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CLASSICAL ZARISKI TOPOLOGY OF MODULES AND SPECTRAL SPACES I

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Abstract. Let R be a ring, M be a left R-module and $Spec({}_RM)$ be the collection of all prime submodules of M. In this paper and its sequel, we introduce and study a generalization of the Zariski topology of rings to modules and call it classical Zariski topology of M. Then we investigate the interplay between the module-theoretic properties of M and the topological properties of $Spec({}_RM)$. Modules whose classical Zariski topology is respectively T_1, Hausdorff or cofinite are studied, and several characterizations of such modules are given. We investigate this topological space from the point of view of spectral spaces (that is, topological spaces homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology). We show that $Spec({}_RM)$ is always a T_0-space and each finite irreducible closed subset of $Spec({}_RM)$ has a generic point. Then by applying Hochster's characterization of a spectral space, we show that for each left R-module M with finite spectrum, $Spec({}_RM)$ is a spectral space. In Part II we shall continue the study of this construction.

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1. Introduction

Throughout, all rings are associative rings with identity elements, and all modules are unital left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If N is a submodule (respectively proper submodule) of a module M we write $N \leq M$ (respectively $N \lneqq M$). We denote the left annihilator of a factor module M/N of M by $(N : M)$. We call M faithful if $(0 : M) = 0$.

Recall that the spectrum Spec(R) of a ring R consists of all prime ideals of R. For every ideal I of R, we set $V(I) = \{\mathcal{P} \in Spec(R) : I \subseteq \mathcal{P}\}$. Then the sets $V(I)$ satisfy the axioms for the closed sets of a topology on Spec(R), called the Zariski

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topology (see for example, [4, 18]). In the literature, there are many different generalizations of the Zariski topology for modules over commutative rings. First, we need to define what we shall mean by a prime submodule of a module.

Recall that a left R-module M is said to be prime if $Ann(N) = Ann(M)$ for every non-zero submodule N of M. Both Goodearl-Warfield [18] and McConnell-Robson [34] use the phrase "prime submodule" to mean "submodule that is prime" (for example, by this notion every $\mathbb{Z}$-submodule of $M := \mathbb{Z}$ is a prime submodule); but Dauns [15] and McCasland-Smith [33] use the phrase "prime submodule" for a submodule P of M, such that M/P is a prime module i.e., for every ideal $\mathcal{A} \subseteq R$ and every submodule $N \subseteq M$, if $\mathcal{A}N \subseteq P$, then either $N \subseteq P$ or $\mathcal{A}M \subseteq P$ (see, for example, [6-12, 15, 23, 24, 25, 27-32, 41, 44]).

In this paper we use the latter notion of prime submodule, and we recall that the spectrum $Spec({}_RM)$ of a module M consists of all (Dauns)-prime submodules of M. As in [29], for any submodule N of a left R-module M we define $V(N)$ to be the set of all prime submodules of M containing N. Of course, $V(M)$ is just the empty set and $V(0)$ is $Spec({}_RM)$. Note that for any family of submodules N_i ($i \in I$) of M, $\bigcap_{i\in I} V(N_i) = V(\sum_{i\in I} N_i)$. Thus if $\mathcal{V}(M)$ denotes the collection of all subsets $V(N)$ of $Spec({}_RM)$, then $\mathcal{V}(M)$ contains the empty set and $Spec({}_RM)$, and $\mathcal{V}(M)$ is closed under arbitrary intersections. Unfortunately, in general, $\mathcal{V}(M)$ is not closed under finite union. A module M is called a top module if $\mathcal{V}(M)$ is closed under finite unions, i.e. for any submodules N and L of M there exists a submodule J of M such that $V(N) \cup V(L) = V(J)$, for in this case $\mathcal{V}(M)$ satisfies the axioms for the closed subsets of a topological space (see [29] for more details). A module M over a commutative ring R is called a multiplication module if each submodule of M is of the form IM, where I is an ideal of R (see for example, [1, 5, 16, 36, 44]). Any multiplication module over a commutative ring R is a top module, and a finitely generated module M over a commutative ring R is a top module if and only if M is a multiplication module (see [29, Theorem 3.5], and see [22], for another generalization of the Zariski topology to modules over commutative rings).

In this article, we introduce and study a new generalization of the Zariski topology of rings to modules. Let M be a left R-module. For each submodule N of M, we define $W(N) = Spec({}_RM) \setminus V(N)$ and put $\mathcal{W}(M) = \{W(N) : N \leq M\}$. Then we define $\mathcal{T}(M)$ to be the topology on $Spec({}_RM)$ by the sub-basis $\mathcal{W}(M)$ and call it the classical Zariski topology of M. In fact $\mathcal{T}(M)$ to be the collection U of all unions of finite intersections of elements of $\mathcal{W}(M)$ (see for example [26, Page 82] for the definition of basis and sub-basis). This notion of classical Zariski topology of a module is analogous to that of the weak Zariski topology of a ring (see [37, 38, 43], for more details; in fact, Sun [37] first investigated the weak Zariski topology on $Spec({}_R R)$). Also, the classical Zariski topology and the Zariski topology considered in [22] agree for all multiplication modules over a commutative ring R. But for a non-commutative ring R, the usual Zariski topology of the ring R is considered in [18] is a subspace of the classical Zariski topology of ${}_R R$

Modules whose classical Zariski topology is respectively T_1, Hausdorff or cofinite are established in the body of Section 2. For example we show that for each R-module M, $Spec({}_R M)$ is a T_1-space if and only if $dim(M) \leq 0$ ($dim(M)$ is prime dimension of M). In particular, if M is a finitely generated module, then $Spec({}_R M)$ is a T_1-space if and only if M is a multiplication module with $dim(M) = 0$. This yields that for a Noetherian left R-module M, $Spec({}_R M)$ is a T_1-space if and only if M is a cyclic Artinian module. In Theorem 2.22, we give a characterization for modules M for which the classical Zariski topology is the cofinite topology. Also, we show that over a commutative ring R, Noetherian modules whose classical Zariski topology is the cofinite topology are precisely the Artinian cyclic modules. In some instance we characterize modules whose classical Zariski topology is Hausdorff. For example, it is shown that for a semisimple module M, $Spec({}_R M)$ is a Hausdorff space if and only if $Spec({}_R M)$ is a T_1-space, if and only if M is a direct sum of non-isomorphic simple modules. In Section 3, we investigate this topological space $Spec({}_R M)$ from the point of view of spectral spaces, topological spaces each of which is homeomorphic to Spec(S) for some commutative ring S. Hochster [20] has characterized spectral spaces as quasi-compact T_0-spaces X such that X has a quasi-compact open basis closed under finite intersection and each irreducible closed subset of X has a generic point. We show that for each left R-module M, $Spec({}_R M)$ is always a T_0-space and each finite irreducible closed subset of $Spec({}_R M)$ has a generic point, but $Spec({}_R M)$ is not quasi-compact in general (Proposition 3.8 and Example 2.23). This yields that for each left R-module M with finite spectrum, $Spec({}_R M)$ is always a spectral space. Finally, in Corollary 3.12 we show that for every finitely generated multiplication module M over a commutative ring R, $Spec({}_R M)$ is a spectral space.

2. Modules whose classical Zariski topology is respectively T_1, Hausdorff or cofinite

Let M be a left R-module. Recall that a proper submodule P of M is called a semiprime submodule if $\mathcal{A}^2 N \subseteq P$, where $N \leq M$ and $\mathcal{A}$ is an ideal of R, then $\mathcal{A}N \subseteq P$. Also, M is called a semiprime module if $(0) \subsetneq M$ is semiprime.

For an element $a \in R$, let us write (a) for the ideal generated by a in R. The following two propositions offer several characterizations of prime submodules and semiprime submodules respectively (see also [6, 8]).

Proposition 2.1. *Let M be a left R-module. For a proper submodule P of M, the following statements are equivalent:*

(1) P is prime.
(2) For $a \in R$ and $m \in M$, $(a)m \subseteq P$ implies that $m \in P$ or $(a)M \subseteq P$.
(3) For $a \in R$ and $m \in M$, $a(Rm) \subseteq P$ implies that $m \in P$ or $aM \subseteq P$.
(4) For a left ideal $\mathcal{A}$ in R and $m \in M$, $\mathcal{A}(Rm) \subseteq P$ implies that $m \in P$ or $\mathcal{A}M \subseteq P$.
(5) For a right ideal $\mathcal{A}$ in R and $m \in M$, $\mathcal{A}m \subseteq P$ implies that $m \in P$ or $\mathcal{A}M \subseteq P$.
(6) for every $0 \neq \overline{m} \in M/P$, $(0 : R\overline{m})$ is a prime ideal and $(0 : R\overline{m}) = (P : M)$.
(7) $(P : M)$ is a prime ideal and the set $\{(0 : R\overline{m}) : \ 0 \neq \overline{m} \in M/P\}$ is singleton.

Proposition 2.2. *Let M be a left R-module. For a proper submodule P of M, the following statements are equivalent:*

(1) P is semiprime.
(2) For $a \in R$ and $m \in M$, $(a)^2 m \subseteq P$ implies that $(a)m \subseteq P$.
(3) For $a \in R$ and $m \in M$, $aRa(Rm) \subseteq P$ implies that $aRm \subseteq P$.
(4) For any left ideal $\mathcal{A}$ in R and $m \in M$, $\mathcal{A}^2(Rm) \subseteq P$ implies that $\mathcal{A}(Rm) \subseteq P$.
(5) For any right ideal $\mathcal{A}$ in R and $m \in M$, $\mathcal{A}^2 m \subseteq P$ implies that $\mathcal{A}m \subseteq P$.
(6) for every $0 \neq \overline{m} \in M/P$, $(0 : R\overline{m})$ is a semiprime ideal.

In this case $(P : M)$ is a semiprime ideal of R.

Let R be a ring and M be a left R-module. A submodule P of M will be called maximal prime if P is a prime submodule of M and there is no prime submodule Q of M such that $P \subset Q$. Also, P is called virtually maximal if the factor module M/P is a homogeneous semisimple module (see for example [6,12], for various other maximality conditions on submodules and relationship between those conditions).

Next, we shall investigate the cases M when satisfies the following condition:

$(*)$ For any submodules N_1, $N_2 \leq M$, $V(N_1) = V(N_2)$ implies that $N_1 = N_2$.

Let R be a simple ring and M be a nonzero R-module. By Proposition 2.1, every proper submodule of M is a prime submodule. Thus one can easily see that M satisfies the $(*)$ condition. In particular every vector space satisfies the $(*)$ condition.

Proposition 2.3. *Let M be a nonzero left R-module. Then the following statements are equivalent:*
(1) M satisfies the $(*)$ condition.
(2) Every proper submodule of M is an intersection of prime submodules.

Proof. (1) $\Rightarrow$ (2) Assume N_1 is a proper submodule of M. We claim that $V(N_1) \neq \emptyset$, for if not, then $V(N_1) = V(M) = \emptyset$ and so $N_1 = M$, a contradiction. Now let $N_2 = \bigcap_{P \in V(N_1)} P$. Clearly $V(N_1) = V(N_2)$, and so by our hypothesis $N_1 = N_2$. It follows that N_1 is an intersection of prime submodules.
(2) $\Rightarrow$ (1) Clearly, a submodule N of M is an intersection of prime submodules if and only if $N = \bigcap_{P \in V(N)} P$. Thus we are thorough. □

We recall that if U, M are $R-$modules, then following Azumaya U is called $M-$injective if for any submodule N of M, each homomorphism $N \longrightarrow U$ can be extended to $M \longrightarrow U$ and, a left R-module M is called co-semisimple if every simple module is $M-$injective i.e., every proper submodule of M is an intersection of maximal submodule (see for example [42]). Every semisimple module is of course co-semisimple. Thus by Proposition 2.3 we have the following corollary.

Corollary 2.4. *Every co-semisimple module M satisfies the $(*)$ condition.*

A prime ring R will be called left bounded if, for each regular element c in R, there exists an ideal A of R and a regular element d such that $Rd \subseteq A \subseteq Rc$. A general ring R will be called left fully bounded if every prime homomorphic image of R is left bounded. A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian. It is well known that if R is a PI-ring (ring with polynomial identity) and P is a prime ideal of R, then the ring R/P is (left and right) bounded and (left and right) Goldie (see [34, 13.6.6]).

Remark 2.5. In general the converse of Corollary 2.4 is not true. For example, any module M over a simple ring R satisfies the $(*)$ condition, but M is not necessarily a co-semisimple R-module. In this section we shall show that if R is a ring such that $R/\mathcal{P}$ is Artinian for every left primitive ideal $\mathcal{P}$, then the converse of Corollary 2.4 is true for all R-modules (see Theorem 2.9 and also Corollary 2.6).

We call an R-module M to be fully prime (respectively fully semiprime) if each proper submodule of M is prime (respectively semiprime). Fully prime and fully semiprime modules over commutative rings are characterized in [9] (for instance, it is shown that a module M over a commutative ring R is fully prime (respectively fully semiprime) if and only if M is a homogeneous semisimple (respectively co-semisimple module). The following corollary shows that over a commutative ring R, the set of all modules with $(*)$ condition and the set of all co-semisimple modules coincide.

Corollary 2.6. *Let R be a commutative ring and M be an R-module. Then the following statements are equivalent:*

(1) M satisfies the $(*)$ condition.
(2) M is a fully semiprime module.
(3) M is a co-semisimple module.

Proof. (1) $\Rightarrow$ (2) is by Proposition 2.3.
(2) $\Rightarrow$ (3) is by [9, Theorem 2.3].
(3) $\Rightarrow$ (1) is by Corollary 2.4. □

In [9, Corollary 1.9], the authors proved that a co-semisimple module M over a commutative ring R is prime if and only if M is a homogeneous semisimple module. Thus we have the following corollary.

Corollary 2.7. *Let R be a commutative ring and M be a prime R-module. Then the following statements are equivalent:*

(1) M satisfies the $(*)$ condition.
(2) M is fully prime.
(3) M is a homogeneous semisimple module.

Lemma 2.8. *Let R be a PI-ring (or an FBN-ring), and let M be a left R-module. If either R is an Artinian ring or M is an Artinian module, then M has a maximal submodule if and only if M has a prime submodule. In addition if M has a prime submodule, then every prime submodule of M is an intersection of maximal submodules.*

Proof. First, we assume that R is a left Artinian PI-ring (or an FBN-ring), M is a left R-module and P is a prime submodule of M. Then $\mathcal{P} = (P : M)$ is a maximal (prime) ideal of R and so the ring $\overline{R} := R/\mathcal{P}$ is simple Artinian. Thus $\overline{M} = M/P$ is a direct sum of isomorphic simple $\overline{R}$-modules. It follows that $\overline{M}$ is a homogeneous semisimple R module. Thus P is an intersection of maximal submodules of M. Now let R be a PI-ring (or an FBN-ring), M be an Artinian module and P be a prime submodule of M.

Suppose that $\mathcal{P} = (P : M)$, $\overline{M} := M/P$ and $\overline{R} := R/\mathcal{P}$. Since M is an Artinian R-module, $\overline{M}$ is an Artinian $\overline{R}$-module. Suppose $\overline{R}\overline{m}$ is a simple submodule of $\overline{M}$. Since $\overline{M}$ is a prime module, $Ann(\overline{R}\overline{m}) = Ann(\overline{M})$ and so $\overline{R}\overline{m}$ is an $\overline{R}$-module. Since $\overline{R}$ is a prime left bounded, left Goldie ring, by [18, Proposition 9.7], $\overline{R}$ embeds as a left R-module in some finite direct sum of copies of R$\overline{m}$. Thus $\overline{R}$ is Artinian and simple, therefore $\overline{R}$-module $\overline{M}$ is a direct sum of copies of isomorphic simple modules. It follows that $\overline{M}$ is a homogeneous semisimple $\overline{R}$-module. Thus P contain in maximal submodule of M. □

Theorem 2.9. *Let R be a ring such that $R/\mathcal{P}$ is Artinian for every left primitive ideal $\mathcal{P}$. Then the left R-module M satisfies condition $(*)$ if and only if M is co-semisimple.*

Proof. ($\Leftarrow$) Clear by Proposition 2.3.
($\Rightarrow$) Suppose that M satisfies $(*)$. Suppose that $Rad(M)$ is non-zero. Let m be any non-zero element of $Rad(M)$. Let K be any maximal submodule of Rm and let $\mathcal{P}$ denote the annihilator in R of M/K. By hypothesis, $R/\mathcal{P}$ is simple Artinian and hence $M/\mathcal{P}M$ is a semisimple R-module. In particular, $Rad(M)$ is contained in PM and hence m belongs to $\mathcal{P}M$. Next K is an intersection of prime submodules P_i where i belongs to some index set I. For each i in I, $\mathcal{P}m$ is contained in P_i and hence $\mathcal{P}M$ is contained in P_i or m belongs to P_i. In any case, m belongs to P_i. Thus the element m belongs to every prime submodule P_i so that m belongs to K, a contradiction. Thus $Rad(M) = (0)$. By Proposition 2.3, every homomorphic image of M also satisfies $(*)$. Thus every homomorphic image of M has zero radical. Therefore M is co-semisimple. □

Let R be a ring. If ${}_RR$ is a co-semisimple module, then the ring R is called left co-semisimple or a left V-ring. If R is a left V-ring, then $J^2 = J$ for every left ideal $J \subseteq R$ and the center $Z(R)$ is a (von Neumann) regular ring, and a commutative ring is a (left) V-ring if and only if it is regular (see [42, 23.5]). Moreover, Armendariz and Fisher show in [3, Theorem 1] that a PI-ring R is a (left) V-ring if and only if $I^2 = I$ for each two-sided ideal I of R. In the next theorem we show that if R is left V-ring, then every R-module satisfies the $(*)$ condition and the converse is also true for PI-rings.

Theorem 2.10. *Consider the following statements for a ring R.*

(1) R is a left V-ring.

(2) Every left R-module satisfies the $(*)$ condition.

(3) The left R-module R satisfies the $(*)$ condition.

(4) Every proper ideal of R is semiprime.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. Moreover, if $R/\mathcal{P}$ is Artinian for every left primitive ideal $\mathcal{P}$, then $(3) \Rightarrow (1)$. Also if R is a PI-ring, then $(4) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ Let R be a left V-ring. Then by [42, 23.5], every R-module is co-semisimple. Thus by Corollary 2.4, every R-module satisfies the $(*)$ condition.
$(2) \Rightarrow (3)$ is clear.
$(3) \Rightarrow (4)$ Let ${}_RR$ satisfy the $(*)$ condition. Then by Proposition 2.3, every proper left ideal of R is semiprime and hence, every proper ideal of R is a semiprime ideal. Now suppose that $R/\mathcal{P}$ is Artinian for every left primitive ideal $\mathcal{P}$. Suppose that (3) holds. Then by Theorem 2.9, the left R-module R is co-semisimple.

Finally, we assume that R is a PI-ring. Suppose that (4) holds. Thus for each two-sided ideal I of R, I^2 is a semiprime ideal and so $I^2 = I$. Therefore R is a left V-ring by Armendariz-Fisher [3, Theorem 1]. □

Proposition 2.11. *Let R be a PI-ring. If every R-module M satisfies the $(*)$ condition, then for every prime ideal P of R the ring R/P is simple Artinian.*

Proof. Assume that R is a PI-ring and every R-module M satisfies the $(*)$ condition. Then by Proposition 2.3, every nonzero R-module has a semiprime submodule. Now by [6, Theorem 5.6], for every prime ideal P of R the ring R/P is simple Artinian. □

One can easily see that for any left R-module M, if $\mathrm{Hom}(M, R/rad(R)) \neq 0$, where $rad(R)$ denote the nil radical of R, then M contains a prime submodule (see also [33, Corollary 1.3]). It is also easy to show that whenever M is a left R-module and $\mathcal{P}$ is a maximal ideal of R with $M \neq \mathcal{P}M$, then each proper submodule of M containing $\mathcal{P}M$ is a prime submodule. Thus, we can naturally provide nontrivial rings over which every module has a prime submodule, simply take a maximal ideal $\mathcal{P}$ in any ring R, then the rings $R/\mathcal{P}^n$, n = 1, 2, 3, ..., give us some natural examples. We recall that a ring R is called a P-ring if every nonzero R-module has a prime submodule. In [6, Theorem 5.6], it is shown that if R is a PI-ring, then R is a P-ring if and only if R is a Max-ring i.e., every nonzero R-module has a maximal submodule. Commutative Max-rings were first characterized in [19] as rings R such $R/J(R)$ is regular and $J(R)$ is T-nilpotent, where $J(R)$ is the Jacobson radical of R (see also [14, 17, 40]). Thus by Proposition 2.11, we have the following result.

Proposition 2.12. *Let R be a ring. Then:*

(a) If every R-module M satisfies the $(*)$ condition, then R is a P-ring.
(b) If R is a PI-ring and every R-module M satisfies the $(*)$ condition, then R is a Max-ring.

Proof. (a) Assume that every R-module satisfies the $(*)$ condition. Then by Proposition 2.3, every nonzero R-module has a prime submodule i.e., R is a P-ring. (b) follows from (a) and [6, Theorem 5.6]. □

Corollary 2.13. *Let R be a semiprime commutative ring. Then the following statements are equivalents:*

(1) Every R-module satisfies the $(*)$ condition.
(2) R is a Max-ring.
(3) $dim(R) = 0$.

Proof. (1) $\Rightarrow$ (2) is by Proposition 2.12 (ii).
(2) $\Leftrightarrow$ (3) is by [31, Remark 4.37].
(2) $\Leftrightarrow$ (1) Assume that R is a Max-ring. Then by [19], $R/J(R)$ is a regular ring and $J(R)$ is T-nilpotent, where $J(R)$ is the Jacobson radical of R. Since R is semiprime, $J(R) = 0$ and so R is a regular ring. Now by Theorem 2.10, every R-module satisfies the $(*)$ condition. □

In the literature, there are two different generalizations of the classical Krull dimension for modules via prime submodules. In fact, the notion of prime dimension of a module M over a commutative ring R [denoted by $D(M)$ or $dim(M)$], was introduced by Marcelo and Masqué [27], as the maximum length of the chains of prime submodules of M (see also [25]; for some known results about the prime dimension of modules). Also, the classical Krull dimension of rings has been extended to modules $_RM$ by Behboodi [6], as the maximum length of the strong chains of prime submodules of M (allowing infinite ordinal values) and denoted by $Cl.K.dim(M)$. (Note: the chain $N_1 \subset_s N_2 \subset_s N_3 \subset_s \cdots$ of submodules of M is called a strong ascending chain if for each $i \in \mathbb{N}$, $N_i \subsetneqq N_{i+1}$ and also $(N_i : M) \subsetneqq (N_{i+1} : M)$; see also [12]; for definition of a strong descending chain).

Let R be a ring and M be a left R-module. We recall the definition of the prime dimension of M. Let every prime submodule of M is contained in a maximal prime submodule. We define, by transfinite induction, sets X_α of prime submodules of M. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined for all ordinals $\beta < \alpha$, let X_α be the set of those prime submodules P in M such that all prime submodules proper containing P belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of maximal prime submodules of M.) If some X_γ contains all prime submodules of M, we say that $dim(M)$ exists, and we set $dim(M)$-*the prime dimension* of M-equal to the smallest such γ. We write "$dim(M) = \gamma$" as an abbreviation for the statement that $dim(M)$ exists and equal γ. In fact, if $dim(M) = \gamma < \infty$, then

$$dim(M) = sup\{ht(P) \mid P \ is\ a\ prime\ submodule\ of\ M\},$$

where $ht(P)$ is the greatest non-negative integer n such that there exists a chain of prime submodules of M

$$P_0 \subset P_1 \subset \cdots \subset P_n = P,$$

and $ht(P) = \infty$ if no such n exists.

Let X be a topological space and let x and y be points in X. We say that x and y can be separated if each lies in an open set which does not contain the other point. X is a T_1- space if any two distinct points in X can be separated. A topological space X is a T_1-space if and only if all points of X are closed in X (i.e., given any x in X, the singleton set $\{x\}$ is a closed set.

Theorem 2.14. *Let M be a left R-module. Then $Spec(_RM)$ is a T_1-space if and only if $dim(M) \leq 0$.*

Proof. ($\Rightarrow$) Assume that $Spec(_RM)$ is a T_1-space. If $Spec(_RM) = \emptyset$, then $dim(M)$ $= -1$. Let $Spec(_RM) \neq \emptyset$ and $P_1 \in Spec(_RM)$. Then $\{P_1\}$ is a closed set in $Spec(_RM)$. We claim that every prime submodule of M is a maximal prime submodule, for if not, we assume that $P_1 \subsetneqq P_2$, where P_1, P_2 are prime submodules of M. Since $\{P_1\}$ is a closed set, $\{P_1\} = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(N_{i,j}))$, where $N_{i,j} \leq M$ and I is an index set. Thus we conclude that $P_1 \in \bigcup_{j=1}^{n_i} V(N_{i,j})$, for all $i \in I$, and hence, there exists $1 \leq j_i \leq n_i$ such that $P_1 \in V(N_{i,j_i})$. Since $P_1 \subsetneqq P_2$, $P_2 \in V(N_{i,j_i})$ for all $i \in I$. It follows that $P_2 \in \bigcup_{j=1}^{n_i} V(N_{i,j})$, for all $i \in I$. Thus $P_2 \in \{P_1\}$, a contradiction. Thus every prime submodule of M is a maximal prime submodule.

($\Leftarrow$) Suppose that $\dim(M) \leq 0$. If $dim(M) = -1$, then $Spec({}_RM) = \emptyset$ i.e., $Spec({}_RM)$ is trivial space and so it is a T_1-space. Now let $\dim(M) = 0$, i.e., $Spec({}_RM) \neq \emptyset$ and every prime submodule of M is a maximal prime submodule. Thus for each prime submodule P of M, $V(P) = \{P\}$, and so $\{P\}$ is a closed set in $Spec({}_RM)$ i.e., $Spec({}_RM)$ is a T_1-space. □

Let M be a finitely generated (or co-semisimple) module. Since every prime submodule of M is contained in a maximal submodule, by Theorem 2.14, $Spec({}_RM)$ is a T_1-space if and only if $Spec({}_RM) = Max(M)$. Moreover, we have the following interesting result.

Proposition 2.15. *For every finitely generated R-module M, the following statements are equivalent:*

(1) M is a co-semisimple module with $dim(M) = 0$.
(2) $Spec({}_RM)$ is a T_1-space and M satisfies the $(*)$ condition.

Proof. (1) $\Rightarrow$ (2) Since M is co-semisimple, by Corollary 2.4, M satisfies the $(*)$ condition and since $Spec({}_RM) = Max(M)$, by Theorem 2.14, $Spec({}_RM)$ is a T_1-space.
(2) $\Rightarrow$ (1) Since M is finitely generated, every maximal prime submodule of M is a maximal submodule. Thus by Theorem 2.14, $Spec({}_RM) = Max(M)$ and so by Proposition 2.3, every submodule of M is an intersection maximal submodule i.e., M is a co-semisimple module. □

Remark 2.16. The assumption finitely generated is necessary (even if R is commutative) in the corollary above. For example if $R = \mathbb{Z}$ and $M = \mathbb{Q}$, then the zero submodule of M is the only prime submodule. Thus X_M is a T_1-space and M satisfies the $(*)$ condition, but M is not a co-semisimple $\mathbb{Z}$-module.

Theorem 2.17. *Let M be a finitely generated module over a commutative ring R. Then the following statements are equivalent:*

(1) $Spec({}_RM)$ is a T_1-space.
(2) M is a multiplication module with $dim(M) = 0$.

Proof. (1) $\Rightarrow$ (2) Let $Spec({}_RM)$ is a T_1-space. Since M is finitely generated, by Theorem 2.14, $dim(M) = 0$. Thus by [44, Corollary 4.15], M is a multiplication module.
(2) $\Rightarrow$ (1) is by Theorem 2.14. □

Corollary 2.18. *Let M be a Noetherian module over a commutative ring R. Then the following statements are equivalent:*

(1) $Spec(_RM)$ is a T_1-space.
(2) M is a multiplication module with $dim(M) = 0$.
(3) M is a cyclic Artinian module.

Proof. (1) $\Leftrightarrow$ (2) is by Theorem 2.17.
(2) $\Rightarrow$ (3) Let M is a multiplication module with $dim(M) = 0$. Then by [6, Theorem 4.9], M is Artinian. Thus by [16, Corollary 2.9], M is a cyclic Artinian module.
(3) $\Rightarrow$ (1) Since every cyclic module is multiplication, by Theorem 2.17 is clear. □

Proposition 2.19. *Let R be a PI-ring (or an FBN-ring), and let M be a left R-module. If either R is an Artinian ring or M is an Artinian module, then $Spec(_RM)$ is a T_1-space if and only if either $Spec(_RM) = \emptyset$ or $Spec(_RM) = Max(M)$.*

Proof. By Lemma 2.8 and Theorem 2.14. □

Proposition 2.20. *Let M be a semisimple R-module. If $Spec(_RM)$ is a T_1-space, then M is a direct sum of non-isomorphic simple modules.*

Proof. Let $Spec(_RM)$ is a T_1-space i.e., every prime submodule of M is maximal prime. Assume that Rm_1 and Rm_2 are two simple submodule of M such that $Rm_1 \cong Rm_2$. Thus $M = Rm_1 \oplus Rm_2 \oplus K$, for some $K \leq M$. Clearly, K and $Rm_2 \oplus K$ are prime submodules of M and so K is not a maximal prime submodule, a contradiction. □

Next, we show that the converse of Proposition 2.20 is true for modules over a PI-ring (or an FBN-ring).

Theorem 2.21. *Let M be a semisimple module over a PI-ring (or an FBN-ring) R. Then the following statements are equivalents:*

(1) $Spec(_RM)$ is a T_1-space;
(2) M is a direct sum of non-isomorphism simple modules.

Proof. (1) $\Rightarrow$ (2) is by Proposition 2.20.
(2) $\Rightarrow$ (1) Let $P \in Spec(_RM)$ with $\mathcal{P} = (P : M)$. Thus $\overline{M} := M/P$ is a prime $\overline{R} := R/\mathcal{P}$-module. Since M is semisimple, $\overline{M}$ is also a semisimple $\overline{R}$-module. Thus, $\overline{M}$ is a prime $\overline{R}$-module with $soc(\overline{M}) \neq 0$, and hence, by [6, Lemma 1.5], $\overline{M}$ is a homogenous semisimple $\overline{R}$-module. It follows that $\overline{M}$ is a homogenous semisimple R-module. Since $\overline{M} = M/P$ is isomorphic with a submodule of M, and every

submodule M is a direct sum of non-isomorphic simple module, M/P is a simple R-module, i.e. P is a maximal R-submodule of M. Thus $dim(M) = 0$ and so by Theorem 2.14, $Spec(_RM)$ is a T_1-space. □

The cofinite topology (sometimes called the finite complement topology) is a topology which can be defined on every set X. It has precisely the empty set and all cofinite subsets of X as open sets. As a consequence, in the cofinite topology, the only closed subsets are finite sets, or the whole of X. Then X is automatically compact in this topology, since every open set only omits finitely many points of X. Also, the cofinite topology is the smallest topology satisfying the T_1 axiom; i.e., it is the smallest topology for which every singleton set is closed. In fact, an arbitrary topology on X satisfies the T_1 axiom if and only if it contains the cofinite topology. If X is not finite, then this topology is not Hausdorff, regular or normal, since no two open sets in this topology are disjoint. One place where this concept occurs naturally is in the context of the Zariski topology. Since polynomials over a field K are zero on finite sets, or the whole of K, the Zariski topology on K (considered as affine line) is the cofinite topology.

Next, we give a characterization for a module M for which $Spec(_RM)$ is the cofinite topology.

Theorem 2.22. *Let M be a left R-module. Then the following statements are equivalent:*

(1) $Spec(_RM)$ is the cofinite topology.
(2) $dim(M) \leq 0$ and for every submodule N of M either $V(N) = Spec(_RM)$ or $V(N)$ is finite.

Proof. (1) $\Rightarrow$ (2) Suppose that $Spec(_RM)$ is the cofinite topology. Since every cofinite topology satisfies the T_1 axiom, by Theorem 2.14, $dim(M) \leq 0$. Suppose that there exists a submodule N of M such that contained in infinite number of prime submodules of M and $V(N) \neq Spec(_RM)$. Then $W(N)$ is an open set in $Spec(_RM)$ with infinite complement, a contradiction.
(2) $\Rightarrow$ (1) Suppose $dim(M) \leq 0$ and for every submodules N of M, $V(N) = Spec(_RM)$ or $V(N)$ is finite. Thus every finite union $\bigcup_{i=1}^{n} V(N_i)$ of submodules $N_i \leq M$ is also finite or $Spec(_RM)$. It follows that any intersection of finite union $\bigcap_{j\in J}(\bigcup_{i=1}^{n} V(N_{ji}))$ of submodules $N_{ji} \leq M$ is finite or $Spec(_RM)$. Therefore, every closed set in $Spec(_RM)$ is either finite or $Spec(_RM)$ i.e., $Spec(_RM)$ is the cofinite topology. □

Let M be a left R-module. Clearly if $dim(M) = -1$, then $Spec({}_RM)$ is trivial space and so it is cofinite. The following example shows that in general (even over a commutative ring), $dim(M) = 0 \not\Rightarrow Spec({}_RM)$ is cofinite.

Example 2.23. Let $R = \mathbb{Z}$ and

$$M = \sum_{p_i \in P} \mathbb{Z}_{p_i} = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots \oplus \mathbb{Z}_{p_i} \oplus \cdots$$

where P is the set of all non-negative prime numbers. Then by [6, Proposition 1.4], every prime $\mathbb{Z}$-submodule of M is virtually maximal. Thus for each prime submodule P of M, M/P is a homogeneous semisimple. Since M is a direct sum of non-isomorphic simple modules, M/P is simple i.e., P is a maximal submodule of M. Thus $dim(M) = 0$. Now we show that $Spec({}_RM)$ is not the cofinite topology. Clearly for each prime number p_j, $P_j := \sum_{p_j \neq p_i \in P} \mathbb{Z}_{p_i}$ is a maximal submodule of M and so $Spec({}_RM)$ is infinite. Let $N = \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus \cdots \leq M$. One can easily see that $V(N) = Spec({}_RM) \setminus \{P_1\}$. Thus $V(N)$ is infinite and $V(N) \neq Spec({}_RM)$ and hence by Theorem 2.22, $Spec({}_RM)$ is not the cofinite topology. Also, it is easy to check that $Spec({}_RM)$ is not quasi-compact.

Corollary 2.24. *Let M be a finitely generated module over a commutative ring R. If $Spec({}_RM)$ is the cofinite topology, then M is a multiplication module.*

Proof. Assume that $Spec({}_RM)$ is the cofinite topology. By Theorem 2.22, $dim(M) = 0$, since M is finitely generated. Thus by [44, Corollary 4.15], M is a multiplication module. □

Next, we show that over a commutative ring R, Noetherian modules whose classical Zariski topology is the cofinite topology are precisely the Artinian cyclic modules.

Corollary 2.25. *Let M be a module over a commutative ring R. Then M is Noetherian and $Spec({}_RM)$ is the cofinite topology if and only if M is an Artinian cyclic module.*

Proof. ($\Rightarrow$) Assume that M is Noetherian and $Spec({}_RM)$ is the cofinite topology. By Theorem 2.22 and Corollary 2.24, M is a multiplication module with $dim(M) = 0$. Then by [6, Theorem 4.9], M is an Artinian multiplication module. Now by [16, Corollary 2.9], M is cyclic.
($\Leftarrow$) Let M is an Artinian cyclic module. Clearly, M is Noetherian, every prime submodule of M is a maximal submodule and $Spec({}_RM)$ is finite and therefore Theorem 2.21 completes the proof. □

Suppose that X is a topological space. Let x and y be points in X. We say that x and y can be separated by neighborhoods if there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint ($U \bigcap V = \emptyset$). X is a Hausdorff space if any two distinct points of X can be separated by neighborhoods. This is why Hausdorff spaces are also called T_2-spaces or separated spaces.

Let M be a left R-module with $|Spec(_RM)| \leq 1$. Then $Spec(_RM)$ is the trivial space and so it is a Hausdorff space. The following proposition gives some properties of modules M with $|Spec(_RM)| \geq 2$ for which $Spec(_RM)$ is a Hausdorff space.

Proposition 2.26. *Let M be a left R-module with $|Spec(_RM)| \geq 2$. If $Spec(_RM)$ is a Hausdorff space, then $dim(M) = 0$ and there exist submodules N_1, N_2, ..., N_n of M such that $V(N_i) \neq Spec(_RM)$, for all i, and*

$$V(N_1) \cup V(N_2) \cup .... \cup V(N_n) = Spec(_RM).$$

Proof. Assume that $Spec(_RM)$ is a Hausdorff space and P_1, $P_2 \in Spec(_RM)$ such that $P_1 \neq P_2$. Then there exist open sets

$$\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij})), \quad \bigcup_{k\in K}(\bigcap_{l=1}^{m_k} W(N'_{kl})) \in \mathcal{T}(M), \quad N_{ij},\ N_{kl} \leq M$$

such that

$$P_1 \in \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij})),\ \textit{and}\ P_2 \in \bigcup_{k\in K}(\bigcap_{l=1}^{m_k} W(N'_{kl}))$$

and

$$[\ \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))\ ]\ \bigcap\ [\ \bigcup_{k\in K}(\bigcap_{l=1}^{m_k} W(N'_{kl}))\ ] = \emptyset.$$

Thus there exist $s \in I$, $t \in K$ such that

$$P_1 \in (\bigcap_{j=1}^{n_s} W(N_{sj})) \text{ and } P_2 \in (\bigcap_{l=1}^{m_t} W(N'_{tl})),$$

and also

$$(\bigcap_{j=1}^{n_s} W(N_{sj}))\ \bigcap\ (\bigcap_{l=1}^{m_t} W(N'_{tl})) = \emptyset.$$

It follows that $P_1 \not\subseteq P_2$ and $P_2 \not\subseteq P_1$. Thus $dim(M) = 0$ and also

$$(\bigcup_{j=1}^{n_s} V(N_{sj}))\ \bigcup\ (\bigcup_{l=1}^{m_t} V(N'_{tl})) = Spec(_RM).$$

□

It is well-known that if X is a finite space, then X is a T_1-space if and only if X is the discrete space. Thus we have the following corollary.

Corollary 2.27. *Let M be an R-module such that $Spec(_RM)$ is finite. Then the following statements are equivalent:*

(1) $Spec(_RM)$ is a Hausdorff space.
(2) $Spec(_RM)$ is a T_1-space.
(3) $Spec(_RM)$ is the cofinite topology.
(4) $Spec(_RM)$ is discrete .
(5) $dim(M) \leq 0$.

Minimal prime submodules are defined in a natural way. By Zorn's Lemma one can easily see that each prime submodule of a module M contains a minimal prime submodule of M. In [39], it is shown that Noetherian modules contain only finitely many minimal prime submodules.

Corollary 2.28. *Let M be a nonzero Noetherian left R-module. Then the following statements are equivalent:*

(1) $Spec(_RM)$ is a Hausdorff space.
(2) $Spec(_RM)$ is a T_1-space.
(3) $Spec(_RM)$ is the cofinite topology.
(4) $Spec(_RM)$ is discrete .
(5) $Spec(_RM) = Max(M)$.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) Assume that $Spec(_RM)$ is a T_1-space. By Theorem 2.14, $dim(M) \leq 0$. Since M is Noetherian, by [33, Theorem 4.2], $Spec(_RM)$ is finite. Thus $Spec(_RM)$ is a cofinite topology.
(3) $\Rightarrow$ (4) Assume that $Spec(_RM)$ is the cofinite topology. Then by Theorem 2.22, $dim(M) \leq 0$ and so by [33, Theorem 4.2], $Spec(_RM)$ is finite. Now by corollary above $Spec(_RM)$ is discrete .
(4) $\Rightarrow$ (5) Let $Spec(_RM)$ be a discrete space. Since $Spec(_RM)$ is a T_1-space, by Theorem 2.22, $dim(M) = 0$. Since M is Noetherian, every prime submodule of M is maximal i.e., $Spec(_RM) = Max(M)$.
(5) $\Rightarrow$ (1) Since M is Noetherian, M contains only a finite number of minimal prime submodules. Therefore $Spec(_RM) = Max(M)$ implies that $Spec(_RM)$ is finite and $dim(M) \leq 0$. Now by Corollary 2.27, $Spec(_RM)$ is a Hausdorff space. □

Clearly any Noetherian module satisfies ascending chain condition (ACC) on semiprime submodules (so on intersection of prime submodules). We shall show that any Artinian module M over a PI-ring (or an FBN-ring) R satisfies ACC on intersection of prime submodules. Note that any intersection of prime submodules of a module M is a semiprime submodule of M. In general, the converse; even, over a commutative Noetherian ring R with $dim(R) = 1$, is false (see for example [21, Corollary 12] and [8, Lemma 3.8 and Theorem 3.9]).

We recall that for a left R-module M the prime radical $rad_R(M)$ is defined to be the intersection of all prime submodules of M, and in case M has no prime submodule, then $rad_R(M)$ is defined to be M.

Proposition 2.29. *Let R be a ring such that $R/\mathcal{P}$ is Artinian for every left primitive ideal $\mathcal{P}$. Let M be an Artinian R-module. Then $rad_R(M) = M$ or $rad_R(M)$ is a finite intersection of prime submodules and $M/rad_R(M)$ is Noetherian. Consequently, M satisfies ACC on intersections of prime submodules.*

Proof. This follows directly from [35, Theorem 1.5] and its proof. □

The following is now immediate.

Corollary 2.30. *Let M be an Artinian module over a PI-ring (or an FBN-ring) R. Then $rad_R(M) = M$ or $rad_R(M)$ is a finite intersection of prime submodules and $M/rad_R(M)$ is Noetherian. Consequently, M satisfies ACC on intersections of prime submodules.*

Theorem 2.31. *Let M be an Artinian module over a PI-ring (or an FBN-ring) R. Then the following statements are equivalent:*

(1) $Spec(_RM)$ is a Hausdorff space.
(2) $Spec(_RM)$ is a T_1-space.
(3) $Spec(_RM)$ is the cofinite topology.
(4) $Spec(_RM)$ is discrete .
(5) Either $Spec(_RM) = \emptyset$ or $Spec(_RM) = Max(M)$.

Proof. (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (3) Let $Spec(_RM)$ be a T_1-space. By Proposition 2.22, either $Spec(_RM) = \emptyset$ or $Spec(_RM) = Max(M)$. Let $Spec(_RM) \neq \emptyset$. Then by Corollary 2.30, $M/rad_R(M)$ is Noetherian and $\dim(M/rad_R(M)) = 0$. Now by [33, Theorem 4.2], $X_{M/rad_R(M)}$ is finite. Clearly, the map $P \longrightarrow P + rad_R(M)$ is a bijective map from $Spec(_RM)$ to $X_{M/rad_R(M)}$. Thus $Spec(_RM)$ is finite and so by Corollary 2.27, $Spec(_RM)$ is the cofinite topology.
(3) $\Rightarrow$ (4) Assume that $Spec(_RM)$ is the cofinite topology. Then by Theorem 2.22,

$dim(M) \leq 0$. It follows that $Spec(_RM)$ is finite (the proof is similar to the proof of (2) $\Rightarrow$ (3)). Now by Corollary 2.27, $Spec(_RM)$ is discrete .
(4) $\Rightarrow$ (5) is by Proposition 2.22.
(5) $\Rightarrow$ (1) Clearly, $Spec(_RM) = \emptyset$ or $Spec(_RM) = Max(M)$ implies that $X_{M/rad_R(M)} = \emptyset$ or $X_{M/rad_R(M)} = Max(M/rad_R(M))$. On the other hand if $Spec(_RM) \neq \emptyset$, then by Corollary 2.30, $M/rad_R(M)$ is Noetherian and so $X_{M/rad_R(M)}$ is finite. Thus $Spec(_RM)$ is also finite. Now by Corollary 2.27, $Spec(_RM)$ is a Hausdorff space. □

Clearly for every simple module M, $Spec(_RM) = \{(0)\}$ and so $Spec(_RM)$ is the discrete space. Also, if R is an integral domain and Q is the filed of fraction of R, then the zero submodule of Q is the only prime submodule of Q. Thus $Spec(_RQ)$ is the discrete space. We are going to show if R is a commutative Noetherian integral domain with $dim(R) \leq 1$, then simple modules and the filed of fraction of R are the only prime modules for which the classical Zariski topology is a T_1-space (i.e., a discrete space).

Lemma 2.32. *Let M be a prime module. Then $Spec(_RM)$ is a T_1-space if and only if the zero submodule is the only prime submodule of M.*

Proof. It is evident by Theorem 2.14. □

Proposition 2.33. *Let R be a commutative Noetherian integral domain with $dim(R) \leq 1$, and let M be a prime module. Then the following are equivalent:*

(1) $Spec(_RM)$ is a T_1-space.
(2) $Spec(_RM)$ is the discrete space.
(3) M is a simple module or $M \simeq Q$, where Q is the filed of fraction of R.

Proof. By Lemma 2.32 and [9, Theorem 3.6]. □

We conclude this section with the following result for free modules.

Proposition 2.34. *Let R be a ring and M be a free R-module. Then $Spec(_RM)$ is a T_1-space if and only if $M \cong R$ and $dim(R) = 0$.*

Proof. Assume that $M = \bigoplus_{\lambda\in\Lambda} R$ and $Spec(_RM)$ is a T_1-space. Then by Theorem 2.14, $dim(M) \leq 0$. We claim that $|\Lambda| = 1$, for if not, we can write $M = R \oplus R \oplus X$ for some submodule X of M. Now, for each prime ideal P of R, it is easy to see that $P \oplus P \oplus X \subsetneqq P \oplus R \oplus X$ are two prime submodules of M, a contradiction. Thus $M \cong R$ and $dim(R) = 0$. The converse is clear. □

3. Modules whose classical Zariski topologies are spectral spaces

Let M be a left R-module and let $Spec({}_RM)$ be endowed with the classical Zariski topology. For each subset Y of $Spec({}_RM)$, We will denote the closure of Y in $Spec({}_RM)$ by $\overline{Y}$, and intersections of elements of Y by $\Im(Y)$ (note that if $Y = \emptyset$, then $\Im(Y) = M$).

A topological space X is called irreducible if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called an irreducible set if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy satisfy $Y \subseteq Y_1 \cup Y_2$, $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see, for example [13, page 94]).

We know that, for any commutative ring R, Spec(R) is always a T_0-space for the usual Zariski topology. This is not true for $Spec({}_RM)$ with the topology considered by Lu in [22] (see [22, page 429]).

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

Following Hochster [20], we say that a topological space X is a spectral space in case X is homeomorphic to Spec(S), with the Zariski topology, for some commutative ring S. Spectral spaces have been characterized by Hochster [20, p.52, Proposition 4] as the topological spaces X which satisfy the following conditions:

(i) X is a T_0-space;

(ii) X is quasi-compact;

(iii) the quasi-compact open subsets of X are closed under finite intersection and form an open basis;

(iv) each irreducible closed subset of X has a generic point.

For any commutative ring R, Spec(R) is well-known to satisfy these conditions (see [13, Chap II, 401-4.3]). However, for a module M over a commutative ring R, $Spec({}_RM)$ with the topology considered in [22] is not necessarily a spectral space in general. In fact in [22, Theorem 6.5], it is shown that for a module M over a commutative ring R, $Spec({}_RM)$ with the topology considered in [22] is a spectral space if and only if $Spec({}_RM)$ is a T_0-space, if and only if $|Spec_{\mathcal{P}}(M)| \leq 1$ for every $\mathcal{P} \in Spec(R)$, where $Spec_{\mathcal{P}}(M) = \{P \in Spec({}_RM) | (P : M) = \mathcal{P}\}$. This yields that if M is also finitely generated, then $Spec({}_RM)$ is a spectral space if and only if M is a multiplication module (see [22, Corollary 6.6]).

In this section, we will show that for any R-module M, $Spec(_RM)$ with the classical Zariski topology is a T_0-space, and each finite irreducible closed subset of $Spec(_RM)$ has a generic point. Then we observe $Spec(_RM)$ from the point of view of spectral topological spaces.

Proposition 3.1. *Let M be a left R-module, and let Y be a nonempty subset of $Spec(_RM)$. Then*

$$\overline{Y} = \bigcup_{P \in Y} V(P).$$

Proof. Clearly, $\overline{Y} \subseteq \bigcup_{P \in Y} V(P)$. Suppose A is any closed subset of X such that $Y \subseteq A$. Thus $A = \bigcap_{i \in I}(\bigcup_{j=1}^{n_i} V(N_{ij}))$, for some $N_{ij} \leq M$, $i \in I$ and $n_i \in \mathbb{N}$. Let $Q \in \bigcup_{P \in Y} V(P)$. Then there exists $P_0 \in Y$ such that $Q \in V(P_0)$ and so $P_0 \subseteq Q$. Since $P_0 \in A$, for each $i \in I$ there exists $j \in \{1, 2, \cdots, n_i\}$ such that $N_{ij} \subseteq P_0$, and hence $N_{ij} \subseteq P_0 \subseteq Q$. It follows that $Q \in A$. Therefore $\bigcup_{P \in Y} V(P) \subseteq A$. □

Now the above proposition immediately yields the following interesting result.

Corollary 3.2. *Let M be a left R-module. Then*

(a) $\overline{\{P\}} = V(P)$, for all $P \in Spec(_RM)$.
(b) $Q \in \overline{\{P\}}$ if and only if $P \subseteq Q$ if and only if $V(Q) \subseteq V(P)$.
(c) The set $\{P\}$ is closed in $Spec(_RM)$ if and only if P is a maximal prime submodule of M.

In [13, Proposition 14], it is shown that if R is a commutative ring, then a subset Y of $X = Spec(R)$ is irreducible if and only if $\Im(Y)$ is a prime ideal of R. In the next theorem we show that if $Y \subseteq Spec(_RM)$ and Y is irreducible, then $\Im(Y)$ is a prime submodule.

Lemma 3.3. *Let M be a left R-module. Then for each $P \in Spec(_RM)$, $V(P)$ is irreducible.*

Proof. Let $V(P) \subseteq Y_1 \bigcup Y_2$, where Y_1 and Y_2 are closed sets. Since $P \in V(P)$, either $P \in Y_1$ or $P \in Y_2$. Without loss of generality we can assume that $P \in Y_1$. We have $Y_1 = \bigcap_{i \in I}(\bigcup_{j=1}^{n_i} V(N_{ij}))$, for some I, n_i $(i \in I)$, and $N_{ij} \leq M$. Thus

$$P \in \bigcup_{j=1}^{n_i} V(N_{ij}), \text{ for all } i \in I.$$

It follows that

$$V(P) \subseteq \bigcup_{j=1}^{n_i} V(N_{ij}), \text{ for all } i \in I.$$

Therefore, $V(P) \subseteq Y_1$. Thus $V(P)$ is irreducible. □

Theorem 3.4. *Let M be a left R-module and $Y \subseteq Spec(_RM)$.*

(i) If Y is irreducible, then $\Im(Y)$ is a prime submodule.
(ii) If $\Im(Y)$ is a prime submodule and $\Im(Y) \in \overline{Y}$, then Y is irreducible.

Proof. ($\Rightarrow$) Suppose that Y is an irreducible subset of $Spec(_RM)$. Clearly, $\Im(Y) = \bigcap_{P\in Y} P \subsetneqq M$ and $Y \subseteq V(\Im(Y))$. Let $IK \subseteq \Im(Y)$. One can easily check that $Y \subseteq V(IK) \subseteq V(K) \cup V(IM)$. Since Y is irreducible, either $Y \subseteq V(K)$ or $Y \subseteq V(IM)$. If $Y \subseteq V(K)$, then $K \subseteq P$, for all $P \in Y$, i.e., $K \subseteq \Im(Y)$. If $Y \subseteq V(IM)$, then $IM \subseteq P$, for all $P \in Y$, i.e., $IM \subseteq \Im(Y)$. Thus by Proposition 2.1, $\Im(Y)$ is a prime submodule of M.
($\Leftarrow$) Suppose that $P := \Im(Y)$ is a prime submodule of M and $P \in \overline{Y}$. It is easy to check that $\overline{Y} = V(P)$. Now let $Y \subseteq Y_1 \cup Y_2$, where Y_1, Y_2 are closes sets. Thus $\overline{Y} \subseteq Y_1 \cup Y_2$. Since $V(P) \subseteq Y_1 \cup Y_2$ and by Lemma 3.3, $V(P)$ is irreducible, $V(P) \subseteq Y_1$ or $V(P) \subseteq Y_2$. It follows that either $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (since $Y \subseteq V(P)$). Thus Y is irreducible. □

The following example shows that the assumption that $\Im(Y) \in \overline{Y}$ is necessary in Theorem 3.4 (ii).

Example 3.5. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $Y = \{0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_3\}$. It is clear that $\Im(Y) = 0 \oplus 0 \oplus \mathbb{Z}_3$ is a prime submodule of M and $Y = V(0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3) \cup V(\mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_3) = \{0 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3\} \cup \{\mathbb{Z}_2 \oplus 0 \oplus \mathbb{Z}_3\}$. Thus Y is not irreducible.

Let R be a ring and M be a left R-module. For a submodule N of M, if there is a prime submodule containing N, then we define

$$\sqrt{N} = \bigcap\{P : P \text{ is a prime submodule of } M \text{ and } N \subseteq P\}.$$

If there is no prime submodule containing N, then we put $\sqrt{N} = M$. In particular, for any module M, we have $rad_R(M) = \sqrt{(0)}$.

By [13, Proposition 14], for each ideal I of a commutative ring R, $V(I)$ is an irreducible subset of Spec(R) (with the usual topology Zariski) if and only if $\sqrt{I}$ is a prime ideal. In the following corollary we extend this fact to modules over any ring.

Corollary 3.6. *Let M be a left R-module and $N \leq M$. Then the subset $V(N)$ of $Spec(_RM)$ is irreducible if and only if $\sqrt{N}$ is a prime submodule. Consequently, $Spec(_RM)$ is irreducible if and only if $rad_R(M)$ is a prime submodule*

Proof. ($\Rightarrow$) is by Theorem 3.4 (i).
($\Leftarrow$) Clearly for each submodule N of M, $V(N) = V(\sqrt{N})$. Now let $\sqrt{N}$ is a prime

submodule of M. Then $\sqrt{N} \in V(N)$, and hence by Theorem 3.4 (ii), $V(N)$ is irreducible. □

In the next theorem we characterize left Artinian modules M over a PI-ring (or an FBN-ring), modules M over a left Artinian PI-ring (or an FBN-ring) and left semisimple modules M over a PI-ring (or an FBN-ring) for which $Spec(_RM)$ is irreducible.

Theorem 3.7. *Let R be a PI-ring (or an FBN-ring), and let M be a nonzero left R-module. Then:*

(a) If R is an Artinian ring or M is left Artinian, then the following statements are equivalent:

(1) $Spec(_RM)$ is irreducible.

(2) $M/rad_R(M)$ is a nonzero homogeneous semisimple module.

(3) $Spec(_RM) \neq \emptyset$ and for each proper submodule N of M, either $V(N) = \emptyset$ or $V(N)$ is irreducible.

(b) If M is left semisimple, then the following statements are equivalent:

(1) $Spec(_RM)$ is irreducible.

(2) $M/rad_R(M)$ is a homogeneous semisimple module.

(3) For each proper submodule N of M, $V(N)$ is irreducible.

Proof. Part (a): (1) ⇒ (2) Assume that $Spec(_RM)$ is irreducible. Then $Spec(_RM) \neq \emptyset$ and by Corollary 3.6, $rad_R(M)$ is a prime submodule. If M is a left Artinian module, then by [6, Corollary 1.6], $rad_R(M)$ is a virtually maximal submodule of M, i.e., $M/rad_R(M)$ is a homogeneous semisimple module. If R is an Artinian ring, then by the proof of Lemma 2.8, $M/rad_R(M)$ is a homogeneous semisimple module. Now since $Spec(_RM) \neq \emptyset$, $rad_R(M)$ is a proper submodule of M and so $M/rad_R(M)$ is nonzero.

(2) ⇒ (3) Let N be a proper submodule of M. Clearly $rad_R(M) \subseteq \sqrt{N}$, and hence either $\sqrt{N} = M$ or $\sqrt{N}/rad_R(M)$ is a proper submodule of $M/rad_R(M)$. If $\sqrt{N} = M$, then $V(N) = \emptyset$. Thus we assume that $\sqrt{N} \neq M$. Since $M/rad_R(M)$ is a homogeneous semisimple module, $\sqrt{N}/rad_R(M)$ is a prime submodule of $M/rad_R(M)$ i.e., $\sqrt{N}$ is a prime submodule of M. Now by Corollary 3.6, $V(N)$ is irreducible.

(3) ⇒ (1) is clear (since $V(0) = Spec(_RM)$).

Part (b): Let M is left semisimple. Since every proper submodule of M is contained in a maximal submodule, $V(N) \neq \emptyset$, for every proper submodule N. On the other hand by [6, Corollary 1.6], every prime submodule of M is a virtually maximal submodule of M. Now the proof is similar to the proof of Part (a). □

We remark that any closed subset of a spectral space is spectral for the induced topology, and we note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space. The following proposition shows that for any R-module M, $Spec({}_RM)$ is always a T_0-space and every finite irreducible closed subset of $Spec({}_RM)$ has a generic point.

Proposition 3.8. *Let M be a left R-module. Then*

(i) $Spec({}_RM)$ is always a T_0-space.
(ii) Every $P \in Spec({}_RM)$ is a generic point of the irreducible closed subset $V(P)$.
(iii) Every finite irreducible closed subset of $Spec({}_RM)$ has a generic point.

Proof. (i) Let P_1, $P_2 \in Spec({}_RM)$. Then by Corollary 3.2 (a), $\{\overline{P_1}\} = \{\overline{P_2}\}$ if and only if $P_1 = P_2$. Now by the fact that a topological space is a T_0-space if and only if the closures of distinct points are distinct, we conclude that for any R-module M, $Spec({}_RM)$ is a T_0-space.
(ii) is clear by Corollary 3.2 (a).
(iii) Let Y be an irreducible closed subset of $Spec({}_RM)$ and $Y = \{P_1, P_2, \cdots, P_k\}$, where $P_i \in Spec({}_RM)$, $k \in \mathbb{N}$. By Proposition 3.1, $Y = \overline{Y} = V(P_1) \cup V(P_2) \cup .... \cup V(P_k)$. Since Y is irreducible, $Y = V(P_i)$ for some i $(1 \leq i \leq k)$. Now by (ii), P_i is a generic point of Y. □

We have not found any examples of a module M with an irreducible closed subset Y of $Spec({}_RM)$ such that Y has not a generic point. The lack of such counterexamples, together with the fact that every finite irreducible closed subset of $Spec({}_RM)$ has a generic point, motivates the following conjecture:

Conjecture. *For any R-module M, every irreducible closed subset of $Spec({}_RM)$ has a generic point.*

By Proposition 3.8, for any R-module M, $Spec({}_RM)$ is a T_0-space and every finite irreducible closed subset of $Spec({}_RM)$ has a generic point. We note that even if the conjecture above is true, then $Spec({}_RM)$ is not necessarily a spectral space in general; since $Spec({}_RM)$ is not quasi-compact in general (see Example 2.23), but in the main theorem of this section, we show that for each left R-module M with finite spectrum, $Spec({}_RM)$ is a spectral space.

Theorem 3.9. *Let M be a left R-module with finite spectrum. Then $Spec({}_RM)$ is a spectral space. Consequently, for each finite module M, $Spec({}_RM)$ is a spectral space.*

Proof. Since $Spec({}_RM)$ is finite, by Proposition 3.8, $Spec({}_RM)$ is a T_0-space and every irreducible closed subset of $Spec({}_RM)$ has a generic point. Again, finiteness of $Spec({}_RM)$ implies that the quasi-compact open subsets of $Spec({}_RM)$ are closed under finite intersection and form an open basis (note: this basis is $\mathcal{B} = \{W(N_1) \cap W(N_2) \cap \cdots \cap W(N_k) \ : \ N_i \leq M,\ 1 \leq i \leq k,\ for\ some\ k \in \mathbb{N}\}$). Now by the Hochster's characterization of a spectral space we conclude that $Spec({}_RM)$ is a spectral space. □

Remark 3.10. By Theorem 3.9, for any left R-module M with finite spectrum, there exists a commutative ring S such that the classical Zariski topology of M is homeomorphic to the usual Zariski topology of the ring S. This is interesting when $M = R$, R non-commutative ring, that in this case $\mathrm{Spec}({}_RR)$ is homeomorphic to $\mathrm{Spec}(S)$, for some commutative ring S.

It is clear that for a multiplication module M over a commutative ring R, the classical Zariski topology $\mathcal{T}(M)$ of M and the Zariski topology $\mathcal{V}^*(M)$ of M considered in [22], coincide (see [22, Example 1 (c)]). Following proposition shows that the converse is also true for all finitely generated modules over commutative rings.

Proposition 3.11. *Let R be a commutative ring and M be a finitely generated R-module. Then the classical Zariski topology of M and the Zariski topology of M considered in [22], coincide if and only if M is a multiplication module.*

Proof. Assume that the classical Zariski topology of M and the Zariski topology of M considered in [22], coincide. Then by Proposition 3.8, $Spec({}_RM)$ with the topology considered in [22] is a T_0-space. Now by [22, Corollary 6.6], M is a multiplication module. The converse is evident. □

Let R be a commutative ring and M be a finitely generated R-module. Then $Spec({}_RM)$ with the topology considered in [22] is a spectral space if and only if M is a multiplication module (see [22, Theorem 5.6 and Corollary 6.6]). Thus by proposition above we have the following interesting result.

Corollary 3.12. *Let R be a commutative ring. Then for every finitely generated multiplication R-module M, $Spec({}_RM)$ is a spectral space.*

Note: In Part II we shall continue the study of this construction.

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