\alpha} m = m, \text{ for all } m \in M_\alpha.$$

A morphism between two left $\pi$-$\mathcal{R}$-modules $\underline{M}$ and $\underline{N}$ is a family of left $A$-linear maps $\{f_\alpha : M_\alpha \longrightarrow N_\alpha\}_{\alpha\in\pi}$ such that

$$f_{\alpha\beta}(r \rightharpoonup_{\alpha,\beta} m) = r \rightharpoonup_{\alpha,\beta} f_\beta(m),$$

for all $r \in \mathcal{R}_\alpha, m \in M_\beta$. The category of left $\pi$-$\mathcal{R}$-modules will be denoted by ${}_{\mathcal{R}}\mathcal{M}^\pi$.

**Remark 2.3.** It is well-known (and easy to see) that there is a categorical one-to-one correspondence between $\pi$-$A$-rings and $\pi$-graded $A$-rings. By the latter we mean $\pi$-graded rings $\mathcal{R}$ together with a ring morphism $\iota : A \to \mathcal{R}_1$. Under this correspondence ${}_{\mathcal{R}}\mathcal{M}$ and ${}_{\mathcal{R}}\mathcal{M}^\pi$ are nothing else than the category of (usual) left modules respectively the category of $\pi$-graded left modules over the $\pi$-graded $A$-ring $\mathcal{R}$.

**Definition 2.4.** A left unital lax $\pi$-$A$-ring $\mathcal{R}$ is a triple $\mathcal{R} = (\mathcal{R}, \mu, \eta)$ consisting of a family $\mathcal{R} = \{\mathcal{R}_\alpha\}_{\alpha\in\pi}$ of left unital $A$-bimodules, a family $\mu = \{\mu_{\alpha,\beta} : \mathcal{R}_\alpha \otimes_A \mathcal{R}_\beta \longrightarrow \mathcal{R}_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of $A$-bimodule maps and an $A$-bimodule map $\eta : A \to \mathcal{R}_1$, such that (7) holds and

$$r = 1_{\mathcal{R}} r = r 1_{\mathcal{R}}, \tag{9}$$

for all $r \in \underline{\mathcal{R}}_\alpha$ and $\alpha \in \pi$. Here we used the notation $\underline{\mathcal{R}}_\alpha = \mathcal{R}_\alpha 1_A$ and $1_{\mathcal{R}} = \eta(1_A)$. So $\mathcal{R}$ is a left unital lax $\pi$-$A$-ring if and only if $\underline{\mathcal{R}} = (\{\underline{\mathcal{R}}_\alpha\}_{\alpha\in\pi}, \{\underline{\mu}_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, q_1 \circ \eta)$ is a $\pi$-$A$-ring, or a $\pi$-graded ring, in view of the preceding Remark 2.3. Here $\underline{\mu}_{\alpha,\beta}$ is the restriction of $\mu_{\alpha,\beta}$ to $\underline{\mathcal{R}}_\alpha \otimes_A \underline{\mathcal{R}}_\beta$, and $q_1 : \mathcal{R}_1 \to \underline{\mathcal{R}}_1$ the canonical projection.

Let $\mathcal{R} = (\{\mathcal{R}_\alpha\}_{\alpha\in\pi}, \{\mu_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, \eta)$ be a left (resp. right) unital lax $\pi$-$A$-ring. Write $\mu_{\alpha,\beta}(r \otimes_A s) = rs$, for each $r \in \mathcal{R}_\alpha$ and $s \in \mathcal{R}_\beta$. Then it is easy to check that $\mathcal{R}^{op} = (\{\mathcal{R}_{\alpha^{-1}}\}_{\alpha\in\pi}, \{\tilde{\mu}_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, \eta)$ is a right (resp. left) unital lax $\pi$-$A^{op}$-ring, where $\tilde{\mu}_{\alpha,\beta}(r \otimes_A s) = sr \in \mathcal{R}_{(\alpha\beta)^{-1}}$, for each $r \in \mathcal{R}_{\alpha^{-1}}$ and $s \in \mathcal{R}_{\beta^{-1}}$. Also it is straightforward to see that $(\underline{\mathcal{R}})^{op} \cong \underline{\mathcal{R}^{op}}$ as $\pi$-$A^{op}$-rings.

Similarly, we can define right unital lax $\pi$-$A$-rings .

**Remark 2.5.** (1) If all $A$-bimodules $\mathcal{R}_\alpha$ are also right unital, then $\mathcal{R}$ is a $\pi$-$A$-ring. In particular, when $\pi$ is trivial, then $\mathcal{R} = \mathcal{R}_1$ is an $A$-ring.

(2) If $\pi$ is trivial and $\mathcal{R} = \mathcal{R}_1$ satisfies the weak unitary property in the sense that $r1_A = 1_{\mathcal{R}}r = r1_{\mathcal{R}}$ for all $r \in \mathcal{R}$, then we call $\mathcal{R}$ a left unital weak $A$-ring (see Section 1 in [6]).

(3) If $\pi$ is trivial, then $\mathcal{R} = \mathcal{R}_1$ is a left unital lax $A$-ring (see Section 1 in [6]).

In what follows, we will study some duality properties of lax group corings. Let $\mathcal{C} = \{\mathcal{C}_\alpha\}_{\alpha\in\pi}$ be a family of left unital $A$-bimodules, then ${}^*\mathcal{C} = \mathcal{R} = (\{\mathcal{R}_\alpha\}_{\alpha\in\pi})$ is a family of right unital $A$-bimodules, where $\mathcal{R}_\alpha = {}_A\mathrm{Hom}(\mathcal{C}_{\alpha^{-1}}, A)$ with $A$-action,

$$(a \cdot f \cdot b)(c) = f(ca)b,$$

for all $a, b \in A, c \in \mathcal{C}_{\alpha^{-1}}, f \in \mathcal{R}_\alpha$ and $\alpha \in \pi$. If $\Delta_{\alpha,\beta} : \mathcal{C}_{\alpha\beta} \longrightarrow \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta$ is a coassociative $A$-bimodule map, for all $\alpha$, $\beta \in \pi$, then we define $\mu_{\alpha,\beta} : \mathcal{R}_\alpha \otimes_A \mathcal{R}_\beta \longrightarrow \mathcal{R}_{\alpha\beta}$, $\quad \mu_{\alpha,\beta}(f_\alpha \otimes_A g_\beta) = f_\alpha \sharp g_\beta$ by

$$(f_\alpha \sharp g_\beta)(c) = g_\beta(c_{(1,\beta^{-1})} f_\alpha(c_{(2,\alpha^{-1})})),$$

for all $c \in \mathcal{C}_{(\alpha\beta)^{-1}}, f_\alpha \in \mathcal{R}_\alpha, g_\beta \in \mathcal{R}_\beta$, and $\alpha$, $\beta \in \pi$. It is easily verified that $\mu_{\alpha,\beta}$ is an associative $A$-bimodule map and we can compute that

$$(f_\alpha \sharp g_\beta \sharp h_\gamma)(c) = h_\gamma(c_{(1,\gamma^{-1})} g_\beta(c_{(2,\beta^{-1})} f_\alpha(c_{(3,\alpha^{-1})}))),$$

for all $c \in \mathcal{C}_{(\alpha\beta\gamma)^{-1}}, f_\alpha \in \mathcal{R}_\alpha, g_\beta \in \mathcal{R}_\beta$, and $h_\gamma \in \mathcal{R}_\gamma$.

Let $\varepsilon : \mathcal{C}_1 \longrightarrow A$ be an $A$-bimodule map. For all $a \in A$, and $c \in \mathcal{C}_1$, we have

$$(a \cdot \varepsilon)(c) = \varepsilon(c)a = (\varepsilon \cdot a)(c),$$

so $\eta : A \longrightarrow \mathcal{R}_1,\ \eta(a) = \varepsilon \cdot a = a \cdot \varepsilon$ is an $A$-bimodule map.

For all $f \in \mathcal{R}_\alpha, c \in \mathcal{C}_{\alpha^{-1}}$, we compute that

$$\begin{aligned}(\varepsilon\sharp f)(c) &= f(c_{(1,\alpha^{-1})}\varepsilon(c_{(2,1)}));\\ (f\sharp\varepsilon)(c) &= \varepsilon(c_{(1,1)}f(c_{(2,\alpha^{-1})})) = \varepsilon(c_{(1,1)})f(c_{(2,\alpha^{-1})}) = f(\varepsilon(c_{(1,1)})c_{(2,\alpha^{-1})}).\end{aligned}$$

**Proposition 2.6.** *Let $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon)$ be a left unital lax $\pi$-$A$-coring. Then we have that ${}^*\mathcal{C} = \mathcal{R} = (\{\mathcal{R}_\alpha\}_{\alpha\in\pi}, \{\mu_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, \eta)$ is a right unital lax $\pi$-$A$-ring, and the $\pi$-$A$-rings (or $\pi$-graded $A$-rings) $\underline{\mathcal{R}} = \underline{{}^*\mathcal{C}}$ and ${}^*\underline{\mathcal{C}}$ are isomorphic. Here ${}^*\underline{\mathcal{C}} = \bigoplus_{\alpha\in\pi} {}_A Hom(\underline{\mathcal{C}}_{\alpha^{-1}}, A)$ is the left dual $\pi$-graded $A$-ring of the $\pi$-$A$-coring $\underline{\mathcal{C}}$ in the sense of [7], and $\underline{\mathcal{C}}_{\alpha^{-1}} = \mathcal{C}_{\alpha^{-1}} \cdot 1_A$.*

**Proof.** Assume that $\mathcal{C} = \{\mathcal{C}_\alpha, \Delta_{\alpha,\beta}, \varepsilon\}_{\alpha,\beta\in\pi}$ is a left unital lax $\pi$-$A$-coring. We first claim that ${}^*\mathcal{C} = \mathcal{R}$ is a right unital lax $\pi$-$A$-ring. Indeed, for all $\underline{f} = 1_A \cdot f \in \underline{\mathcal{R}}_\alpha$, we have

$$\begin{aligned}(\varepsilon\sharp\underline{f})(c) &= (\varepsilon\sharp 1_A\cdot f)(c) = (1_A\cdot f)(c_{(1,\alpha^{-1})}\varepsilon(c_{(2,1)})) = f(c_{(1,\alpha^{-1})}\varepsilon(c_{(2,1)})1_A)\\ &= f(c_{(1,\alpha^{-1})}\varepsilon(c_{(2,1)})) \overset{(6)}{=} f(c1_A) = (1_A\cdot f)(c),\end{aligned}$$

and

$$\begin{aligned}(\underline{f}\sharp\varepsilon)(c) &= \varepsilon(c_{(1,1)}(1_A\cdot f)(c_{(2,\alpha^{-1})})) = (1_A\cdot f)(\varepsilon(c_{(1,1)})c_{(2,\alpha^{-1})})\\ &= f(\varepsilon(c_{(1,1)})c_{(2,\alpha^{-1})}1_A) \overset{(6)}{=} f(c1_A) = (1_A\cdot f)(c).\end{aligned}$$

Therefore, one gets $\varepsilon\sharp\underline{f} = \underline{f}\sharp\varepsilon = \underline{f}$, and so (9) holds. This completes the first claim.

To prove the second statement, we notice that $f \in 1_A\cdot\mathcal{R}_\alpha$ if and only if $1_A\cdot f = f$, or $f(c1_A) = f(c)$, for all $c \in \mathcal{C}_{\alpha^{-1}}, \alpha \in \pi$. Define a map

$$F_\alpha : 1_A\cdot\mathcal{R}_\alpha \longrightarrow \underline{\mathcal{R}}_\alpha,\ F_\alpha(f) = f_{|\mathcal{C}_{\alpha^{-1}}1_A}$$

with an inverse $G_\alpha$, given by the formula

$$G_\alpha(g)(c) = g(c1_A),$$

for all $g \in \underline{\mathcal{R}}_\alpha$.

It is easy to check that $F = \{F_\alpha\}_{\alpha\in\pi}$ gives rise to an isomorphism between the $\pi$-$A$-rings $\underline{{}^*\mathcal{C}}$ and ${}^*\underline{\mathcal{C}}$. □

## 3. Lax Group Entwining Structures

In this section we will provide a class of examples of a lax group coring by introducing lax group entwining structures and study their structural properties.

**Definition 3.1.** Let $A$ be a $k$-algebra, $C = \{C_\alpha, \delta_{\alpha,\beta}, \epsilon\}_{\alpha,\beta\in\pi}$ be a $\pi$-coalgebra over $k$ and $\psi = \{\psi_\alpha : C_\alpha \otimes A \longrightarrow A \otimes C_\alpha\}_{\alpha\in\pi}$ be a family of $k$-linear maps satisfying the following relations:

$$(ab)_{\psi_\alpha} \otimes c^{\psi_\alpha} = a_{\psi_\alpha} b_{\phi_\alpha} \otimes c^{\psi_\alpha\phi_\alpha}; \tag{10}$$

$$a_{\psi_{\alpha\beta}} 1_{A\phi_\alpha} \otimes {c^{\psi_{\alpha\beta}}{}_{(1,\alpha)}}^{\phi_\alpha} \otimes c^{\psi_{\alpha\beta}}{}_{(2,\beta)} = a_{\psi_\beta\phi_\alpha} \otimes {c_{(1,\alpha)}}^{\phi_\alpha} \otimes {c_{(2,\beta)}}^{\psi_\beta}; \tag{11}$$

$$\epsilon(c^{\psi_1}) 1_{\psi_1} a = \epsilon(c^{\psi_1}) a_{\psi_1}; \tag{12}$$

$$1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = \epsilon({c_{(1,1)}}^{\phi_1}) 1_{A\psi_\alpha\phi_1} \otimes {c_{(2,\alpha)}}^{\psi_\alpha}, \tag{13}$$

where we use the notation $\psi_\alpha(c \otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha}$. Then we say that $(A, C, \psi)$ is a lax $\pi$-entwining structure. It is not hard to verify that this quadruple of conditions is equivalent to the quadruple of conditions (10), (11), (12) and (14).

$$1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = \epsilon({c_{(1,1)}}^{\phi_1}) 1_{A\phi_1} 1_{A\psi_\alpha} \otimes {c_{(2,\alpha)}}^{\psi_\alpha}. \tag{14}$$

**Remark 3.2.** (1) If $\psi$ satisfies the relations (10), (11), (12) and

$$1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = \epsilon({c_{(1,1)}}^{\psi_1}) 1_{A\psi_1} \otimes c_{(2,\alpha)}, \tag{15}$$

or $\psi$ satisfies the relations (10), (15), (12) and

$$a_{\psi_{\alpha\beta}} \otimes \delta_{\alpha,\beta}(c^{\psi_{\alpha\beta}}) = a_{\psi_\beta\phi_\alpha} \otimes {c_{(1,\alpha)}}^{\phi_\alpha} \otimes {c_{(2,\beta)}}^{\psi_\beta}, \tag{16}$$

then we call $(A, C, \psi)$ a weak group entwining structure.

(2) If $\psi$ satisfies the relations (10), (16) and

$$1_{A\psi_\alpha} \otimes c^{\psi_\alpha} = 1 \otimes c, \tag{17}$$

$$\epsilon(c)a = \epsilon(c^{\psi_1}) a_{\psi_1}, \tag{18}$$

then we call $(A, C, \psi)$ a group entwining structure.

(3) If $(A, C, \psi)$ is a lax group entwining structure and satisfies

$$\epsilon(c^{\psi_1}) 1_{A\psi_1} = \epsilon(c) 1_A, \tag{19}$$

then we call $(A, C, \psi)$ a partial group entwining structure. We can show that $(A, C, \psi)$ is a partial group entwining structure if and only if $\psi$ satisfies the relations (10), (11) and (18).

(4) If $\pi$ is trivial, one recovers the notions introduced in [6].

The following proposition is easy to prove:

**Proposition 3.3.** *$(A, C, \psi)$ is a group entwining structure if and only if it is at the same time a partial and weak group entwining structure.*

**Theorem 3.4.** *Let $A$ be a $k$-algebra, $C = \{C_\alpha, \delta_{\alpha,\beta}, \epsilon\}_{\alpha,\beta\in\pi}$ be a $\pi$-coalgebra over $k$. Assume that $A\otimes C = \{A\otimes C_\alpha\}_{\alpha\in\pi}$ is a family of $A$-bimodules such that the left actions are canonical, for all $\alpha\in\pi$. Consider the following left $A$-linear maps:*

$$\Delta_{\alpha,\beta} : A\otimes C_{\alpha\beta} \longrightarrow A\otimes C_\alpha \otimes_A A\otimes C_\beta \cong \underline{A\otimes C_\alpha}\otimes C_\beta,$$
$$\Delta_{\alpha,\beta}(a\otimes c) = (a\otimes c_{(1,\alpha)})\otimes_A (1_A\otimes c_{(2,\beta)}) \cong (a\otimes c_{(1,\alpha)})\cdot 1_A\otimes c_{(2,\beta)};$$
$$\varepsilon : A\otimes C_1 \longrightarrow (A\otimes C_1)\cdot 1_A \longrightarrow A,$$
$$\varepsilon(a\otimes c) = (A\otimes\epsilon)((a\otimes c)\cdot 1_A).$$

*where $\underline{A\otimes C_\alpha} = (A\otimes C_\alpha)\cdot 1_A$. Then the following assertions are equivalent:*

*(1) $(A\otimes C, \Delta = \{\Delta_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, \varepsilon)$ is a left unital lax $\pi$-$A$-coring;*

*(2) There exists a family of $k$-linear maps $\psi = \{\psi_\alpha : C_\alpha\otimes A \longrightarrow A\otimes C_\alpha\}_{\alpha\in\pi}$ such that $(A, C, \psi)$ is a lax group entwining structure.*

**Proof.** (1) $\Rightarrow$ (2). Define a family of $k$-linear maps:

$$\psi = \{\psi_\alpha : C_\alpha\otimes A \longrightarrow A\otimes C_\alpha\}_{\alpha\in\pi}, \psi_\alpha(c\otimes a) = (1_A\otimes c)\cdot a = a_{\psi_\alpha}\otimes c^{\psi_\alpha}.$$

We have to prove that (10), (11), (12) and (13) hold. By the associativity of the right $A$-action, one has

$$(ab)_{\psi_\alpha}\otimes c^{\psi_\alpha} = (1\otimes c)\cdot ab = ((1\otimes c)\cdot a)\cdot b = a_{\psi_\alpha} b_{\phi_\alpha}\otimes c^{\psi_\alpha\phi_\alpha}.$$

$\Delta_{\alpha,\beta}$ is right $A$-linear if and only if $\Delta_{\alpha,\beta}(1_A\otimes c)a = \Delta_{\alpha,\beta}((1_A\otimes c)a)$. Indeed, we have

$$\Delta_{\alpha,\beta}(1_A\otimes c)a = (1_A\otimes c_{(1,\alpha)})\otimes_A (a_{\psi_\beta}\otimes {c_{(2,\beta)}}^{\psi_\beta}) \cong a_{\psi_\beta\phi_\alpha}\otimes {c_{(1,\alpha)}}^{\phi_\alpha}\otimes {c_{(2,\beta)}}^{\psi_\beta},$$

and

$$\begin{aligned}
\Delta_{\alpha,\beta}((1_A\otimes c)a) &= \Delta_{\alpha,\beta}(a_{\psi_{\alpha\beta}}\otimes c^{\psi_{\alpha\beta}})\\
&= (a_{\psi_{\alpha\beta}}\otimes_A {c^{\psi_{\alpha\beta}}}_{(1,\alpha)})\otimes_A (1_A\otimes_A {c^{\psi_{\alpha\beta}}}_{(2,\beta)})\\
&\cong a_{\psi_{\alpha\beta}} 1_{A\phi_\alpha}\otimes {{c^{\psi_{\alpha\beta}}}_{(1,\alpha)}}^{\phi_\alpha}\otimes {c^{\psi_{\alpha\beta}}}_{(2,\beta)}.
\end{aligned}$$

$\varepsilon$ is right $A$-linear if and only if $\varepsilon(b\otimes c)a = \varepsilon((b\otimes c)a)$, for all $a, b\in A$ and $c\in C_1$. This is obvious.

Take $1_{\psi_{\alpha\beta}}\otimes c^{\psi_{\alpha\beta}} \in \underline{A\otimes C_{\alpha\beta}}$, then

$$\begin{aligned}
\underline{\Delta}_{\alpha,\beta}(1_{\psi_{\alpha\beta}}\otimes c^{\psi_{\alpha\beta}}) &= (1_{\psi_{\alpha\beta}}\otimes {c^{\psi_{\alpha\beta}}}_{(1,\alpha)})\otimes_A (1\otimes {c^{\psi_{\alpha\beta}}}_{(2,\beta)})\\
&= (1_{\psi_{\alpha\beta}} 1_{\phi_\alpha}\otimes {{c^{\psi_{\alpha\beta}}}_{(1,\alpha)}}^{\phi_\alpha})\otimes_A (1\otimes {c^{\psi_{\alpha\beta}}}_{(2,\beta)}).
\end{aligned}$$

From the counit property (6) of $\varepsilon$, it follows that

$$\begin{aligned} 1_{\psi_\alpha} \otimes c^{\psi_\alpha} &= ((\varepsilon \circ \iota_1) \otimes_A \underline{A \otimes C_\alpha})\underline{\Delta}_{1,\,\alpha}(1_{\psi_\alpha} \otimes c^{\psi_\alpha}) \\ &= (A \otimes \epsilon)(1_{\psi_\alpha} 1_{\phi_1} \otimes {c^{\psi_\alpha}{}_{(1,\,1)}}^{\phi_1})(1 \otimes c^{\psi_\alpha}{}_{(2,\,\alpha)}) \\ &= \epsilon({c^{\psi_\alpha}{}_{(1,\,1)}}^{\phi_1}) 1_{\psi_\alpha} 1_{\phi_1} \otimes c^{\psi_\alpha}{}_{(2,\,\alpha)} \\ &\overset{(11)}{=} \epsilon({c_{(1,\,1)}}^{\phi_1}) 1_{\psi_\alpha \phi_1} \otimes {c_{(2,\,\alpha)}}^{\psi_\alpha}, \end{aligned}$$

and so (13) is proved.

(2) $\Rightarrow$ (1). Define the right $A$-module structures of $A \otimes C_\alpha$ as $(a \otimes c) \cdot b = ab_{\psi_\alpha} \otimes c^{\psi_\alpha}$, for all $c \in C_\alpha, \alpha \in \pi$. A straightforward computation shows that $(A \otimes C, \Delta, \varepsilon)$ is a left unital lax $\pi$-$A$-coring. $\square$

Let $(A, C, \psi)$ be a lax group entwining structure, $\mathcal{C} = A \otimes C = \{\mathcal{C}_\alpha = A \otimes C_\alpha, \Delta_{\alpha,\,\beta}, \varepsilon\}_{\alpha,\,\beta \in \pi}$ the associated lax $\pi$-$A$-coring, and $\underline{\mathcal{C}} = (A \otimes C) \cdot 1_A = \{\underline{\mathcal{C}}_\alpha = \underline{A \otimes C_\alpha}, \Delta_{\alpha,\,\beta}, \varepsilon \circ \iota_1\}_{\alpha,\,\beta \in \pi}$ the associated $\pi$-$A$-coring.

Over a lax group entwining structure, we can define the following two different notions of lax entwined modules.

For a family of $k$-linear maps $\rho = \{\rho_{\alpha,\,\beta} : M_{\alpha\beta} \longrightarrow M_\alpha \otimes C_\beta\}_{\alpha,\,\beta \in \pi}$, we will adopt the notation $\rho_{\alpha,\,\beta}(m) = m_{[0,\,\alpha]} \otimes m_{[1,\,\beta]}$, $(\rho_{\alpha,\,\beta} \otimes C_\gamma)(\rho_{\alpha\beta,\,\gamma}(m)) = m_{[0,\,\alpha]} \otimes m_{[1,\,\beta]} \otimes m_{[2,\,\gamma]}$, for all $m \in M_{\alpha\beta\gamma}, \alpha,\, \beta,\, \gamma \in \pi$. We don't assume that $\rho$ is coassociative.

A lax $\pi$-entwined module $\underline{M}$ is a family of right $A$-modules $(M_\alpha)_{\alpha \in \pi}$, together with a family of $k$-linear maps $\rho = \{\rho_{\alpha,\,\beta} : M_{\alpha\beta} \longrightarrow M_\alpha \otimes C_\beta\}_{\alpha,\,\beta \in \pi}$ such that the following conditions are satisfied:

$$m_{[0,\,\alpha]} \epsilon(m_{[1,\,1]}) = m, \tag{20}$$

for all $m \in M_\alpha$;

$$(\rho_{\alpha,\,\beta} \otimes C_\gamma)(\rho_{\alpha\beta,\,\gamma}(m)) = m_{[0,\,\alpha]} 1_{A_{\psi_\gamma \phi_\beta}} \otimes {m_{[1,\,\beta\gamma](1,\,\beta)}}^{\phi_\beta} \otimes {m_{[1,\,\beta\gamma](2,\,\gamma)}}^{\psi_\gamma}, \tag{21}$$

for all $m \in M_{\alpha\beta\gamma}$;

$$\rho_{\alpha,\,\beta}(ma) = m_{[0,\,\alpha]} a_{\psi_\beta} \otimes {m_{[1,\,\beta]}}^{\psi_\beta}, \tag{22}$$

for all $m \in M_{\alpha\beta}$.

A morphism between two lax $\pi$-entwined modules $\underline{M}$ and $\underline{N}$ is a family of right $A$-linear maps $f_\alpha : M_\alpha \longrightarrow N_\alpha$ such that $f_\alpha(m_{[0,\,\alpha]}) \otimes m_{[1,\,\beta]} = f_{\alpha\beta}(m)_{[0,\,\alpha]} \otimes f_{\alpha\beta}(m)_{[1,\,\beta]}$, for all $m \in M_{\alpha\beta}, \alpha,\, \beta \in \pi$. $\mathcal{M}(\psi)_A^{\pi - C}$ will denote the category of lax $\pi$-entwined modules.

For a family of $k$-linear maps $\rho = \{\rho_\alpha : M \longrightarrow M \otimes C_\alpha\}_{\alpha\in\pi}$, we will adopt the notation $\rho_\alpha(m) = m_{[0]} \otimes m_{[1,\,\alpha]}$, $(\rho_\alpha \otimes C_\beta)(\rho_\beta(m)) = m_{[0]} \otimes m_{[1,\,\alpha]} \otimes m_{[2,\,\beta]}$, for all $m \in M, \alpha,\ \beta \in \pi$. We don't assume that $\rho$ is coassociative.

A lax entwined module $M$ is a right $A$-module together with a family of $k$-linear maps $\rho = \{\rho_\alpha : M \longrightarrow M \otimes C_\alpha\}_{\alpha\in\pi}$ such that the following conditions are satisfied, for all $m \in M, \alpha,\ \beta \in \pi$,

$$\begin{aligned} m_{[0]}\epsilon(m_{[1,\,1]}) &= m, \\ (\rho_\alpha \otimes C_\beta)(\rho_\beta(m)) &= m_{[0]} 1_{A\psi_\beta\phi_\alpha} \otimes m_{[1,\,\alpha\beta](1,\,\alpha)}{}^{\phi_\alpha} \otimes m_{[1,\,\alpha\beta](2,\,\beta)}{}^{\psi_\beta}, \\ \rho_\alpha(ma) &= m_{[0]} a_{\psi_\alpha} \otimes m_{[1,\,\alpha]}{}^{\psi_\alpha}. \end{aligned}$$

A morphism between two lax entwined modules $M$ and $N$ is a right $A$-linear map $f : M \longrightarrow N$ such that $f(m_{[0]}) \otimes m_{[1,\,\alpha]} = f(m)_{[0]} \otimes f(m)_{[1,\,\alpha]}$ , for all $m \in M, \alpha \in \pi$. $\mathcal{M}(\psi)_A^C$ will denote the category of lax entwined modules.

**Proposition 3.5.** *For a lax $\pi$-entwining structure $(A, C, \psi)$, we have two different isomorphisms of categories, namely:*

*(1) the categories $\mathcal{M}^{\pi-\underline{\mathcal{C}}}$ and $\mathcal{M}(\psi)_A^{\pi-C}$ are isomorphic;*

*(2) the categories $\mathcal{M}^{\underline{\mathcal{C}}}$ and $\mathcal{M}(\psi)_A^C$ are isomorphic.*

**Proof.** We will just show that (1) holds, and in a similar way we can prove (2).

Let $\underline{M} = (M_\alpha)_{\alpha\in\pi}$ be a family of right $A$-modules. We will first show that for all $\alpha, \beta \in \pi$, $Hom_A(M_{\alpha\beta}, M_\alpha \otimes_A \underline{\mathcal{C}}_\beta)$ is isomorphic to the submodule of $\mathrm{Hom}(M_{\alpha\beta}, M_\alpha \otimes C_\beta)$ consisting of maps $\rho_{\alpha,\,\beta}$ satisfying (22). Take $\rho_{\alpha,\,\beta} : M_{\alpha\beta} \longrightarrow M_\alpha \otimes C_\beta$ satisfying (22), and define $F(\rho_{\alpha,\,\beta}) : M_{\alpha\beta} \longrightarrow M_\alpha \otimes_A \underline{\mathcal{C}}_\beta$ as follows:

$$F(\rho_{\alpha,\,\beta})(m) = m_{[0,\,\alpha]} \otimes_A (1_{A\psi_\beta} \otimes m_{[1,\,\beta]}{}^{\psi_\beta}).$$

We can check that $F(\rho_{\alpha,\,\beta})$ is right $A$-linear as follows:

$$\begin{aligned} F(\rho_{\alpha,\,\beta})(ma) &= m_{[0,\,\alpha]} a_{\psi_\beta} \otimes_A (1_{A\phi_\beta} \otimes m_{[1,\,\beta]}{}^{\psi_\beta\phi_\beta}) \\ &= m_{[0,\,\alpha]} \otimes_A (a_{\psi_\beta} 1_{A\phi_\beta} \otimes m_{[1,\,\beta]}{}^{\psi_\beta\phi_\beta}) \\ &\overset{(10)}{=} m_{[0,\,\alpha]} \otimes_A (a_{\psi_\beta} \otimes m_{[1,\,\beta]}{}^{\psi_\beta}) \\ &= m_{[0,\,\alpha]} \otimes_A (1_{A\psi_\beta} \otimes m_{[1,\,\beta]}{}^{\psi_\beta})a \\ &= F(\rho_{\alpha,\,\beta})(m)a \end{aligned}$$

Conversely, take $\widetilde{\rho}_{\alpha,\,\beta} \in \mathrm{Hom}_A(M_{\alpha\beta}, M_\alpha \otimes_A \underline{\mathcal{C}}_\beta)$, and define $G(\widetilde{\rho}_{\alpha,\,\beta}) \in \mathrm{Hom}(M_{\alpha\beta}, M_\alpha \otimes C_\beta)$ as follows: for $m \in M_{\alpha\beta}$, there exist (a finite number of) $m_i \in M_\alpha$ and $c_i \in C_\beta$

such that $\widetilde{\rho}_{\alpha,\beta}(m)=\sum_i m_i\otimes_A(1_{A\psi_\beta}\otimes {c_i}^{\psi_\beta})$; then we define

$$G(\widetilde{\rho}_{\alpha,\beta})(m)=\sum_i m_i 1_{A\psi_\beta}\otimes {c_i}^{\psi_\beta}.$$

$G(\widetilde{\rho}_{\alpha,\beta})$ satisfies (22), since

$$\widetilde{\rho}_{\alpha,\beta}(ma)=\sum_i m_i\otimes_A(a_{\psi_\beta}\otimes {c_i}^{\psi_\beta})=\sum_i m_i a_{\psi_\beta}\otimes_A(1_{A\phi_\beta}\otimes {c_i}^{\psi_\beta\phi_\beta}),$$

hence

$$G(\widetilde{\rho}_{\alpha,\beta})(ma)=\sum_i m_i a_{\psi_\beta}1_{A\phi_\beta}\otimes {c_i}^{\psi_\beta\phi_\beta}=\sum_i m_i 1_{A\psi_\beta}a_{\phi_\beta}\otimes {c_i}^{\psi_\beta\phi_\beta}.$$

$F$ and $G$ are mutually inverse:

$$\begin{aligned}
F(G(\widetilde{\rho}_{\alpha,\beta}))(m) &= \sum_i m_i 1_{A\psi_\beta}\otimes_A(1_{A\phi_\beta}\otimes {c_i}^{\psi_\beta\phi_\beta})\\
&= \sum_i m_i\otimes_A(1_{A\psi_\beta}\otimes {c_i}^{\psi_\beta})=\widetilde{\rho}_{\alpha,\beta}(m).\\
G(F(\rho_{\alpha,\beta}))(m) &= m_{[0,\alpha]}1_{A\psi_\beta}\otimes {m_{[1,\beta]}}^{\psi_\beta}\overset{(22)}{=}\rho_{\alpha,\beta}(m).
\end{aligned}$$

Now take $\rho_{\alpha,\beta}:M_{\alpha\beta}\longrightarrow M_\alpha\otimes C_\beta$ satisfying (22) and the corresponding right $A$-linear map $\widetilde{\rho}_{\alpha,\beta}$. We claim that $\widetilde{\rho}_{\alpha,\beta}$ is coassociative if and only if $\rho_{\alpha,\beta}$ satisfies (21). First we compute

$$(\widetilde{\rho}_{\alpha,\beta}\otimes\underline{\mathcal{C}}_\gamma)\widetilde{\rho}_{\alpha\beta,\gamma}(m) = m_{[0,\alpha]}\otimes_A(1_{A\psi_\beta}\otimes {m_{[1,\beta]}}^{\psi_\beta})\otimes_A(1_{A\phi_\gamma}\otimes {m_{[2,\gamma]}}^{\phi_\gamma}),$$

and

$$((M_\alpha\otimes_A\underline{\Delta}_{\beta,\gamma})\circ\widetilde{\rho}_{\alpha,\beta\gamma})(m) = m_{[0,\alpha]}\otimes_A(1_{A\psi_{\beta\gamma}}\otimes {m_{[1,\beta\gamma]}}^{\psi_{\beta\gamma}}{}_{(1,\beta)})\otimes_A(1_A\otimes {m_{[1,\beta\gamma]}}^{\psi_{\beta\gamma}}{}_{(2,\gamma)}).$$

If $\widetilde{\rho}$ is coassociative, then it follows that

$$\begin{aligned}
&m_{[0,\alpha]}1_{A\psi_\beta}1_{A\phi_\gamma\varphi_\beta}\otimes {m_{[1,\beta]}}^{\psi_\beta\varphi_\beta}\otimes {m_{[2,\gamma]}}^{\phi_\gamma}\\
&\quad\overset{(10)}{=} m_{[0,\alpha]}1_{A\phi_\gamma\psi_\beta}\otimes {m_{[1,\beta]}}^{\psi_\beta}\otimes {m_{[2,\gamma]}}^{\phi_\gamma}\\
&\quad\overset{(22)}{=} \rho_{\alpha,\beta}(m_{[0,\alpha\beta]}1_{A\phi_\gamma})\otimes {m_{[1,\gamma]}}^{\phi_\gamma}\overset{(22)}{=}\rho_{\alpha,\beta}(m_{[0,\alpha\beta]})\otimes m_{[1,\gamma]},
\end{aligned}$$

equals

$$\begin{aligned}
&m_{[0,\alpha]}1_{A\psi_{\beta\gamma}}1_{A\phi_\beta}\otimes {{m_{[1,\beta\gamma]}}^{\psi_{\beta\gamma}}{}_{(1,\beta)}}^{\phi_\beta}\otimes {m_{[1,\beta\gamma]}}^{\psi_{\beta\gamma}}{}_{(2,\gamma)}\\
&\quad\overset{(11)}{=} m_{[0,\alpha]}1_{A\psi_\gamma\phi_\beta}\otimes {m_{[1,\beta\gamma](1,\beta)}}^{\phi_\beta}\otimes {m_{[1,\beta\gamma](2,\gamma)}}^{\psi_\gamma},
\end{aligned}$$

and (21) follows for $\rho_{\alpha,\beta}$. Conversely, if (21) holds, then

$$\begin{aligned}
&((M_\alpha \otimes_A \underline{\Delta}_{\beta,\gamma}) \circ \widetilde{\rho}_{\alpha,\beta\gamma})(m) \\
&\quad m_{[0,\alpha]} \otimes_A (1_{A\psi_{\beta\gamma}} 1_{A\phi_\beta} \otimes {m_{[1,\beta\gamma]}{}^{\psi_{\beta\gamma}}{}_{(1,\beta)}}^{\phi_\beta}) \otimes_A (1_A \otimes m_{[1,\beta\gamma]}{}^{\psi_{\beta\gamma}}{}_{(2,\gamma)}) \cdot 1_A \\
&\quad \overset{(11)}{=} m_{[0,\alpha]} \otimes_A (1_{A\psi_\gamma} 1_{A\phi_\beta} \otimes m_{[1,\beta\gamma](1,\beta)}{}^{\phi_\beta}) \otimes_A (1_A \otimes m_{[1,\beta\gamma](2,\gamma)}{}^{\psi_\gamma}) \cdot 1_A \\
&\quad = m_{[0,\alpha]} 1_{A\psi_\gamma} 1_{A\phi_\beta} \otimes_A (1_A \otimes m_{[1,\beta\gamma](1,\beta)}{}^{\phi_\beta}) \cdot 1_A \otimes_A (1_A \otimes m_{[1,\beta\gamma](2,\gamma)}{}^{\psi_\gamma}) \cdot 1_A \\
&\quad \overset{(21)}{=} m_{[0,\alpha]} \otimes_A (1_A \otimes m_{[1,\beta]}) \cdot 1_A \otimes_A (1_A \otimes m_{[2,\gamma]}) \cdot 1_A \\
&\quad = (\widetilde{\rho}_{\alpha,\beta} \otimes \underline{\mathcal{C}}_\gamma)\widetilde{\rho}_{\alpha\beta,\gamma}(m),
\end{aligned}$$

so $\widetilde{\rho}$ is coassociative. Finally, $\widetilde{\rho}$ satisfies the counit property if and only if

$$m = m_{[0,\alpha]} 1_{\psi_1} \epsilon(m_{[1,1]}{}^{\psi_1}) \overset{(22)}{=} m_{[0,\alpha]} \epsilon(m_{[1,1]}).$$

□

**Remark 3.6.** Let $(A, C, \psi)$ be a weak $\pi$-entwining structure. Using (16), we find that (21) is equivalent to

$$(\rho_{\alpha,\beta} \otimes \mathcal{C}_\gamma)\rho_{\alpha\beta,\gamma}(m) = m_{[0,\alpha]} 1_{\psi_{\beta\gamma}} \otimes \delta_{\beta,\gamma}(m_{[1,\beta\gamma]}{}^{\psi_{\beta\gamma}}) = m_{[0,\alpha]} \otimes \delta_{\beta,\gamma}(m_{[1,\beta\gamma]}),$$

so (20), (21) are equivalent to saying that $\underline{M}$ is a right $\pi$-$C$-comodule.

Let $A$ be a $k$-algebra and $H = \{H_\alpha, m_\alpha, 1_\alpha, \delta_{\alpha,\beta}, \epsilon\}_{\alpha,\beta\in\pi}$ a semi-Hopf $\pi$-coalgebra. Consider a family of $k$-linear maps

$$\rho = \{\rho_\alpha : A \longrightarrow A \otimes H_\alpha, \rho_\alpha(a) = a_{[0]} \otimes a_{[1,\alpha]}\}_{\alpha\in\pi}.$$

To $\rho$, we associate a family of $k$-linear maps

$$\psi = \{\psi_\alpha : H_\alpha \otimes A \longrightarrow A \otimes H_\alpha\},$$

$\psi_\alpha(h \otimes a) = a_{[0]} \otimes h a_{[1,\alpha]} = a_{\psi_\alpha} \otimes h^{\psi_\alpha}$, for all $a \in A, h \in H$.

**Lemma 3.7.** *$\psi$ satisfies* (10) *if and only if*

$$\rho_\alpha(ab) = a_{[0]} b_{[0]} \otimes a_{[1,\alpha]} b_{[1,\alpha]}. \tag{23}$$

*$\psi$ satisfies* (17) *if and only if*

$$\rho_\alpha(1_A) = 1_A \otimes 1_{H_\alpha}. \tag{24}$$

*$\psi$ satisfies* (11) *if and only if*

$$\rho_\alpha(a_{[0]}) \otimes a_{[1,\beta]} = a_{[0]} 1_{A0} \otimes a_{[1,\alpha\beta](1,\alpha)} 1_{A[1,\alpha]} \otimes a_{[1,\alpha\beta](2,\beta)}. \tag{25}$$

*$\psi$ satisfies* (12) *if and only if*

$$\epsilon(a_{[1,\,1]})a_{[0]} = \epsilon(1_{A\,[1,\,1]})1_{A\,[0]}a. \tag{26}$$

*$\psi$ satisfies* (18) *if and only if*

$$\epsilon(a_{[1,\,1]})a_{[0]} = a. \tag{27}$$

*$\psi$ satisfies* (15) *if and only if*

$$\rho_1(1_A) = \epsilon(1_{A\,[1,\,1]})1_{A\,[0]} \otimes 1_{H_1}. \tag{28}$$

*$\psi$ satisfies* (16) *if and only if*

$$\rho_\alpha(a_{[0]}) \otimes a_{[1,\,\beta]} = a_{[0]} \otimes \delta_{\alpha,\,\beta}(a_{[1,\,\alpha\beta]}). \tag{29}$$

*$\psi$ satisfies* (13) *if and only if*

$$\rho_1(1_A) = \epsilon(1_{A\,[0][1,\,1]})1_{A\,[0][0]} \otimes 1_{A\,[1,\,1]}. \tag{30}$$

*$\psi$ satisfies* (14) *if and only if*

$$\rho_1(1_A) = \epsilon(1_{A\,[1',\,1]})1_{A\,[0']}1_{A\,[0]} \otimes 1_{A\,[1,\,1]}. \tag{31}$$

*$\psi$ satisfies* (19) *if and only if*

$$\epsilon(1_{A\,[1,\,1]})1_{A\,[0]} = 1_A. \tag{32}$$

**Proof.** Straightforward. □

**Example 3.8.** It follows immediately from Lemma 3.7 that

1. $(A, H, \psi)$ is a $\pi$-entwining structure if and only if$A$ is a right $\pi$-$H$-comodule algebra;

2. $(A, H, \psi)$ is a partial $\pi$-entwining structure if and only if (23), (25), (27) hold. We will then say that $H$ coacts $\pi$-partially on $A$, or that $A$ is a right partial $\pi$-$H$-comodule algebra;

3. $(A, H, \psi)$ is a lax $\pi$-entwining structure if and only if (23), (25), (26) and (30) or (31) are satisfied, we then say that $H$ is a right lax $\pi$-$H$-comodule algebra;

4. $(A, H, \psi)$ is a weak $\pi$-entwining structure if and only if (23), (26), (28) and (25) or (29) are satisfied, we then say that $H$ is a right weak $\pi$-$H$-comodule algebra.

**Example 3.9.** Let $H = \{H_\alpha, m_\alpha, 1_\alpha, \delta_{\alpha,\,\beta}, \epsilon\}_{\alpha,\,\beta\in\pi}$ be a semi-Hopf $\pi$-coalgebra, $e = (e_\alpha)_{\alpha\in\pi} \in \prod_{\alpha\in\pi} H_\alpha$ such that $e_\alpha \otimes e_\beta = \delta_{\alpha,\,\beta}(e_{\alpha\beta})(e_\alpha \otimes 1)$, $e_\alpha^2 = e_\alpha$, $\epsilon(e_1) = 1$. Then we can define the following partial $\pi$-$H$-coaction on an algebra $A$:

$$\rho = \{\rho_\alpha : \rho_\alpha(a) = a \otimes e_\alpha \in A \otimes H_\alpha\}_{\alpha\in\pi}.$$

It is straightforward to verify that (23), (25), and (27) hold :

$$\begin{aligned}\rho_\alpha(a)\rho_\alpha(b) &= ab\otimes e_\alpha^2 = ab\otimes e_\alpha = \rho_\alpha(ab),\\ \rho_\alpha(a_0)\otimes a_{[1,\beta]} &= a\otimes e_\alpha\otimes e_\beta = a\otimes e_{\alpha\beta_{(1,\,\alpha)}}e_\alpha\otimes e_{\alpha\beta_{(2,\,\beta)}},\\ &= a_0 1_{A_o}\otimes a_{[1,\,\alpha\beta](1,\,\alpha)}1_{A[1,\,\alpha]}\otimes a_{[1,\,\alpha\beta](2,\,\beta)},\\ a\epsilon(e_1) &= a.\end{aligned}$$

Such an idempotent $e$ exists in a semisimple Hopf $\pi$-coalgebra of finite type (see [13]) and the proof is left to the reader.

## 4. Partial Group Smash Products

In this section, we introduce and investigate the notion of a lax $\pi$-smash product.

**Definition 4.1.** Let $A$ be an algebra with unit $1_A$, $B=\bigoplus_{\alpha\in\pi}B_\alpha$ a $\pi$-graded algebra with unit $1_B$ and $R=\{R_\alpha : B_\alpha\otimes A\longrightarrow A\otimes B_\alpha\}_{\alpha\in\pi}$ a family of $k$-linear maps satisfying the relations:

$$(ac)_{R_\alpha}\otimes b^{R_\alpha} = a_{R_\alpha}c_{r_\alpha}\otimes b^{R_\alpha r_\alpha}, \tag{33}$$

$$(ac_{R_\gamma})_{r_\beta}\otimes b^{r_\beta}d^{R_\gamma} = a_{R_\beta}c_{r_{\beta\gamma}}\otimes (b^{R_\beta}d)^{r_{\beta\gamma}}, \tag{34}$$

$$a1_{A R_1}\otimes 1_B{}^{R_1} = a_{R_1}\otimes 1_B{}^{R_1}, \tag{35}$$

$$a1_{A R_\beta}\otimes b^{R_\beta} = (a1_{A r_\beta})_{R_1}\otimes 1_B{}^{R_1}b^{r_\beta}, \tag{36}$$

where we use the notation $R_\alpha(b\otimes a)=a_{R_\alpha}\otimes b^{R_\alpha}=a_{r_\alpha}\otimes b^{r_\alpha}$. Then we say that $(A,B,R)$ is a lax $\pi$-smash product structure. It is not hard to verify that this quadruple of conditions is equivalent to the quadruple of conditions (33), (34), (35) and

$$a1_{A R_\beta}\otimes b^{R_\beta} = a_{R_1}1_{A r_\beta}\otimes (1_B{}^{R_1}b)^{r_\beta}, \tag{37}$$

**Remark 4.2.** (1) If $R$ satisfies (33), (34), (35) and

$$1_{A R_\beta}\otimes b^{R_\beta} = 1_{A R_1}\otimes 1_B{}^{R_1}b, \tag{38}$$

or $R$ satisfies (33), (35), (38) and

$$a_{R_\gamma r_\beta}\otimes b^{r_\beta}d^{R_\gamma} = a_{R_{\beta\gamma}}\otimes (bd)^{R_{\beta\gamma}}, \tag{39}$$

then we call $(A,B,R)$ a weak $\pi$-smash product structure.

(2) If $R$ satisfies (33), (39) and

$$a\otimes 1_B = a_{R_1}\otimes 1_B{}^{R_1}, \tag{40}$$

$$1\otimes b = 1_{R_\alpha}\otimes b^{R_\alpha}, \tag{41}$$

then we call $(A,B,R)$ a $\pi$-smash product structure.

(3) If $(A,B,R)$ a lax $\pi$-smash product structure and $R$ satisfies

$$1_A \otimes 1_B = 1_{A R_1} \otimes 1_B{}^{R_1}, \tag{42}$$

or $R$ satisfies (33), (34) and (40), then we call $(A,B,R)$ a partial $\pi$-smash product structure.

It is easy to obtain the following proposition.

**Proposition 4.3.** *$(A,B,R)$ is a $\pi$-smash product structure if and only if it is at the same time a weak and a partial $\pi$-smash product structure.*

Let $A$ be an algebra with unit $1_A$, $B = \bigoplus_{\alpha\in\pi} B_\alpha$ a $\pi$-graded algebra with unit $1_B$. We define $\mathcal{R} = A\sharp B = \{A\sharp B_\alpha\}_{\alpha\in\pi}$, where $A\sharp B_\alpha := A\otimes B_\alpha$ as a $k$-module. The multiplication $\mu_{\alpha,\beta} : (A\otimes B_\alpha)\otimes_A (A\otimes B_\beta) \to A\otimes B_{\alpha\beta}$ of $\mathcal{R}$ is defined as

$$\mu_{\alpha,\beta}((a\otimes b)\otimes_A (c\otimes d)) = (a\sharp b)(c\sharp d),$$

for all $a, c\in A,\ b\in B_\alpha,\ d\in B_\beta$.

**Theorem 4.4.** *Let $A$ be an algebra with unit $1_A$, $B = \bigoplus_{\alpha\in\pi} B_\alpha$ a $\pi$-graded algebra with unit $1_B$ and $R = \{R_\alpha : B_\alpha\otimes A \longrightarrow A\otimes B_\alpha\}_{\alpha\in\pi}$ a family of $k$-linear maps. Assume that $A\sharp B = \{A\sharp B_\alpha\}_{\alpha\in\pi}$ is a family of $A$-bimodules with canonical left $A$-action. Consider the maps*

$$\begin{aligned}
&\mu = \{\mu_{\alpha,\beta} : (A\otimes B_\alpha)\otimes_A (A\otimes B_\beta) \longrightarrow (A\otimes B_{\alpha\beta})\},\\
&\mu_{\alpha,\beta}((a\otimes b)\otimes_A (c\otimes d)) = ac_{R_\alpha}\otimes b^{R_\alpha} d,\\
&\underline{\eta} : A \longrightarrow \underline{A\otimes B_1},\ \underline{\eta}(a) = (a\otimes 1_B)\cdot 1_A = a1_{A R_1}\otimes 1_B{}^{R_1},
\end{aligned}$$

*where $\underline{A\otimes B_1} = (A\otimes B_1)1_A$, then the following assertions are equivalent:*

*(1) $(A\sharp B, \mu, \underline{\eta})$ is a left unital lax $\pi$-$A$-ring.*

*(2) $(A,B,R)$ is a lax $\pi$-smash product structure.*

**Proof.** The proof is dual to the proof of Theorem 3.4 and left to the reader. $\square$

**Theorem 4.5.** *Let $A$ be a $k$-algebra and $C = \{C_\alpha, \Delta_{\alpha,\beta}, \varepsilon\}_{\alpha,\beta\in\pi}$ a $\pi$-coalgebra of finite type. Then there is a one-to-one correspondence between lax group entwining structures of the form $(A, C, \psi)$ and lax $\pi$-smash product structures of the form $(A^{op}, C^*, R)$.*

**Proof.** For all $\alpha \in \pi$, let $\{(e_{\alpha,i}, \xi^{\alpha,i}) \mid i = 1, ..., n_\alpha\} \subset C_\alpha \times C_\alpha^*$ be a dual basis of $C_\alpha$. Then it is easy to check that:

$$\sum_i \Delta_{\alpha,\beta}(e_{\alpha\beta,i}) \otimes \xi^{\alpha\beta,i} = \sum_{i,j} e_{\alpha,i} \otimes e_{\beta,j} \otimes \xi^{\alpha,i} * \xi^{\beta,j},$$

for all $\alpha, \beta \in \pi$.

Let $(A, C, \psi)$ be a lax group entwining structure. We define $R = \{R_\beta : C_\beta^* \otimes A \longrightarrow A \otimes C_\beta^*\}_{\beta\in\pi}$ by the formula

$$R_\beta(c^* \otimes a) = a_{R_\beta} \otimes (c^*)^{R_\beta} = \sum_i \langle c^*, e_{\beta,i}{}^{\psi_\beta}\rangle a_{\psi_\beta} \otimes \xi^{\beta,i}, \tag{43}$$

for any $c^* \in C_\beta^*$ and $a \in A$. A routine computation shows that the $(A^{op}, C^*, R)$ is a lax (resp. partial) $\pi$-smash product structure.

Conversely, if $(A^{op}, C^*, R)$ is a lax (resp. partial) $\pi$-smash product structure, then one defines $\psi = \{\psi_\alpha : C_\alpha \otimes A \longrightarrow A \otimes C_\alpha\}_{\alpha\in\pi}$ by

$$\psi_\alpha(c \otimes a) = a_{\psi_\alpha} \otimes c^{\psi_\alpha} = \sum_i \langle (\xi^{\alpha,i})^{R_\alpha}, c\rangle a_{R_\alpha} \otimes e_{\alpha,i}, \tag{44}$$

for any $a \in A, c \in C_\alpha$ and $\alpha \in \pi$. We can prove that $(A, C, \psi)$ is a lax (resp. partial) $\pi$-entwining structure, although the calculations are quite tedious. Eqs.(43) − (44) define the required one-to-one correspondence. $\square$

**Remark 4.6.** Theorem 4.5 also holds for weak group entwining structures versus weak group smash product structures.

Let $(A, C, \psi)$ be a lax group entwining structure, i.e. $A\otimes C = \{A\otimes C_\alpha, \Delta_{\alpha,\beta}, \varepsilon\}_{\alpha\in\pi}$ is a left unital lax $\pi$-$A$-coring. Set ${}^*(A\otimes C) = \{{}_A\mathrm{Hom}(A \otimes C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$. Consider the family of $k$-module isomorphisms

$${}_A\mathrm{Hom}(A \otimes C_{\alpha^{-1}}, A) \cong \mathrm{Hom}(C_{\alpha^{-1}}, A), f \mapsto f \circ (\eta_A \otimes C_{\alpha^{-1}})$$

for all $f \in {}_A\mathrm{Hom}(A \otimes C_{\alpha^{-1}}, A), \alpha \in \pi$.

The right unital lax $\pi$-$A$-ring structure on ${}^*(A\otimes C)$ induces a right unital lax $\pi$-$A$-ring structure on $\mathcal{R}' = \{\mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$. It is given by the following formulas,

$$(afb)(c) = a_{\psi_{\alpha^{-1}}} f(c^{\psi_{\alpha^{-1}}})b,$$

for all $a, b \in A, c \in C_\alpha$, and $f \in {}_A\mathrm{Hom}(C_{\alpha^{-1}}, A)$.

$$(f\sharp g)(c) = f(c_{(2,\,\alpha^{-1})})_{\psi_{\beta^{-1}}} g(c_{(1,\,\beta^{-1})}{}^{\psi_{\beta^{-1}}}),$$

for all $f \in {}_A\mathrm{Hom}(C_{\alpha^{-1}}, A), g \in {}_A\mathrm{Hom}(C_{\beta^{-1}}, A)$, and $c \in C_{(\alpha\beta)^{-1}}$.

$$\eta : A \longrightarrow {}_A\mathrm{Hom}(C_1, A), \eta(a)(c) = \epsilon(c^{\psi_1})a_{\psi_1},$$

for $a \in A, c \in C_1$.

If $\mathcal{R}' = \{\mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ is equipped with the above right unital lax $A$-ring structure, then we call it a lax Koppinen $\pi$-smash product and denote it by $\sharp(C, A)$.

The left dual of the corresponding $\pi$-$A$-coring $\underline{\mathcal{C}} = \mathcal{C}1_A$, regarded as a $\pi$-$A$-ring, is then isomorphic to the $\pi$-$A$-ring

$$\underline{\sharp}(C, A) = 1_A\sharp(C, A) = \{1_A\sharp(C, A)_\alpha\}_{\alpha\in\pi}$$

where we have

$$1_A\sharp(C, A)_\alpha := \{f \in \mathrm{Hom}(C_{\alpha^{-1}}, A) \mid f(c) = 1_{A\psi_{\alpha^{-1}}} f(c^{\psi_{\alpha^{-1}}}),\ \text{for all } c \in C_{\alpha^{-1}}\}$$

for any $\alpha \in \pi$.

**Proposition 4.7.** *Assume that $A$ is a $k$-algebra and $C = \{C_\alpha, \Delta_{\alpha,\beta}, \varepsilon\}_{\alpha,\beta\in\pi}$ a $\pi$-coalgebra of finite type. Let $(A, C, \psi)$ be a lax $\pi$-entwining structure, and $(A^{op}, C^*, R)$ the corresponding lax $\pi$-smash product structure. Then ${}^*(A \otimes C)^{op}$ is isomorphic to $A^{op}\sharp C^*$ as left unital lax $\pi$-$A^{op}$-rings, and ${}^*(\underline{A \otimes C})^{op}$ is isomorphic to $\underline{A^{op}\sharp C^*}$ as $\pi$-$A^{op}$-rings (or $\pi$-graded $A^{op}$-rings). Consequently we have that the categories ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}^\pi$ and $\mathcal{M}(\psi)_A^{\pi-C}$ are isomorphic. Moreover, if $\pi$ is a finite group, then we have an equivalence of categories between ${}_{\underline{A^{op}\sharp C^*}}\mathcal{M}$ and $\mathcal{M}(\psi)_A^C$.*

**Proof.** Since ${}^*(A \otimes C)$ is a right unital lax $\pi$-$A$-ring, we know

$${}^*(A \otimes C)^{op} = (\{\mathrm{Hom}(A \otimes C_\alpha, A)\}_{\alpha\in\pi}) \cong (\{\mathrm{Hom}(C_\alpha, A)\}_{\alpha\in\pi}) = \sharp(C, A)^{op}$$

is left unital lax $\pi$-$A^{op}$-rings, with multiplication

$$(f \bullet g)(c) = g(c_{(2,\beta)})_{\psi_\alpha} f({c_{(1,\alpha)}}^{\psi_\alpha})$$

for all $f \in \mathrm{Hom}(C_\alpha, A), g \in \mathrm{Hom}(C_\beta, A), c \in C_{\alpha\beta}, \alpha, \beta \in \pi$. Since $C$ is a $\pi$-coalgebra of finite type, the map $F$ is an isomorphism of $k$-module defined as

$$F : A^{op} \otimes C^*_\alpha \longrightarrow \mathrm{Hom}(C_\alpha, A), F(a\sharp c^*)(c) = a < c^*, c >,$$

for all $a \in A, c^* \in C^*_\alpha$, and $c \in C_\alpha$. It is straightforward to check that $F$ preserves the multiplication. Then the first statement holds.

Applying Proposition 2.6, we see that

$${}^*(\underline{A \otimes C})^{op} \cong (1_A \cdot {}^*(A \otimes C))^{op} = {}^*(A \otimes C)^{op} \cdot 1_A \cong (A^{op}\sharp C^*) \cdot 1_A = \underline{A^{op}\sharp C^*}.$$

From Proposition 3.5, we know $\mathcal{M}(\psi)_A^{\pi-C} \cong \mathcal{M}^{\pi-\underline{A\otimes C}}$, and $\mathcal{M}(\psi)_A^C \cong \mathcal{M}^{\underline{A\otimes C}}$, so the last two statements follow immediately from Propositions 5.1 and 5.4 of [7]. □

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