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CLASSICAL ZARISKI TOPOLOGY OF MODULES AND SPECTRAL SPACES II

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Abstract. In this paper we continue our study of classical Zariski topology of modules, that was introduced in Part I (see [2]). For a left R-module M, the prime spectrum $Spec({}_RM)$ of M is the collection of all prime submodules. First, we study some continuous mappings which are induced from some natural homomorphisms. Then we generalize the patch topology of rings to modules, and show that for every left R-module M, $Spec({}_RM)$ with the patch topology is Hausdorf and it is disconnected provided $|Spec({}_RM)| > 1$. Next, by applying Hochster's characterization of a spectral space, we show that if M is a left R-module such that M has ascending chain condition (ACC) on intersection of prime submodules, then $Spec({}_RM)$ is a spectral space, i.e., $Spec({}_RM)$ is homeomorphic to $Spec(S)$ for some commutative ring S. This yields if M is a Noetherian left R-module or R is a PI-ring (or an FBN-ring) and M is an Artinian left R-module, then $Spec({}_RM)$ is a spectral space. Finally, we show that for every Noetherian left R-module M, $Max(M)$ (with the classical Zariski topology) is homeomorphic with the maximal ideal space of some commutative ring S.

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1. Introduction

The present paper is a sequel to [2] and so the notations introduced in Introduction of [2] will remain in force. In particular, all rings are associative rings with identity elements, and all modules are unitary left modules. Let M be a left R-module. If N is a submodule (respectively proper submodule) of M we write $N \leq M$ (respectively $N < M$). We denote the left annihilator of a factor module M/N of M by $(N :_R M)$. We call M faithful if $(0 :_R M) = 0$.

Let M be a left R-module. A proper submodule P of M is called a prime submodule if for every ideal $\mathcal{A}$ of R and every submodule $N \subseteq M$, if $\mathcal{A}N \subseteq P$, then

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either $N \subseteq P$ or $\mathcal{A}N \subseteq P$. We recall that the spectrum $Spec(_RM)$ of a module M consists of all prime submodules of M. For any submodule N of M we define $V(N)$ to be the set of all prime submodules of M containing N and call $V(N)$ the variety of N . Of course, $V(M)$ is just the empty set and $V(0)$ is Spec(M). Note that for any family of submodules N_i $(i \in I)$ of M, $\bigcap_{i\in I} V(N_i) = V(\sum_{i\in I} N_i)$. Thus if $\mathcal{V}(M))$ denotes the collection of all varieties $V(N)$ of Spec(M), then $\mathcal{V}(M)$ contains the empty set and Spec(M), and $\mathcal{V}(M)$ is closed under arbitrary intersections. Unfortunately, in general, $\mathcal{V}(M)$ is not closed under finite union (see for example [9]).

For a left R-module M we denote $Spec(_RM)$ by X_M. As in [2], we put $\mathcal{W}(M) = \{W(N) : W(N) = X_M - V(N)\ for\ some\ N \leq M\}$ and we define $\mathcal{T}(M)$ to be the collection U of all unions of finite intersections of elements of $\mathcal{W}(M)$. In fact, by [7, page 82], $\mathcal{T}(M)$ is a topology on X_M by the sub-basis $\mathcal{W}(M)$. We say that $\mathcal{T}(M)$ is the classical Zariski topology of M. Clearly, $\mathfrak{B} = \{W(N_1)\cap W(N_2)\cap\cdots\cap W(N_k)\ :\ N_i \leq M,\ 1 \leq i \leq k,\ for\ some\ k \in \mathbb{N}\}$ is a basis for this topology. In [2], it is shown that for each left R-module M, X_M is always a T_0-space and every finite irreducible closed subset Y of X_M has a generic point (i.e., Y is the closure of a unique point). In particular, for each left R-module M with finite spectrum, $Spec(_RM)$ is a spectral space, i.e., $Spec(_RM)$ is homeomorphic to Spec(S) for some commutative ring S (see [2, Theorem 3.9]).

In this article, we continue the study of this construction. In Section 1, we study some continuous mappings which are induced from some natural homomorphisms. It is shown that if $f : M \longrightarrow M'$ is an R-module epimorphism, then neutral map $\upsilon : X_{M'} \longrightarrow X_M$, given by $\upsilon(P) = f^{-1}(P)$ $(P \in X_{M'})$ is a continuous map. This yields that for each left R-module M, the natural map $\Phi_M : X_M \to X_{M/rad_R(M)}$ defined by $\Phi(P) = \overline{P}$, where $P \in X_M$ and $\overline{P} = P/rad(M)$, is a homeomorphism, and hence, $X_M \cong X_{M/rad_R(M)}$, where $rad_R(M)$ is the prime radical of M. Let $I_{M/rad_R(M)} = Ann(M/rad_R(M))$ and $\widehat{R} := R/I_{M/rad_R(M)}$. In Proposition 1.9, it is shown that for any left R-module M, the natural map $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$, given by $\Psi_M(P) = (P :_R M)/I_{M/rad_R(M)}$ is also a continuous map. Moreover, $\Psi_M = \Psi_{M/Rad_M} \circ \Phi_M$. In particular, if M is a Noetherian module over a commutative ring R, then Ψ_M is surjective, and if M is an Artinian module over a PI-ring (or an FBN-ring), then Ψ_M is surjective, both closed and open. Also, in Theorem 1.16, we give several characterizations of an Artinian module M over a PI-ring (or an FBN-ring), for which X_M is connected. In Section 2, we generalize the patch topology of rings to modules, and show that for any Noetherian left R-module M,

$Spec(_RM)$ with the patch topology is Hausdorff and compact. Moreover, if also $|Spec(_RM)| > 1$, then $Spec(_RM)$ is disconnected.

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. M. Hochster [5] has characterized spectral spaces as quasi-compact T_0-spaces X such that X has a quasi-compact open basis closed under finite intersection and each irreducible closed subset of X has a generic point. By [2, Proposition 3.8], for each left R-module M, X_M is always a T_0-space. In Theorem 2.4, we show that if M is a left R-module with ACC on intersection of prime submodules, then X_M is quasi-compact and has a basis of quasi-compact open subsets. Also, in Section 3, we show that every irreducible closed subset of X_M (where M has ACC on intersection of prime submodules), has a generic point, and hence, by applying Hochster's characterization of a spectral space, we conclude that; if M has ACC on intersection of prime submodules, then $Spec(_RM)$ is a spectral space, i.e., $Spec(_RM)$ is homeomorphic to $\mathrm{Spec}(S)$ for some commutative ring S. Also, we show that this fact is true for Artinian modules over a PI-ring (or an FBN-ring). Finally, we show that for every Noetherian left R-module M, $Max(M)$ (with the classical Zariski topology) is homeomorphic with the maximal ideal space of some commutative ring S (with the topology inherited from $\mathrm{Spec}(S)$).

1. Continuous mapping and homomorphisms

Let R be a ring and M, M' be R-modules. The following assertions are routine to check:

1. P is a prime submodule of M if and only if for each $a \in R$ and $m \in M$, $aRm \subseteq P$ implies that $m \in P$ or $aM \subseteq P$.

2. If P is a prime submodule of M, then for each submodule B of M, either $B \subseteq P$ or $P \cap B$ is prime in B (see also, [8, Lemma 3]).

3. If $f : M \longrightarrow M'$ is an R-module epimorphism, then there exists a bijection between X'_M and the set of all prime submodules of M containing kerf.

Proposition 1.1. *Let $f : M \longrightarrow M'$ be an R-module epimorphism, and let N be a submodule of M such that $ker f \subseteq N$.Then $V(N) \to V(f(N))$, given by $P \to f(P)$ is a bijection, unless $V(N)$ is the empty set, in which case so is $V(f(N))$.*

Proposition 1.2. *Let $f : M \longrightarrow M'$ be an R-module homomorphism. Define $\varphi : \mathcal{T}(M) \to \mathcal{T}(M')$ by $\varphi(\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(N_{ij}))) = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij})))$ where I is an index set, $n_i \in \mathbb{N}$ and $N_{ij} \leq M$. Then φ is a well-defined map.*

Proof. Assume $\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(N_{ij})) = \bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(K_{tj}))$ where $N_{ij}, K_{tj} \leq M$ and I, T are index set. We shall show that;

$$\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij}))) = \bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(f(K_{tj}))) \ (*)$$

Let $P \in \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij})))$. Then for each $i \in I$, there exists j_i $(1 \leq j_i \leq n_i)$ such that $P \in V(f(N_{ij_i}))$. If $P \supseteq f(M)$, then for each $t \in T$, and each j $(1 \leq j \leq n_t)$ we have $P \supseteq f(K_{tj})$. It follows that $P \in \bigcap_{i\in I}(\bigcup_{j=1}^{m} V(f(K_{tj})))$. Now let $P \not\supseteq f(M)$. Then $P \bigcap f(M)$ is prime submodule of $f(M)$. Thus for each $i \in I$, $f^{-1}(P \bigcap f(M)) \in V(N_{ij_i})$, and hence, $f^{-1}(P \bigcap f(M)) \in \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(N_{ij}))$. Therefore, $f^{-1}(P \bigcap f(M)) \in \bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(K_{tj}))$ and hence for each $t \in T$, there exists j_t $(1 \leq j_t \leq n_t)$ such that $f^{-1}(P \bigcap f(M)) \in V(K_{tj_t})$. It follows that for each $t \in T$, $f(K_{tj_t}) \subseteq P \bigcap f(M)$, i.e. $P \in V(f(K_{tj_t}))$. Consequently, we have $P \in \bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(f(K_{tj})))$. Thus $\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij}))) \subseteq \bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(f(K_{tj})))$. By a similar argument we can show that

$$\bigcap_{t\in T}(\bigcup_{j=1}^{n_t} V(f(K_{tj}))) \subseteq \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij}))).$$

Thus $(*)$ holds and this completes the proof. □

Theorem 1.3. *Let $f : M \longrightarrow M'$ be an R-module epimorphism such that M' is not primeless (i.e., $X_{M'} = Spec(M') \neq \emptyset$). Define $\upsilon : X_{M'} \longrightarrow X_M$ by $\upsilon(P) = f^{-1}(P)$ for each $P \in X_{M'}$. Then υ is a continuous map.*

Proof. Clearly, υ is well-defined. Let $V = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(N_{i,j}))$ be a closed set in X_M. We show that $\upsilon^{-1}(V) = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij})))$ and so it is a closed set in X'_M. Let $P \in \upsilon^{-1}(V)$. Then $f^{-1}(P) = \upsilon(P) \in V$. Thus for each $i \in I$ we have $f^{-1}(P) \in \bigcup_{j=1}^{n_i} V(N_{ij})$, and hence, for each $i \in I$ there exists j_i $(1 \leq j_i \leq n_i)$ such that $f^{-1}(P) \in V(N_{ij_i})$. Now by Proposition 1.1, $P \in V(f(N_{ij_i}))$. It follows that $P \in \bigcap_{i\in I} \bigcup_{j=1}^{n_i} V(f(N_{ij}))$. By a similar argument we can show that $\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(f(N_{ij}))) \subseteq \upsilon^{-1}(V)$ and so we are through. □

Recall that a function Φ between two topological spaces X and Y is called an open map if, for any open set U in X, the image $\Phi(U)$ is open in Y. Also, Φ is called a homeomorphism if it has the following properties

(i) Φ is a bijection;
(ii) Φ is continuous;
(iii) Φ is an open map (i.e., the inverse function Φ^{-1} is continuous).

If such a function exists we write $X \cong Y$ and we say X and Y are homeomorphic. The homeomorphisms form an equivalence relation on the class of all topological spaces. The resulting equivalence classes are called homeomorphism classes.

For a left R-module M, the prime radical of M (denoted by $rad_R(M)$) is defined to be the intersection of all prime submodules of M (note that, if M has no any prime submodule, then $rad_R(M) := M$). We recall that a proper submodule P of M is called a semiprime submodule of M if, for every ideal $\mathcal{A} \subseteq R$ and every submodule $N \subseteq M$, if $\mathcal{A}^2 N \subseteq P$, then $\mathcal{A}N \subseteq P$. Also, a left R-module M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. It is clear that for a submodule P of M, M/P is a semiprime module if and only if P is a semiprime submodule of M. Clearly, for each left R-module M, either $rad_R(M) = M$ or $rad_R(M)$ is a semiprime submodule of M.

The following proposition shows that the study of classical Zariski topology of general modules reduces to that of semiprime modules.

Proposition 1.4. *Let M be a left R-module. Then*

$$X_M \overset{\Phi_M}{\cong} X_{M/rad_R(M)}$$

where $\Phi_M : X_M \to X_{M/rad_R(M)}$ defined by $\Phi(P) = \overline{P}$, where $P \in X_M$ and $\overline{P} = P/rad_R(M)$.

Proof. We define $f : M \to M/rad_R(M)$ by $f(N_{ij}) = \overline{N}_{ij}$, where $N_{ij} \leq M$ and $\overline{N}_{ij} := N_{ij}/rad_R(M)$. Clearly,

$$Spec(M/rad_R(M)) = \{P/rad_R(M) : P \in Spec({}_R M)\},$$

and hence $\Phi_M : X_M \to X_{M/rad_R(M)}$ defined by $\Phi(P) = \overline{P}$ is a bijection. Now by Theorem 1.3, Φ^{-1} is continuous. We claim that Φ is a continuous map. Suppose $U = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\overline{N}_{ij}))$, where $N_{ij} \leq M$. We show that

$$\Phi^{-1}(U) = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij})).$$

Let $P \in \Phi^{-1}(U)$. Thus there exists $\overline{P} \in U$ such that $P = \Phi^{-1}(\overline{P})$, and hence, there exists $k \in I$ such that for each j $(1 \leq j \leq n_i)$ $\overline{P} \in W(\overline{N}_{kj})$, i.e. $\overline{N}_{kj} \nsubseteq \overline{P}$. It follows that $N_{kj} \nsubseteq P$ for each j $(1 \leq j \leq n_i)$. Thus $P \in \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))$. Therefore, $\Phi^{-1}(U) \subseteq \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))$. By a similar argument we can show that $\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij})) \subseteq \Phi^{-1}(U)$. Therefore we have $\Phi^{-1}(U) = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))$ and so Φ is continuous. $\square$

In topology, a subspace of a topological space X is a subset Y of X which is equipped with a natural topology induced from that of X called the subspace topology (or the relative topology, or the induced topology). It is well-known that if $Y \subseteq X$ and $\mathfrak{B}$ is a basis for X then $\mathfrak{B}_Y = \{Y \cap U | U \in \mathfrak{B}\}$ is a basis for Y.

Lemma 1.5. *Let M be a left R-module and $P \in Spec({}_RM)$. Let $V(P)$ be endowed with the induced topology of X_M. Then $V(P) \cong X_{M/P}$.*

Proof. Suppose P is a prime submodule of M. We define $\Phi : V(P) \to X_{M/P}$ with $\Phi(Q) = \overline{Q}$ for each $Q \in V(P)$, where $\overline{Q} = Q/P$. Clearly Φ is a bijection map. First we show that Φ is continuous. Let $U = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\overline{N}_{ij}))$ is an open subset of $X_{M/P}$, where $P \subseteq N_{ij} \leq M$ and $\overline{N}_{ij} = N_{ij}/P$. We claim that

$$\Phi^{-1}(U) = V(P)\bigcap[\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))]. \qquad (*)$$

Let $Q \in \Phi^{-1}(U)$. Thus $\overline{Q} \in U$ and so there exists $t \in I$ such that for each j $(1 \leq j \leq n_t)$ $\overline{Q} \in W(\overline{N}_{tj})$, i.e. $\overline{N}_{tj} \nsubseteq \overline{Q}$. It follows that $N_{tj} \nsubseteq Q$, for each j $(1 \leq j \leq n_t)$, and hence $Q \in V(P)\bigcap[\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))]$. Therefore $\Phi^{-1}(U) \subseteq V(P)\bigcap[\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))]$. By a similar argument we can show that $V(P)\bigcap[\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))] \subseteq \Phi^{-1}(U)$. Thus $(*)$ holds, and hence, since $\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))$ is an open subset of X_M, $\Phi^{-1}(U)$ is an open subset of $V(P)$. Thus Φ is continuous.

Now we show that Φ is open. Suppose U is an open subset of $V(P)$. Then $U = V(P)\bigcap[\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))]$, where $n_i \in \mathbb{N}$ and $N_{ij} \leq M$. We will show that

$$\Phi(U) = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\overline{P + N}_{ij})),$$

where $\overline{P + N}_{ij} = (P + N_{ij})/P$. Let $\overline{Q} \in \Phi(U)$, where $Q \leq M$ and $\overline{Q} = Q/P$. It follows that $Q \in U$ and so there exists $k \in I$ such that for each j $(1 \leq j \leq n_k)$ $Q \in W(N_{kj})\bigcap V(P)$, i.e. $N_{kj} \nsubseteq Q$ and $P \subseteq Q$. Therefore $\overline{P + N}_{ij} \subseteq \overline{Q}$ for each j $(1 \leq j \leq n_k)$. Thus $Q \in \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\overline{P + N}_{ij}))$, and hence $\Phi(U) \subseteq \bigcup_{i\in I}\bigcap_{j=1}^{n_i} W((\overline{P + N}_{ij}))$. By a similar argument we can show that

$$\bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\overline{P + N}_{ij})) \subseteq \Phi(U)$$

and this means that Φ is open. □

Proposition 1.6. *Let M be a Noetherian left R-module. Then*

$$X_M = V(P_1) \cup V(P_2) \cup ... \cup V(P_n),$$

where $P_1, P_2, ..., P_n$ are all minimal prime submodules of M. Moreover, for each i $(1 \leq i \leq n)$, $V(P_i)$ is a subspace of X_M such that $V(P_i) \cong X_{M/P_i}$.

Proof. Since M is a Noetherian left R-module, by [10, Theorem 4.2], M has a finite number of minimal prime submodules, and hence $X_M = V(P_1) \cup ... \cup V(P_n)$, where $P_1, P_2, ..., P_n$ are all minimal prime submodules of M. The "moreover" statement is clear by Lemma 1.5. □

A prime ring R will be called left bounded if, for each regular element c in R, there exists an ideal A of R and a regular element d such that $Rd \subseteq A \subseteq Rc$. A general ring R will be called left fully bounded if every prime homomorphic image of R is left bounded. A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian. It is well known that if R is a PI-ring (ring with polynomial identity) and P is a prime ideal of R, then the ring R/P is (left and right) bounded and (left and right) Goldie [11, 13.6.6].

We need the following Lemma of [2].

Lemma 1.7. ([2, Proposition 2.17]). *Let M be an Artinian module over a PI-ring (or an FBN-ring) R. Then $rad_R(M) = M$ or M has $rad_R(M)$ is a finite intersection of prime submodules and $M/rad_R(M)$ is a Noetherian left R-module.*

The following proposition shows that the facts of Proposition 1.6 are also true for Artinian modules over a PI-ring (or an FBN-ring).

Proposition 1.8. *Let R be a PI-ring (or an FBN-ring), and let M be an Artinian left R-module. Then $X_M = \emptyset$ or M has a finite number of minimal prime submodules. Consequently, $X_M = V(P_1) \cup V(P_2) \cup ... \cup V(P_n)$, where $P_1, P_2, ..., P_n$ are all minimal prime submodules of M and for each i $(1 \leq i \leq n)$, $V(P_i)$ is a subspace of X_M such that $V(P_i) \cong X_{M/P_i}$.*

Proof. Let $rad(M) \neq M$. Thus by Lemma 1.7, $M/rad_R(M)$ is a Noetherian left R-module, and hence by [10, Theorem 4.2], $M/rad_R(M)$ has a finite number of minimal prime submodules. It follows that M has a finite number of minimal prime submodules (see also the proof of Proposition 1.4). Let P_1, P_2, ..., P_n are all minimal prime submodules of M. Clearly, $X_M = V(P_1) \cup V(P_2) \cup ... \cup V(P_n)$. Now by Lemma 1.5, for each i $(1 \leq i \leq n)$, $V(P_i)$ is a subspace of X_M such that $V(P_i) \cong X_{M/P_i}$. □

Let M be a left R-module. Then by Proposition 1.4, $X_M \cong X_{M/rad_R(M)}$. Clearly for each prime submodule P of M, $rad_R(M) \subseteq P$ and so $P/rad_R(M)$ is a prime R-submodule of $M/rad_R(M)$. Thus

$$I_{M/rad_R(M))} := Ann(M/rad_R(M)) \subseteq (P/rad_R(M) : M/rad_R(M)) = (P : M),$$

and hence, $(\overline{P : M}) := (P : M)/I_{M/rad_R(M))}$ is a prime ideal of $R/I_{M/rad_R(M))}$. Now we assume that $\widehat{R} := R/I_{M/rad_R(M))}$ and we will use $X^{\widehat{R}}$ to represent Spec$(\widehat{R})$ with the usual Zariski topology of rings considered in [4, Chapter 16]. From the definition of the classical Zariski topology of M and by Proposition 1.4, it is evident that in some instance the topological space X_M is closely related to $X^{\widehat{R}}$, particularly, under the correspondence $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ defined by $\Psi_M(P) = (\overline{P : M})$ for every $P \in X_M$. For a module M, the above notations and the notion of Φ_M ($X_M \overset{\Phi_M}{\cong} X_{M/rad_R(M)}$) considered in Proposition 1.4, are fixed for this section.

Proposition 1.9. *For any left R-module M, the natural map $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ is a continuous map. Moreover, $\Psi_M = \Psi_{M/Rad_M} \circ \Phi_M$.*

Proof. Let A be a closed subset of $X^{\widehat{R}}$. Then $A = V(\overline{I})$ where $\overline{I} = I/I_{M/rad_R(M))}$ is an ideal of $\widehat{R}$. We claim that $\Psi_M^{-1}(V(\overline{I})) = V(IM)$. If $P \in \Psi_M^{-1}(V(\overline{I}))$, then $\overline{\mathcal{P}} \in V(\overline{I})$, where $\mathcal{P} = (P : M)$. This means that $\overline{I} \subseteq \overline{\mathcal{P}}$ i.e., $I \subseteq \mathcal{P}$, and hence $IM \subseteq P$ i,e., $P \in V(IM)$. Therefore, $\Psi_M^{-1}(V(\overline{I})) \subseteq V(IM)$. One can easily see that, $P \in V(IM)$ follows that $P \in \Psi_M^{-1}(V(\overline{I}))$. Thus $\Psi_M^{-1}(V(\overline{I})) = V(IM)$ for each ideal I of R such that $I_M \subseteq I$, i.e., the natural map Ψ_M is continues. For "moreover" statement we note that $X_{M/rad_R(M)} = \{P/rad_R(M) : P \in Spec({}_RM)\}$, and by Proposition 1.4, $X_M \cong X_{M/rad_R(M)}$ by the natural map

$$\Phi : X_{M/rad_R(M)} \to X_M$$

with $\Phi(\overline{P}) = P$ where $P \in X_M$ and $\overline{P} := P/rad_R(M)$. It is clear that for each $P \in X_M$, $(P :_R M) = (P/rad_R(M) :_R M/rad_R(M))$, i.e., $(\overline{P :_R M}) = (\overline{P/Rad_R(M) :_R M/rad_R(M)})$. This means that $\psi_{M/rad_R(M)}(\Phi^{-1}(P)) = \Psi_M(P)$, for each $P \in X_M$, i.e., $\Psi_{M/rad_R(M)} \circ \Phi = \Psi_M$. $\square$

Lemma 1.10. *Let R be a ring and M be a left R-module. Then for each maximal ideal $\mathcal{P}$ of R, either $\mathcal{P}M = M$ or $\mathcal{P}M$ is a prime submodule of M.*

Proof. Let $\mathcal{P}M \neq M$. Since $\mathcal{P}$ is maximal and $\mathcal{P} \subseteq (\mathcal{P}M : M)$, $(\mathcal{P}M :_R M) = \mathcal{P}$. Thus $R/\mathcal{P}$ is a simple ring and $M/\mathcal{P}M$ is a left $R/\mathcal{P}$-module. Clearly, every module over a simple ring is prime. Thus $M/\mathcal{P}M$ is a prime left $R/\mathcal{P}$-module. It follows that $\mathcal{P}M$ is a prime left R-submodule of M. $\square$

Lemma 1.11. *Let R be a PI-ring (or an FBN-ring), and let M be a homogeneous semisimple left R-module with $\mathcal{P} = Ann(M)$. Then $\mathcal{P}$ is a maximal ideal of R. Moreover, the ring $R/\mathcal{P}$ is simple Artinian.*

Proof. Since M is semisimple and homogeneous, $Ann(M) = Ann(Rm) = \mathcal{P}$, for all $0 \neq m \in M$. Let $0 \neq m \in M$. Then Rm is a simple $R/\mathcal{P}$-module. Since R is a PI-ring (or an FBN-ring), the ring $R/\mathcal{P}$ is a left bounded, left Goldie ring. Now [4, Proposition 9.7] gives that $R/\mathcal{P}$ embeds as a left R-module in a finite direct sum of copies of Rm. Thus $R/\mathcal{P}$ is a left Artinian ring, and hence, $R/\mathcal{P}$ is simple Artinian. □

We need the following lemma of [6].

Lemma 1.12. ([6, Theorem 2]). *Let R be a commutative ring, and let M a faithful Noetherian R-module. Then for each prime ideal $\mathcal{P}$ of R, there exists a prime submodule P of M such that $(P :_R M) = \mathcal{P}$.*

Corollary 1.13. *Let R be a commutative ring, and let M be a Noetherian R-module. Then then the natural map $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ is surjective.*

Proof. By Proposition 1.9, we have $\Psi_M = \Psi_{M/Rad_M} \circ \Phi_M$. Now apply Lemma 1.12. □

Let M be a left R-module. A submodule P of M will be called maximal prime if P is a prime submodule of M and there is no prime submodule Q of M such that $P \subset Q$. Also, P is virtually maximal if the factor module M/P is a homogeneous semisimple module (see for example [1] and [2] for others various maximality conditions on submodules and relationship between those conditions).

Lemma 1.14. *Let R be a PI-ring (or an FBN-ring), and let M be an Artinian left R-module $X_M \neq \emptyset$. Then $X^{\widehat{R}}$ is a discrete space.*

Proof. Let R be a PI-ring (or an FBN-ring), and let M be an Artinian left R-module. By Proposition 1.8, $rad_R(M) = P_1 \cap \cdots \cap P_n$ where $P_1, \cdots, P_n$ are all minimal prime submodules of M. It follows that

$$Ann(M/rad_R(M)) = \bigcap_{i=1}^{n} Ann(M/P_i) = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_n,$$

where $\mathcal{P}_i = (P_i :_R M)$ for each i $(1 \leq i \leq n)$. By [1, Corollary 1.6], for each i, P_i is a virtually maximal submodule of M, i.e., M/P_i is a homogeneous semisimple R-module. Thus by Lemma 1.11, each $\mathcal{P}_i$ is a maximal ideal of R and so $V(\mathcal{P}_i) = \{\mathcal{P}_i\}$, i.e. $X^{\widehat{R}}$ is T_1-space. Also, since $P_1, \cdots, P_n$ are all minimal prime submodules of M, $X^{\widehat{R}} = \{\overline{\mathcal{P}}_1, \cdots, \overline{\mathcal{P}}_n\}$. Therefore $X^{\widehat{R}}$ is a discrete space. $\square$

Theorem 1.15. *Let R be a PI-ring (or an FBN-ring), and let M be an Artinian left R-module with $X_M \neq \emptyset$. Then the natural map $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ is surjective, both closed and open.*

Proof. Since $X_M \neq \emptyset$, by Proposition 1.8, $rad_R(M) = P_1 \cap P_2 \cap ... \cap P_n$ where P_1, P_2, ..., P_n are all minimal prime submodules of M. It follows that $Ann(M/rad_R(M)) = \bigcap_{i=1}^{n} Ann(M/P_i) = \mathcal{P}_1 \cap \mathcal{P}_2 \cap ... \cap \mathcal{P}_n$, where $\mathcal{P}_i = (P_i :_R M)$ for each i $(1 \leq i \leq n)$. By [1, Corollary 1.6], for each i, P_i is a virtually maximal submodule of M, i.e., M/P_i is a homogeneous semisimple R-module. Thus by Lemma 1.11, each $\mathcal{P}_i$ is a maximal ideal of R. This yields that $X^{\widehat{R}} = \{\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2, ...\overline{\mathcal{P}}_n\}$, and hence the natural map $\Psi_{M/rad_R(M)} : X_{M/rad_R(M)} \longrightarrow X^{\widehat{R}}$ is surjective. On the other hand by Proposition 1.4, $X_M \overset{\Phi_M}{\cong} X_{M/rad_R(M)}$ and by Proposition 1.9, $\Psi_{M/rad_R(M)} \circ \Phi_M = \Psi_M$. This yields that the natural map $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ is also surjective. Moreover, by Lemma 1.14, $X^{\widehat{R}}$ is a discrete space. Thus $\Psi_M : X_M \longrightarrow X^{\widehat{R}}$ is both closed and open. $\square$

We are in position to give several characterizations of an Artinian module M over a PI-ring (or an FBN-ring), for which X_M is connected.

Theorem 1.16. *Let R be a PI-ring (or an FBN-ring), and let M be an Artinian left R-module with $X_M \neq \emptyset$. Then the following statements are equivalent:*

(1) X_M is connected.
(2) $X^{\widehat{R}}$ is connected.
(3) The ring $\widehat{R}$ contains no idempotent other than $\overline{0}$, $\overline{1}$.
(4) $\widehat{R}$ is a simple Artinian ring.
(5) $rad_R(M)$ is a prime submodule of M.
(6) $M/rad_R(M)$ is a homogeneous semisimple R-module.
(7) $X_M = V(P)$ for some prime submodule P of M.

Proof. $(2 \Leftrightarrow 3)$ is well-known (see for example [3]).
$(1 \Rightarrow 2)$ Assume that X_M is connected. Since the natural map Ψ is surjective, $X^{\widehat{R}}$ is also connected.
$(2 \Rightarrow 4)$ Assume that $X^{\widehat{R}}$ is connected. If $|X^{\widehat{R}}| = n$, where $n > 1$, then there exist distinct prime ideals $\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2, ..., \overline{\mathcal{P}}_n \in X^{\widehat{R}}$. Since by Lemma 1.14, $X^{\widehat{R}}$ is a discrete space and each $\overline{\mathcal{P}}_i$ is maximal, we have $V(\overline{\mathcal{P}}_1) \bigcap V(\cap_{i=2}^{n} \overline{\mathcal{P}}_i) = \emptyset$ and

$V(\overline{\mathcal{P}}_1)\bigcup V(\cap_{i=2}^{n}\overline{\mathcal{P}}_i) = X^{\widehat{R}}$. It follows that

$$W(\overline{\mathcal{P}}_1)\bigcap W(\cap_{i=2}^{n}\overline{\mathcal{P}}_i) = \emptyset \text{ and } W(\overline{\mathcal{P}}_1)\bigcup W(\cap_{i=2}^{n}\overline{\mathcal{P}}_i) = X^{\widehat{R}},$$

a contradiction (since $X^{\widehat{R}}$ is connected). Thus $X^{\widehat{R}} = \{\overline{\mathcal{P}}\}$ for some prime (maximal) ideal $\mathcal{P}$ of R. Since $M/rad_R(M)$ is a semiprime module, $I_{M/rad_R(M)}$ is a semiprime ideal of R i.e., $\widehat{R}$ is a semiprime ring. This yields that $\overline{\mathcal{P}} = (0)$. Thus $\overline{M} := M/rad_R(M)$ is a faithful Artinian left $\widehat{R}$-module, and hence, by Lemma 1.11, $\widehat{R}$ is simple Artinian.

$(4 \Rightarrow 5)$ is clear (since every module over a simple ring is prime).

$(5 \Rightarrow 6)$ Since $rad_R(M)$ is a prime submodule of M, by [1, Corollary 1.6], $rad_R(M)$ is a virtually maximal submodule of M i.e., $M/rad_R(M)$ is a homogeneous semisimple R-module.

$(6 \Rightarrow 7)$ It is clear that for each left R-module M, $X_M = V(rad_R(M))$. Since $rad_R(M) = P$ is a prime submodule of M, $X_M = V(P)$.

$(7 \Rightarrow 1)$ Suppose $X_M = V(P)$ for some prime submodule P of M. If X_M is disconnected, then there exist nonempty open subsets Y_1 and Y_2 such that $X_M = Y_1\bigcup Y_2$ and $Y_1\bigcap Y_2 = \emptyset$. Since $P \in X_M$, without loss of generality we can assume that $P \in Y_1$ and $P \notin Y_2$. Since Y_2 is open subset, $Y_2 = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(N_{ij}))$, where $n_i \in \mathbb{N}$ $(i \in I)$ and $N_{ij} \leq M$. Thus there exists $k \in I$ such that for each j $(1 \leq j \leq n_k)$, $P \in W(N_{kj})$ i.e. $N_{kj} \nsubseteq P$. Since $X_M = V(P)$, for each $Q \in X_M$ we have $N_{kj} \nsubseteq Q$ and so $Q \notin Y_2$. Therefore $Y_2 = \emptyset$, a contradiction. Thus X_M is connected. □

2. Patch topologies associated to the classical Zariski topology of a module

We need to recall the patch topology (see for example [4,9] for definition and more details). Let X be a topological space. By the patch topology on X, we mean the topology which has as a sub-basis for its closed sets the closed sets and compact open sets of the original space. By a patch we mean a set closed in the patch topology. The patch topology associated to a spectral space is compact and Hausdorff [5]. Also, the patch topology associated to the Zariski topology of a ring R (R is not necessarily commutative) with ACC on ideals is compact and Hausdorff (see, [4, Proposition 16.1]).

Definition 1. Let M be a left R-module, and let $\mathcal{U}(M)$ be the family of all subsets of X_M of the form $V(N)\bigcap W(N)$ where $N \leq M$ such that $W(N)$ is a classical Zariski-quasi-compact subset of X_M. Clearly $\mathcal{U}(M)$ contains X_M and the empty set, since X_M equals $V(0)\bigcap W(M)$ and the empty set equals $V(0)\bigcap W(0)$. Let $\mathcal{T}_p(M)$ to be the collection U of all unions of finite intersections of elements of $\mathcal{U}(M)$. Then $\mathcal{T}_p(M)$ is a topology on X_M and is called the patch topology or constructible topology (in fact, $\mathcal{U}(M)$ is a sub-basis for the patch topology of M).

Definition 2. Let M be a left R-module, and let $\widetilde{\mathcal{U}}(M)$ be the family of all subsets of X_M of the form $V(N)\bigcap W(K)$ where N, $K \leq M$. Clearly $\mathcal{U}(M)$ contains X_M and the empty set, since X_M equals $V(0)\bigcap W(M)$ and the empty set equals $V(0)\bigcap W(0)$. Let $\mathcal{T}_p(M)$ to be the collection $\widetilde{U}$ of all unions of finite intersections of elements of $\widetilde{\mathcal{U}}(M)$. Then $\widetilde{\mathcal{T}_p}(M)$ is a topology on X_M and is called the finer patch topology or finer constructible topology (in fact, $\widetilde{\mathcal{U}}(M)$ is a sub-basis for the finer patch topology of M).

Clearly, if $M = R$, R commutative, then the patch topology on R as an R-module coincides with the patch topology of R as a ring. But, in general for a non-commutative ring R, $\mathcal{T}_p({}_RR)$ is not equal to the patch topology of R as a ring, since not necessarily prime left ideals of R and prime two-sided ideals of R coincide. Later, in Corollary 2.6, we show that for a Noetherian left R-module M, the patch topology and the finer patch topology of M coincide.

Proposition 2.1. *Let M be a left R-module. Then X_M with the finer patch topology is Hausdorff. Moreover, X_M with this topology is disconnected if and only if $|X_M| > 1$.*

Proof. Suppose distinct points P, $Q \in X_M$. Since $P \neq Q$, either $P \nsubseteq Q$ or $Q \nsubseteq P$. Assume that $P \nsubseteq Q$. By Definition 2, $U_1 := W(M)\bigcap V(P)$ is a finer patch-neighborhood of P and $U_2 := W(P)\bigcap V(Q)$ is a finer patch-neighborhood of Q. Clearly $W(P)\bigcap V(P) = \emptyset$ and hence $U_1 \cap U_2 = \emptyset$. Thus X_M is a Hausdorff space. On the other hand for every submodule N of M, observe that the sets $W(N)$ and $V(N)$ are open in finer patch topology, since $V(N) = W(M)\bigcap V(N)$ and $W(N) = W(M)\bigcap V(0)$. Since $W(N)$ and $V(N)$ are complements of each other, these are both closed as well. It follows that "moreover" statement. $\square$

We recall the prime radical of a submodule. Let R be a ring and M be a left R-module. For a submodule N of M, if there is a prime submodule containing N, then we define

$$\sqrt{N} = \bigcap\{P : P \text{ is a prime submodule of } M \text{ and } N \subseteq P\}.$$

If there is no prime submodule containing N, then we put $\sqrt{N} = M$. In particular, for any module M, we have $rad_R(M) = \sqrt{(0)}$. It is easy to see that for a submodule N of M, $V(N) = V(\sqrt{N})$. In particular, for every left R-module M, we have $V(0) = V(\sqrt{(0)}) = X_M$.

Theorem 2.2. *Let M be a left R-module such that M has ACC on intersection of prime submodules. Then X_M with the finer patch topology is a compact space.*

Proof. Suppose M is a left R-module such that M has ACC on intersection of prime submodules. Let $\mathcal{A}$ be a family of finer patch-open sets covering X_M, and suppose that no finite subfamily of $\mathcal{A}$ covers X_M. Since $V(\sqrt{0}) = V(0) = X_M$, we may use the ACC on intersection of prime submodules to choose a submodule N maximal with respect to the property that no finite subfamily of $\mathcal{A}$ covers $V(N)$ (note that e have $N = \sqrt{N}$, since $V(N) = V(\sqrt{N})$). We claim that N is a prime submodule of M, for if not, then there exist $m \in M$ and $a \in R$, such that $aRm \subseteq N$ and $m \notin N$ and $aM \nsubseteq N$. Thus $N \subsetneqq N + Rm \subseteq \sqrt{N + Rm}$ and $N \subsetneqq N + aM \subseteq \sqrt{N + aM}$. Hence without loss of generality, there must be a finite subfamily $\mathcal{A}'$ of $\mathcal{A}$ that covers both $V(N + aM)$ and $V(N + Rm)$. Let $P \in V(N)$. Since $aRm \subseteq N$, $aRm \subseteq P$ and since submodule P is prime, $Rm \subseteq P$ or $aM \subseteq P$. Thus either $P \in V(N + Rm)$ or $P \in V(N + aM)$, therefore

$$V(N) \subseteq V(N + aM) \bigcup V(N + Rm).$$

Thus $V(N)$ cover with the finite subfamily $\mathcal{A}'$, a contradiction. Therefore, N is a prime submodule of M.

Now choose $U \in \mathcal{A}$ such that $N \in U$. Thus N must have a patch-neighborhood $\bigcap_{i=1}^{n}(W(K_i) \bigcap V(N_i))$, for some K_i, $N_i \leq M$, $n \in \mathbb{N}$, such that

$$\bigcap_{i=1}^{n}[W(K_i) \bigcap V(N_i)] \subseteq U.$$

We claim that for each i $(1 \leq i \leq n)$,

$$N \in W(K_i + N) \bigcap V(N) \subseteq W(K_i) \bigcap V(N_i).$$

For see this, assume that $P \in W(K_i + N) \bigcap V(N)$ i.e., $K_i + N \nsubseteq P$ and $N \subseteq P$. Thus $K_i \nsubseteq P$ i.e., $P \in W(K_i)$. On the other hand, $N \in V(N_i)$ i.e., $N_i \subseteq N$. Therefore, $N_i \subseteq P$ i.e., $P \in V(N)$. Consequently,

$$N \in \bigcap_{i=1}^{n}[W(K_i + N) \bigcap V(N)] \subseteq \bigcap_{i=1}^{n}[W(K_i) \bigcap V(N_i)] \subseteq U.$$

Thus $[\bigcap_{i=1}^{n}[W(K'_i)]\bigcap V(N)$, where $N \subsetneqq K'_i := K_i + N$, is a neighborhood of N, such that $[\bigcap_{i=1}^{n}[W(K'_i)]\bigcap V(N) \subseteq U$. Since for each i $(1 \leq i \leq n)$, $N \subsetneqq K'_i$, $V(K'_i)$ can be covered by some finite subfamily $\mathcal{A}'_i$ of $\mathcal{A}$. But

$$V(N)\setminus[\bigcup_{i=1}^{n} V(K'_i)] = V(N)\setminus[\bigcap_{i=1}^{n} W(K'_i)]^c = [\bigcap_{i=1}^{n}[W(K'_i)]\bigcap V(N) \subseteq U.$$

and so $V(N)$ can be covered by $\mathcal{A}'_i \bigcup \mathcal{A}'_2 \bigcup ... \bigcup \mathcal{A}'_n \bigcup\{U\}$, contrary to our choice of N. Thus there must exist a finite subfamily of $\mathcal{A}$ which covers X_M. Therefore X_M is compact in the finer patch topology of M. □

We need the following evident lemma.

Lemma 2.3. *Assume τ and τ^* are two topologies on X such that $\tau \subseteq \tau^*$. If X is quasi-compact (i.e., any open cover of it has a finite subcover) in τ^*, then X is also quasi-compact in τ.*

Theorem 2.4. *Let M be a left R-module such that M has ACC on intersection of prime submodules. Then for each $n \in \mathbb{N}$ and submodules N_i $(1 \leq i \leq n)$ of M, $W(N_1) \cap W(N_2) \cap \cdots \cap W(N_n)$ is a quasi-compact subset of X_M with the classical Zariski topology. Consequently, X_M is quasi-compact and has a basis of quasi-compact open subsets.*

Proof. By Definition 2, for each submodule N of M, $V(N) = V(N)\bigcap W(M)$ is an open subset of X_M with finer patch topology, and hence, for each submodule N of M, $W(N)$ is a closed subset in X_M. Thus, for each $n \in \mathbb{N}$ and $N_i \leq M$ $(1 \leq i \leq n)$, $W(N_1) \cap W(N_2) \cap \cdots \cap W(N_n)$ is also a closed subset in X_M with finer patch topology. By Theorem 2.2, X_M is a compact space with the finer patch topology and since every closed subset of a compact space is compact, $W(N_1)\cap W(N_2)\cap\cdots\cap W(N_n)$ is compact in X_M with finer patch topology and so by Lemma 2.3, it is quasi-compact in X_M with the classical Zariski topology. Now, since $X_M = W(M)$, and

$$\mathfrak{B} = \{W(N_1)\cap W(N_2)\cap\cdots\cap W(N_n) \ : \ N_i \leq M,\ 1 \leq i \leq n,\ for\ some\ n \in \mathbb{N}\}$$

is a basis for the classical Zariski topology of M, the proof is complete. □

We conclude this section with the following evident corollaries.

Corollary 2.5. *Let M be left R-module such that M has ACC on intersection of prime submodules. If U is a classical Zariski-quasi-compact open subset of X_M,* *then* $U = \bigcup_{i=1}^{m} (\bigcap_{j=1}^{n_i} W(N_j))$. *Consequently, the family of classical Zariski-quasi-compact open subsets of* X_M *is closed under finite intersections.*

Corollary 2.6. *Let* M *be left* R*-module such that* M *has ACC on intersection of prime submodules. Then the finer patch topology and the patch topology of* M *coincide.*

Also, by applying Proposition 1.4, Lemma 1.7 and Theorem 2.4, we have the following corollary.

Corollary 2.7. *For each Artinian left* R*-module* M *over a PI-ring (or an FBN-ring), the finer patch topology and the patch topology of* M *coincide.*

3. Modules whose classical Zariski topology are spectral space

A topological space X is called irreducible if $X \neq \emptyset$ and every finite intersection of non-empty open sets of X is non-empty. A (non-empty) subset Y of a topology space X is called an irreducible set if the subspace Y of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets Y_1, Y_2 which are closed in X and satisfy satisfy $Y \subseteq Y_1 \cup Y_2$, $Y \subseteq Y_1$ or $Y \subseteq Y_2$ (see, for example [3, page 94]).

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a generic point of Y if $Y = \overline{\{y\}}$. Note that a generic point of the irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology. By Hochster's characterization [5], a topology $\mathcal{T}$ on a set X is spectral if and only if the following axioms hold:

(i) X is a T_0-space.
(ii) X is quasi-compact and has a basis of quasi-compact open subsets.
(iii) The family of quasi-compact open subsets of X is closed under finite intersections.
(iv) X_M is a sober space (i.e., every irreducible closed subset of X_M has a generic point).

Proposition 3.1. *Let* M *be a left* R*-module such that* M *has ACC on intersection of prime submodules. Then every irreducible closed subset of* X_M *(with the classical Zariski topology) has a generic point.*

Proof. Let Y be an irreducible closed subset of X_M. First, we show that $Y = \bigcup_{P\in Y} V(P)$. Clearly $Y \subseteq \bigcup_{P\in Y} V(P)$. By [2, Corollary 3.2(a)], for each $P \in Y$ we have $V(P) = \overline{\{P\}} \subseteq \overline{Y}$, and since $\overline{Y} = Y$, $\bigcup_{P\in Y} V(P) \subseteq Y$. Thus $Y = \bigcup_{P\in Y} V(P)$. By Definition 2, for each $P \in Y$, $V(P)$ is an open subset of X_M with the finer patch topology. On the other hand since $Y \subseteq X_M$ is closed with the classical Zariski topology, the complement of Y is open by this topology. This yields that the complement of Y is open with the finer patch topology i.e., $Y \subseteq X_M$ is closed with the finer patch topology. By Theorem 2.2, X_M is compact with the finer patch topology and since $Y \subseteq X_M$ is closed, Y is also compact. Now since $Y = \bigcup_{P\in Y} V(P)$ and each $V(P)$ is finer patch-open, there exists a finite subset Y' of Y such that $Y = \bigcup_{P\in Y'} V(P)$. Now since Y is irreducible $Y = V(P)$ for some $P \in Y'$. Therefore, we have $Y = V(P) = \overline{\{P\}}$ for some $P \in Y$ i.e., P is a generic point for Y. □

In [2, Theorem 3.9], it is shown that for every left R-module M with finite spectrum, X_M is a spectral space. We are in position to give a nice generalization of this result, which yields directly for every Noetherian left R-module M, X_M is a spectral space.

Theorem 3.2. *Let M be a left R-module such that M has ACC on intersection of prime submodules. Then X_M (with the classical Zariski topology) is spectral spaces.*

Proof. By [2, Proposition 3.8.], X_M is a T_0-space. Since M has ACC on intersections of prime submodules, by Theorem 2.4, X_M is quasi-compact and has a basis of quasi-compact open subsets. Also, by Corollary 2.5, the family of quasi-compact open subsets of X_M is closed under finite intersections. Finally, by Proposition 3.1, every irreducible closed subset of X_M has generic point. Thus by Hochster's characterization, X_M is spectral spaces. □

Corollary 3.3. *Let M be a Noetherian left R-module. Then X_M (with the classical Zariski topology) is spectral spaces.*

Theorem 3.4. *Let R be a PI-ring (or an FBN-ring), and let M be an Artinian R-module. Then X_M (with the classical Zariski topology) is spectral spaces.*

Proof. By Lemma 1.7, $M/rad_R(M)$ is a Noetherian left R-module and by Proposition 1.4, $X_M \cong X_{M/rad_R(M))}$. Now apply Theorem 3.2. □

Looking at Theorems 3.2, 3.4, the following questions arise in a natural way.

Question 3.5. *Let M be a left R-module with Noetherian spectrum. Is $Spec({}_RM)$ a spectral space?*

Question 3.6. *Let M be an Artinian left R-module. Is $Spec({}_RM)$ a spectral space?*

For a left R-module M we denote the set of all maximal prime submodules of M by X^m_M. We will show that for a each left R-module M, the subspace X^m_M (with the topology inherited from X_M) is homeomorphic with the maximal ideal space of some commutative ring S if and only if $X^m_M \neq \emptyset$ and X^m_M is quasi-compact.

Lemma 3.7. *Let M be a left R-module. Then X^m_M (with the topology inherited from X_M the classical Zariski topology) is a T_1-space.*

Proof. Clearly for each $P \in X^m_M$, $V(P) = \{P\}$. Thus $V(P) \bigcap X^m_M = \{P\}$ is a closed set in X^m_M, i.e., X^m_M is a T_1-space. □ □

Hochster in [5, Proposition 11], proved that a topological space X ($X \neq \emptyset$) is homeomorphic with the maximal ideal space of some commutative ring S (with the topology inherited from Spec(S)) if and only if X is T_1 and quasi-compact. Thus we have the following proposition.

Proposition 3.8. *Let M be a left R-module. Then X^m_M (with the topology inherited from X_M with the classical Zariski topology) is homeomorphic with the maximal ideal space of some commutative ring S if and only if $X^m_M \neq \emptyset$ and X^m_M is quasi-compact.*

Theorem 3.9. *Let M be a left R-module such that $X_M \neq \emptyset$ and M has ACC on intersections of prime submodules. Then X^m_M (with the classical Zariski topology) is homeomorphic with the maximal ideal space of some commutative ring S.*

Proof. Since $X_M \neq \emptyset$ and M has ACC on intersections of prime submodules, $X^m_M \neq \emptyset$. Clearly for each $P \in X^m_M$, $V(P) = \{P\}$ and hence by [2, Proposition 3.1], $\overline{X^m_M} = \bigcup_{P \in X^m_M} V(P) = \bigcup_{P \in X^m_M} \{P\} = X^m_M$, i.e., X^m_M is closed in X_M. Since X_M is quasi-compact (see Theorem 2.4), X^m_M is also quasi-compact. Now apply Proposition 3.8. □

We conclude this paper with the following interesting result.

Corollary 3.10. *Let M be a Noetherian left R-module. Then $Max(M)$ (with the classical Zariski topology) is homeomorphic with the maximal ideal space of some commutative ring S.*

Proof. Since M is Noetherian, $X^m_M = Max(M) \neq \emptyset$ and also M has ACC on intersections of prime submodules. Now apply Theorem 3.9. □

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