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GENERALISED ASSOCIATED PRIMES AND RADICALS OF SUBMODULES

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Abstract. A challenging problem in recent years has been to find a good description of the radical of a submodule N of a (Noetherian) module M over a commutative ring, where the radical of N is the intersection of all prime submodules of M which contain N. In this paper we give a description of the radical of N in a Noetherian module M which is amenable to computation either by hand in simple cases or by using a computer algebra system in other cases, and illustrate this by examples.

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Since the notion of the radical (sometimes referred to as the M-radical) of a submodule was introduced in the early 1980's [10], a number of authors have tried to describe this radical, either in terms of its elements, or as some sort of decomposition. As for the former, most of the efforts were directed towards finding a description similar to the well-known formula for the elements of the radical of an ideal; namely, for an ideal I of a ring (commutative, with identity) R, then $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$. A method for computing the radical of a submodule N of a free module F was given in [9], using the symmetric algebra of F. In other special cases, a "radical formula" was shown to hold, where all the computations remained within the original module. See, for example, [5], [6], [11] and [12].

In this paper, we seek a decomposition of the radical of a submodule N of a module M, as an intersection of (finitely many) known prime submodules lying over N. One advantage of such a representation is that this allows one to compute the uniform dimension of $M/\operatorname{rad} N$ (see [16] for details).

Throughout this work, R denotes a commutative ring with identity and M denotes a unital R-module. For a submodule N of M we let $(N : M)$ denote the ideal $\{r \in R : rM \subseteq N\}$. Similarly, for an element $s \in R$, we let $(N :_M s) = \{m \in M : sm \in N\}$. We remark that this paper owes much to [14].

1. Associated Primes

Associated primes have been studied rather extensively, with several different notions of these primes occurring in the literature. For a comparison of (and references to) some of the more common definitions of primes associated to an ideal, see [4].

In this paper, we are interested in primes associated to submodules, and in particular to primes associated to the radical of a given submodule. It is well-known that in the Noetherian ring case, the associated primes to a given ideal coincide with the associated primes to the radical of the ideal, and indeed provide the decomposition we seek. However, in the module case, things are not nearly so simple, as we shall show.

Let $\mathfrak{p}$ be a prime ideal of R and let N be a proper submodule of M. We say that $\mathfrak{p}$ is an *associated prime* of N if $(N : m) = \mathfrak{p}$ for some $m \in M \setminus N$. Following [1], we write $AP(N)$ to denote the set of all associated primes of N. Note that $AP(N)$ depends on M but it will always be clear which module M we are considering. We begin with a few elementary results about associated primes.

Proposition 1.1. *Let N be a proper submodule of M. Then for any $\mathfrak{p} \in AP(N)$, we have $(N : M) \subseteq \mathfrak{p}$.*

Proof. Clear. □

A submodule N of an R-module M is said to be *prime* if $N \neq M$ and whenever $rm \in N$ (where $r \in R$ and $m \in M$) then $r \in (N : M)$ or $m \in N$. If N is prime, then the ideal $\mathfrak{p} = (N : M)$ is a prime ideal of R, and N is said to be $\mathfrak{p}$-prime. Alternatively, a submodule Q of M is said to be *primary* if $Q \neq M$ and if $rm \in Q$ (where $r \in R$ and $m \in M$) implies that either $m \in Q$ or $r \in \sqrt{(Q : M)}$. If Q is primary, then $(Q : M)$ is a primary ideal of R, and in this case we say that Q is $\mathfrak{q}$-primary, where $\mathfrak{q} = \sqrt{(Q : M)}$ is a prime ideal of R. It is worth noting that if Q is $\mathfrak{q}$-primary and $(Q : M) = \mathfrak{q}$, then Q is in fact a prime submodule (see [7, Proposition 1]). As for the next result, while it does not pertain directly to associated primes, it will prove useful to refer back to it in the sequel.

Proposition 1.2. *The intersection of a non-empty collection of $\mathfrak{p}$-prime (resp., $\mathfrak{p}$-primary) submodules is itself a $\mathfrak{p}$-prime (resp., $\mathfrak{p}$-primary) submodule.*

Proof. Elementary. □

A submodule N of M has a *primary decomposition* if $N = Q_1 \bigcap \cdots \bigcap Q_n$, where Q_i is a primary submodule for each i $(1 \leq i \leq n)$. Moreover, we say that an intersection $N = Q_1 \bigcap \cdots \bigcap Q_n$ is *irredundant* if $N \neq Q_1 \bigcap \cdots \bigcap Q_{i-1} \bigcap Q_{i+1} \cdots \bigcap Q_n$ for each i $(1 \leq i \leq n)$. An irredundant, primary decomposition $N = Q_1 \bigcap \cdots \bigcap Q_n$ is a *normal primary decomposition* if $\sqrt{(Q_i : M)} \neq \sqrt{(Q_j : M)}$ whenever $i \neq j$. We remark that 'normal' primary decompositions are sometimes instead referred to in the literature as either 'minimal' or 'reduced'. It is well known that every proper submodule of a Noetherian module has a normal primary decomposition.

Proposition 1.3. *(See* [2, Theorem 3.10]*.) Let R be Noetherian, let N be a proper submodule of a finitely generated R-module M, and let $N = \bigcap_{i=1}^{k} Q_i$ be a normal primary decomposition, where each Q_i is $\mathfrak{p}_i$-primary $(1 \leq i \leq k)$. Then $AP(N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$.*

Let N be a submodule of M. The *radical* of N, denoted $\operatorname{rad} N$, is the intersection of all prime submodules of M that contain N, unless there are no such prime submodules, in which case $\operatorname{rad} N = M$. Note that this is consistent with the usual definition of the radical of an ideal of R, in case $M = R$. It is clear that $\operatorname{rad} N$ can be obtained by intersecting only the minimal prime submodules over N – where P is a *minimal prime submodule* over N if P is a minimal element of the set of all prime submodules of M that contain N. In certain cases, $\operatorname{rad} N$ can be expressed as a finite intersection of prime submodules. Whenever this holds, we make the obvious modifications to the terminology for a primary decomposition of N, applied to $\operatorname{rad} N$. Hence, by a *prime decomposition* of rad N we mean a decomposition $\operatorname{rad} N = P_1 \bigcap \cdots \bigcap P_n$, where n is a positive integer and P_i is a prime submodule of M for each i $(1 \leq i \leq n)$. Letting $(P_i : M) = \mathfrak{p}_i$ for each i $(1 \leq i \leq n)$, we say that an irredundant, prime decomposition $\operatorname{rad} N = P_1 \bigcap \cdots \bigcap P_n$ is a *normal prime decomposition* if $\mathfrak{p}_i \neq \mathfrak{p}_j$ whenever $i \neq j$.

Since for our purposes, we will often require $\operatorname{rad} N$ to be a finite intersection of primes, then throughout much of this work, we will focus on the Noetherian case, where for every proper submodule N, $\operatorname{rad} N$ indeed has a prime decomposition (see [8, Theorem 4]). One exception to this rule is the following result.

Proposition 1.4. *(See* [16, Lemma 2.1]*.) Let N be a submodule of M such that $\operatorname{rad} N$ has a prime decomposition. Then $\operatorname{rad} N$ has a normal prime decomposition. Moreover, if $\operatorname{rad} N = P_1 \bigcap \cdots \bigcap P_n$ and $\operatorname{rad} N = Q_1 \bigcap \cdots \bigcap Q_m$ are both normal prime decompositions, with $(P_i : M) = \mathfrak{p}_i$ for each i $(1 \leq i \leq n)$ and $(Q_j : M) = \mathfrak{q}_j$*

for each j $(1 \leq j \leq m)$, *then* $n = m$ *and* $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_m\}$. *Therefore,* $AP(\operatorname{rad} N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

To reiterate, having a normal prime decomposition of $\operatorname{rad} N$ is sufficient for determining the associated primes to $\operatorname{rad} N$. Conversely, at least in the Noetherian setting, once we have in hand the (finitely many) members of $AP(\operatorname{rad} N)$, then it is not terribly difficult to obtain a normal prime decomposition of $\operatorname{rad} N$ – as Theorem 2.7 testifies.

However, it is important to note that, even in the Noetherian case, the sets $AP(N)$ and $AP(\operatorname{rad} N)$ are not necessarily equal. Indeed, as Example 1.6 below shows, there exists a primary submodule Q such that $\operatorname{rad} Q$ is not prime, and in this case, $AP(Q) \neq AP(\operatorname{rad} Q)$. One of the main objects of this paper therefore is to compare, for a given submodule N of M, the sets $AP(N)$ and $AP(\operatorname{rad} N)$. As noted earlier, there is at least one case where these two sets coincide (Proposition 1.5).

Proposition 1.5. *Let* R *be a Noetherian ring. Then for any ideal* I *of* R, *we have* $AP(I) = AP(\sqrt{I})$.

Proof. This is an easy exercise. □

Example 1.6. *(See* [18, Example 1.11]*.) Let* R *be the polynomial ring* $\mathbb{Z}[x]$ *and let* M *be the* R*-module* $R \oplus R$. *Then the submodule* $Q = R(2, x) + R(x, 0)$ *is a* $\mathfrak{p}$*-primary submodule of* M, *where* $\mathfrak{p}$ *is the prime ideal* Rx. *However,* $\operatorname{rad} Q = K \bigcap \mathfrak{m}M$ *is a normal prime decomposition, where* K *is the prime submodule* $R \oplus Rx$ *and* $\mathfrak{m}$ *is the maximal ideal* $R2 + Rx$. *Hence,* $AP(Q) = \{\mathfrak{p}\}$, *but* $AP(\operatorname{rad} Q) = \{\mathfrak{p}, \mathfrak{m}\}$.

The situation is made even more difficult by the fact that $AP(N)$ is not necessarily contained in $AP(\operatorname{rad} N)$, as Example 1.7 shows.

Example 1.7. *Let* $R = \mathbb{Z}[x]$, *let* p *be a prime in* $\mathbb{Z}$, *let* $M = R \oplus R$ *and let* $\mathfrak{m}$ *be the ideal* $Rp + Rx$. *Consider the submodule* $N = \mathfrak{m}(p, x)$ *of* M. *We have* $\mathfrak{m} \in AP(N)$, *since* $(p, x) \notin N$ *and* $\mathfrak{m} = (N : (p, x))$. *However,* $\mathfrak{m} \notin AP(\operatorname{rad} N)$. *To see this, first note that for any prime submodule* P *of* M *such that* $N \subseteq P$, *either* $(p, x) \in P$ *or* $(P : M) = \mathfrak{m}$. *Now since* $R(p, x)$ *is* 0*-prime and is contained in* $\mathfrak{m}M$, *it follows that* $\operatorname{rad} N = R(p, x)$, *and thus* $AP(\operatorname{rad} N) = \{0\}$.

To complicate matters still further, consider a minimal prime submodule P over a submodule N of M. It turns out that $(P : M)$ need not belong to either $AP(N)$ or $AP(\operatorname{rad} N)$, as one can see in Example 1.9 below. Compare this with Proposition 1.8.

Proposition 1.8. *(See* [2, Theorem 3.1]*.) Let R be Noetherian and let M be finitely generated. Then $AP(0)$ contains every minimal prime ideal over ann$M$.*

Example 1.9. *Let $R = \mathbb{Z}[x]$, let p be a prime in $\mathbb{Z}$, let $M = R \oplus R$ and let $N = Rp \oplus Rx$. Then the minimal prime submodules of N are $P_1 = Rp \oplus R$, $P_2 = R \oplus Rx$ and $P_3 = \mathfrak{m}M$, where $\mathfrak{m}$ is the ideal $Rp + Rx$. However, $\mathfrak{m} = (P_3 : M)$ is not an associated prime ideal either of N or of* $\operatorname{rad} N$*, since $N = \operatorname{rad} N = P_1 \bigcap P_2$ is a normal prime decomposition of N.*

While several results pertaining to prime ideals do indeed generalise to prime submodules, many other such results (e.g., Propositions 1.5 and 1.8) do not. Amongst those cases which do not generalise, many fail to do so primarily because of the following: a prime submodule can in fact contain the intersection of two submodules, without it containing either individually (whereas, whenever a prime ideal contains an intersection of two ideals, it must contain one of them). This difficulty arises even in a simple case such as a 2-dimensional vector space, where every proper subspace is a prime submodule. Note that any two distinct 1-dimensional subspaces have trivial intersection, and hence any third (distinct from the other two) 1-dimensional subspace contains their intersection.

2. Minimal Primes

Throughout this section, let N be a proper submodule of M. Before approaching the problem of determining the members of $AP(\operatorname{rad} N)$, we first turn our attention to finding, for each prime ideal $\mathfrak{p}$ belonging to $AP(\operatorname{rad} N)$, a minimal prime submodule P over N such that $(P : M) = \mathfrak{p}$. From Proposition 1.2 we see that if P exists, then it is in fact unique, subject to these constraints. Importantly, once we have such a minimal prime submodule for each member of $AP(\operatorname{rad} N)$, then Theorem 2.7 shows that the intersection of these minimal primes gives a normal prime decomposition of $\operatorname{rad} N$. Meanwhile, we begin by showing the existence of such a minimal prime submodule P, in the Noetherian case. Note that the converse of the next result does not hold (see Example 1.9).

Lemma 2.1. *Let R be Noetherian, let N be a proper submodule of a finitely generated R-module M, and let $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p} \in AP(\operatorname{rad} N)$ then there exists a $\mathfrak{p}$-prime submodule P of M which is a minimal prime over N.*

Proof. By Proposition 1.4 and the remarks immediately preceding it, $\operatorname{rad} N$ has a normal prime decomposition, say $\operatorname{rad} N = P_1 \bigcap \cdots \bigcap P_n$, where n is a positive integer and P_i is a $\mathfrak{p}_i$-prime submodule of M for each i $(1 \leq i \leq n)$. Since

$\mathfrak{p} \in AP(\operatorname{rad} N)$, then without loss of generality we may assume that $\mathfrak{p}_1 = \mathfrak{p}$. Now take the intersection of all $\mathfrak{p}$-prime submodules containing N, and apply Proposition 1.2, to obtain a prime submodule P which is minimal amongst all $\mathfrak{p}$-prime submodules containing N. In particular, note that $P \subseteq P_1$. We now show that P is actually a minimal prime over N (not just minimal amongst $\mathfrak{p}$-primes). Suppose there exists a prime submodule Q of M such that $N \subseteq Q \subseteq P$. Then $\operatorname{rad} N = Q \bigcap P_2 \bigcap \cdots \bigcap P_n$ is a prime decomposition of $\operatorname{rad} N$, and since there are n components in this decomposition, it is in fact normal (otherwise we could obtain a normal prime decomposition from this one, with fewer than n components – but this would contradict Proposition 1.4). Moreover, from Proposition 1.4 we see that $\{(Q : M), \mathfrak{p}_2, \ldots, \mathfrak{p}_n\} = AP(\operatorname{rad} N) = \{\mathfrak{p}, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$, and hence $(Q : M) = \mathfrak{p}$. It follows that $Q = P$, and the proof is complete. $\square$

Bearing Lemma 2.1 in mind, let $\mathfrak{p}$ be any prime ideal of R. Following [15], we let $cl_{\mathfrak{p}}(N)$ denote the $\mathfrak{p}$-*closure* of N, as defined by

$$cl_{\mathfrak{p}}(N) = \{m \in M : rm \in N \text{ for some } r \in R \setminus \mathfrak{p}\}.$$

It is clear that $cl_{\mathfrak{p}}(N) = \bigcup_{r \in R \setminus \mathfrak{p}} (N :_M r)$ and that $N \subseteq cl_{\mathfrak{p}}(N)$. For our purposes, the most interesting case is where $(N : M) \subseteq \mathfrak{p}$.

We have seen (Example 1.6) that the radical of a primary submodule need not be prime – there can in fact exist more than one minimal prime submodule over a primary submodule. The following result is interesting because, while it is fairly straightforward, it suggests a possible approach to finding minimal primes in the general case.

Lemma 2.2. *If a submodule Q of M is $\mathfrak{p}$-primary, then* $\operatorname{rad} Q = \operatorname{rad}(Q + \mathfrak{p}M)$.

Proof. Clearly $\operatorname{rad} Q \subseteq \operatorname{rad}(Q + \mathfrak{p}M)$. If P is a prime submodule of M containing Q, then $(P : M) \supseteq \sqrt{(Q : M)} = \mathfrak{p}$, and thus $Q + \mathfrak{p}M \subseteq P$. $\square$

Our approach then will be to consider submodules of the form $N + \mathfrak{p}M$, for any submodule N of M and prime ideal $\mathfrak{p}$ of R such that $(N : M)$ is contained in $\mathfrak{p}$. The next two results will prove quite useful in this endeavour.

Lemma 2.3. *(See* [13, Corollary 3.4]*.) Let M be finitely generated and let N be a submodule of M. For every prime ideal $\mathfrak{p}$ of R such that $(N : M) \subseteq \mathfrak{p}$, then $(N + \mathfrak{p}M : M) = \mathfrak{p}$.*

Corollary 2.4. *Let N be a submodule of a finitely generated module M and let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p} \in AP(N + \mathfrak{p}M)$ if and only if $(N : M) \subseteq \mathfrak{p}$.*

Proof. Suppose first that $(N : M) \subseteq \mathfrak{p}$. There exist a positive integer k and elements $m_i \in M$ $(1 \leq i \leq k)$ such that $M = Rm_1 + \cdots + Rm_k$. By Lemma 2.3,

$$\mathfrak{p} = (N + \mathfrak{p}M : M) = \bigcap_{i=1}^{k} (N + \mathfrak{p}M : m_i),$$

so that $\mathfrak{p} = (N + \mathfrak{p}M : m_i)$ for some i $(1 \leq i \leq k)$. Conversely suppose that $\mathfrak{p} \in AP(N + \mathfrak{p}M)$. By Proposition 1.1, $(N : M) \subseteq (N + \mathfrak{p}M : M) \subseteq \mathfrak{p}$. □

Theorem 2.5. *Let M be finitely generated and let N be a submodule of M such that $(N : M) = \mathfrak{p}$ is a prime ideal of R. Then $cl_{\mathfrak{p}}(N)$ is a minimal prime submodule (of M) over N and $(cl_{\mathfrak{p}}(N) : M) = \mathfrak{p}$.*

Proof. It is clear that $\mathfrak{p} \subseteq (cl_{\mathfrak{p}}(N) : M)$. Now let $r \in (cl_{\mathfrak{p}}(N) : M)$, and let $M = Rm_1 + \cdots + Rm_n$. Then there exist $s_1, \ldots, s_n \in R \setminus \mathfrak{p}$ such that $s_i(rm_i) \in N$ for each i $(1 \leq i \leq n)$. Let $s = \Pi_{i=1}^{n} s_i$ and note that $rsM \subseteq N$, which implies that $rs \in (N : M) = \mathfrak{p}$. Since $s \notin \mathfrak{p}$, we must have $r \in \mathfrak{p}$; hence $(cl_{\mathfrak{p}}(N) : M) = \mathfrak{p}$.

It is now easy to see that $cl_{\mathfrak{p}}(N)$ is a prime submodule of M, because if $rm \in cl_{\mathfrak{p}}(N)$ $(m \in M$ and $r \in R)$, and $r \notin (cl_{\mathfrak{p}}(N) : M) = \mathfrak{p}$, then $(tr)m = t(rm) \in N$ for some $t \in R \setminus \mathfrak{p}$, which implies that $m \in cl_{\mathfrak{p}}(N)$. Finally, suppose that $cl_{\mathfrak{p}}(N)$ is not a minimal prime over N. Then there exists a prime submodule P of M that contains N, with P contained in $cl_{\mathfrak{p}}(N)$. Then $\mathfrak{p} = (N : M) \subseteq (P : M) \subseteq (cl_{\mathfrak{p}}(N) : M) = \mathfrak{p}$. Now if $m \in cl_{\mathfrak{p}}(N)$ then there exists some $r \in R \setminus \mathfrak{p}$ such that $rm \in N \subseteq P$. But this implies that $m \in P$, and the proof is complete. □

Suppose M is finitely generated. Given Lemma 2.3 and Theorem 2.5, in the case $(N : M) \subseteq \mathfrak{p}$, the $\mathfrak{p}$-prime submodule $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is of particular interest. Of course if $(N : M) = \mathfrak{p}$, then $N = N + \mathfrak{p}M$ and we have that $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is a minimal prime submodule over N. On the other hand, if $(N : M)$ is properly contained in $\mathfrak{p}$, then $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is not necessarily a minimal prime over N. However, it is the case that any $\mathfrak{p}$-prime submodule P of M that contains N must likewise contain $N + \mathfrak{p}M$, and $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is minimal amongst these. We summarise these remarks in the following

Corollary 2.6. *Let N be a submodule of a finitely generated R-module M and let $\mathfrak{p}$ be a prime ideal of R such that $(N : M) \subseteq \mathfrak{p}$. Then $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is minimal amongst those $\mathfrak{p}$-prime submodules of M that contain N.*

We are now ready to show that, at least in some cases, the radical of N can be expressed in terms of the submodules $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$, where $\mathfrak{p} \in AP(\operatorname{rad} N)$.

Theorem 2.7. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M (so that $AP(\operatorname{rad} N)$ is finite). If $AP(\operatorname{rad} N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ then $\operatorname{rad} N = \bigcap_{i=1}^n cl_{\mathfrak{p}_i}(N + \mathfrak{p}_i M)$. Moreover, this intersection is a normal prime decomposition of $\operatorname{rad} N$.*

Proof. To see that $\operatorname{rad} N \subseteq \bigcap_{i=1}^n cl_{\mathfrak{p}_i}(N + \mathfrak{p}_i M)$, apply Proposition 1.1 and Corollary 2.6. Now as in the proof of Lemma 2.1, $\operatorname{rad} N$ has a normal prime decomposition, say $\operatorname{rad} N = \bigcap_{j=1}^m Q_j$, where for each j $(1 \leq j \leq m)$, Q_j is $\mathfrak{q}_j$-prime. By Proposition 1.4, $\{\mathfrak{q}_1, \dots, \mathfrak{q}_m\} = AP(\operatorname{rad} N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and $n = m$. Without loss of generality, we may take $\mathfrak{q}_i = \mathfrak{p}_i$ for each i $(1 \leq i \leq n)$. By Corollary 2.6 we have $cl_{\mathfrak{p}_i}(N + \mathfrak{p}_i M) \subseteq Q_i$ for each i $(1 \leq i \leq n)$. It follows that $\operatorname{rad} N = \bigcap_{i=1}^n cl_{\mathfrak{p}_i}(N + \mathfrak{p}_i M)$, and indeed that this is a normal prime decomposition. $\square$

In the remainder of this section we consider some computational aspects of $cl_{\mathfrak{p}}(N)$. Firstly, recall that $cl_{\mathfrak{p}}(N)$ can be described as the union of submodules of the form $(N :_M r)$, where r ranges over the elements of $R \setminus \mathfrak{p}$. Proposition 2.10 simplifies this considerably, at least in certain cases (e.g., M is Noetherian). Furthermore, for a submodule N such that $(N : M)$ is a prime ideal, it would be convenient to have some equivalent conditions for N to be prime.

Proposition 2.8. *(See [19, Lemma 3.5].) Let $\mathfrak{p}$ be a prime ideal of R and let N be a submodule of M such that $(N : M) = \mathfrak{p}$. The following are equivalent:*

(i) N is a $\mathfrak{p}$-prime submodule of M;

(ii) $cl_{\mathfrak{p}}(N) = N$;

(iii) $(N :_M s) = N$ for all $s \in R \setminus \mathfrak{p}$.

We leave the proofs of the next two results to the reader.

Proposition 2.9. *Let N be a submodule of M and let $r, s \in R$. Then*

(i) $((N :_M r) :_M s) = (N :_M rs)$ and

(ii) $(N :_M r) \subseteq (N :_M rs)$.

Proposition 2.10. *Let $\mathfrak{p}$ be a prime ideal of R and let N be a submodule of M such that $(N : M) = \mathfrak{p}$. If $cl_{\mathfrak{p}}(N)$ is finitely generated, then there exists $s \in R \setminus \mathfrak{p}$ such that $cl_{\mathfrak{p}}(N) = (N :_M s)$.*

Let R be Noetherian and let N be a submodule of a finitely generated R-module M such that $(N : M) = \mathfrak{p}$ is a prime ideal of R. The preceding three Propositions collectively tell us that not only does there exist a ring element $r \in R \setminus \mathfrak{p}$ such that

$cl_{\mathfrak{p}}(N) = (N :_M r)$, but that for any $s \in R \setminus \mathfrak{p}$, rs will also work. The next result tells how to find such a ring element r – provided that one is able (either by hand or with the aid of a computer algebra system) to compute $\operatorname{ann}\operatorname{Ext}^k(M/N, R)$. Note that in this result, we require the ring R to have finite dimension. In Section 3 we show that this condition is actually unnecessary.

Theorem 2.11. *Let R be Noetherian with finite (Krull) dimension, and let N be a submodule of a finitely generated R-module M such that $(N : M) = \mathfrak{p}$ is a prime ideal of R. For each positive integer k such that $ht(\mathfrak{p}) < k \leq \dim R$, there exists an element $r_k \in \operatorname{ann}\operatorname{Ext}^k(M/N, R) \setminus \mathfrak{p}$. Let $r = \Pi_{k=ht(\mathfrak{p})+1}^{\dim R} r_k$. Then there exists a positive integer j such that $(N :_M r^j) = cl_{\mathfrak{p}}(N)$.*

Proof. If N is prime, the result follows from the fact that $\mathfrak{p} \subsetneq \operatorname{ann}\operatorname{Ext}^k(M/N, R)$ for all $ht(\mathfrak{p}) < k \leq \dim R$ [3, Theorem 1.1] and from Proposition 2.8. If N is not prime, N has a non-trivial (i.e., $n \geq 2$) normal primary decomposition, say $N = \bigcap_{i=1}^n Q_i$, where for each i $(1 \leq i \leq n)$, Q_i is $\mathfrak{p}_i$-primary. Let $(Q_i : M) = \mathfrak{q}_i$ and note that $\mathfrak{p} = (N : M) \subseteq \mathfrak{q}_i$ for each i $(1 \leq i \leq n)$. On the other hand, since $\bigcap_{i=1}^n \mathfrak{q}_i = (N : M) \subseteq \mathfrak{p}$, then $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some i $(1 \leq i \leq n)$. Hence, without loss of generality, we may assume that $\mathfrak{q}_1 = \mathfrak{p} = \mathfrak{p}_1$. This implies that Q_1 is actually $\mathfrak{p}$-prime, and hence $Q_1 \supseteq cl_{\mathfrak{p}}(N)$.

Note that $\mathfrak{p} \subsetneq \bigcap_{j=2}^n \mathfrak{q}_j$ and so we may choose $r \in \left(\bigcap_{j=2}^n \mathfrak{q}_j\right) \setminus \mathfrak{p}$. It turns out that $cl_{\mathfrak{p}}(N) = (N :_M r)$, since for any $m \in cl_{\mathfrak{p}}(N)$, we have $rm \in cl_{\mathfrak{p}}(N) \bigcap \left(\bigcap_{j=2}^n Q_j\right) = N$. Now by [3, Theorem 1.1] we see that $\bigcap_{ht(\mathfrak{p})<k\leq \dim R} \left(\operatorname{ann}\operatorname{Ext}^k(M/N, R)\right) \subseteq \mathfrak{p}_i$ for every i such that $2 \leq i \leq n$, and since every primary ideal contains a power of its radical, the result follows. □

3. Generalised Associated Primes

We now return to the problem of determining the associated prime ideals of (the radical of) a submodule N of M. From Theorem 2.7 we have a description, in the Noetherian case, of the components of a normal prime decomposition of the radical of N – provided, that is, we know the associated primes of $\operatorname{rad} N$. We have seen, however, that $AP(N)$ can be quite different from $AP(\operatorname{rad} N)$. Nevertheless, one might hope that there is some sort of connection between these two collections of associated primes. The results of this section demonstrate one such connection, at least in the Noetherian case.

Despite the remarks at the end of Section 1, there are in fact some cases where, if a prime submodule contains an intersection of submodules, then it must contain one of these submodules. Note that this result was anticipated by the proof of Theorem 2.11.

Lemma 3.1. *Let N be a proper submodule of a finitely generated R-module M such that N has a normal primary decomposition $N = \bigcap_{i=1}^{n} Q_i$, where Q_i is $\mathfrak{p}_i$-primary for each i $(1 \leq i \leq n)$. Then for every minimal prime ideal $\mathfrak{p}$ to $(N : M)$, every $\mathfrak{p}$-prime submodule P of M which contains N must also contain Q_i for some i $(1 \leq i \leq n)$.*

Proof. Let $\mathfrak{q}_i = (Q_i : M)$ for each i $(1 \leq i \leq n)$, and note that $\bigcap_{i=1}^{n} \mathfrak{q}_i = (N : M) \subseteq \mathfrak{p}$. Hence $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some i $(1 \leq i \leq n)$, and since $\mathfrak{p}$ is a minimal prime to $(N : M)$, then $\mathfrak{q}_j \nsubseteq \mathfrak{p}$ for all $j \neq i$. Since $\left(\bigcap_{j \neq i} \mathfrak{q}_j\right) Q_i \subseteq N \subseteq P$ and $\bigcap_{j \neq i} \mathfrak{q}_j \nsubseteq \mathfrak{p}$, it follows that $Q_i \subseteq P$. $\square$

Compare the following result with Proposition 1.8, and contrast it with Example 1.9.

Corollary 3.2. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M. Then for every minimal prime ideal $\mathfrak{p}$ over $(N : M)$, $\mathfrak{p} \in AP(N) \bigcap AP(\operatorname{rad} N)$.*

Proof. Let $\mathfrak{p}$ be a minimal prime over $(N : M)$. It is well-known that, in this setting, $\mathfrak{p} \in AP(N)$. We also know that $\operatorname{rad} N$ has a normal prime decomposition, say $\operatorname{rad} N = \bigcap_{i=1}^{n} P_i$, where P_i is $\mathfrak{p}_i$-prime for each i $(1 \leq i \leq n)$. (Note that $AP(\operatorname{rad} N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ by Proposition 1.4). Now for any prime ideal $\mathfrak{q}$ (including $\mathfrak{q} = \mathfrak{p}$) containing $(N : M)$, the existence of a $\mathfrak{q}$-prime submodule containing N is given by [13, Theorem 3.3] (even if R is not Noetherian). Hence there exists a $\mathfrak{p}$-prime submodule P that contains $\operatorname{rad} N$, and we see from Lemma 3.1 that P must contain P_i for some i $(1 \leq i \leq n)$, and thus $\mathfrak{p} \supseteq \mathfrak{p}_i$. Since $\mathfrak{p}$ is minimal over $(N : M)$, it follows that $\mathfrak{p} \in AP(\operatorname{rad} N)$. $\square$

Let N be a proper submodule of a finitely generated module M and let $\mathfrak{p}$ be a prime ideal of R such that $(N : M) \subseteq \mathfrak{p}$. By Corollary 2.4, $\mathfrak{p} \in AP(N + \mathfrak{p}M)$. For a prime ideal $\mathfrak{q}$ of R we write $\mathfrak{p} \underset{N}{\longrightarrow} \mathfrak{q}$ provided $\mathfrak{q} \in AP(N + \mathfrak{p}M)$. Note that if $\mathfrak{p} \underset{N}{\longrightarrow} \mathfrak{q}$ then in particular $\mathfrak{p} \subseteq \mathfrak{q}$.

Definition 3.3. *Let N be a submodule of a finitely generated module M. A prime ideal $\mathfrak{p}$ of R is a **generalised associated prime** of N if there exists a positive integer n and there exist prime ideals $\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R such that $\mathfrak{p}_0 \in AP(N)$ and*

$$\mathfrak{p}_0 \underset{N}{\longrightarrow} \mathfrak{p}_1 \underset{N}{\longrightarrow} \mathfrak{p}_2 \underset{N}{\longrightarrow} \cdots \underset{N}{\longrightarrow} \mathfrak{p}_n = \mathfrak{p}.$$

Note that $\mathfrak{p}_0 \in AP(N)$ implies that $(N : M) \subseteq \mathfrak{p}_0$ so that $\mathfrak{p}_0 \underset{N}{\longrightarrow} \mathfrak{p}_1$ makes sense. Note also that for each i $(1 \leq i \leq n-1)$, $\mathfrak{p}_i \in AP(N + \mathfrak{p}_{i-1}M)$. So $(N : M) \subseteq (N + \mathfrak{p}_{i-1}M : M) \subseteq \mathfrak{p}_i$ and hence $\mathfrak{p}_i \underset{N}{\longrightarrow} \mathfrak{p}_{i+1}$ makes sense.

For a submodule N of a finitely generated module M, we let $GAP(N)$ denote the collection of generalised associated primes of N. It is clear that $AP(N) \subseteq GAP(N)$. We remark that we could have chosen to write $\mathfrak{p}_0 \underset{N}{\longrightarrow} \mathfrak{p}_1 \underset{N+\mathfrak{p}_0M}{\longrightarrow} \mathfrak{p}_2 \underset{N+\mathfrak{p}_1M}{\longrightarrow} \cdots \underset{N+\mathfrak{p}_{n-2}M}{\longrightarrow} \mathfrak{p}_n = \mathfrak{p}$, instead of the above definition. However, nothing would have been gained by doing so, since under this alternative we would have had, for each i $(2 \leq i \leq n)$, $\mathfrak{p}_i \in AP(N + \mathfrak{p}_0M + \mathfrak{p}_1M + \cdots + \mathfrak{p}_{i-1}M)$. But as noted above, $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$, so clearly $N + \mathfrak{p}_0M + \mathfrak{p}_1M + \cdots + \mathfrak{p}_{i-1}M = N + \mathfrak{p}_{i-1}M$.

Considering that for each link in the chain given in Definition 3.3, we have $\mathfrak{p}_i \in AP(N + \mathfrak{p}_{i-1}M)$, the next result is perhaps not terribly surprising.

Lemma 3.4. *Let N be a submodule of a finitely generated module M, let $\mathfrak{p} \in GAP(N)$ and let n be a positive integer. If $\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals satisfying $\mathfrak{p}_0 \in AP(N)$ and $\mathfrak{p}_0 \underset{N}{\longrightarrow} \mathfrak{p}_1 \underset{N}{\longrightarrow} \mathfrak{p}_2 \underset{N}{\longrightarrow} \cdots \underset{N}{\longrightarrow} \mathfrak{p}_n = \mathfrak{p}$, then for each i $(0 \leq i \leq n-1)$, $GAP(N + \mathfrak{p}_iM) \subseteq GAP(N)$ and moreover, for each j $(0 \leq i \leq j \leq n)$, $\mathfrak{p}_j \in GAP(N + \mathfrak{p}_iM)$.*

Proof. Let $\mathfrak{q} \in GAP(N + \mathfrak{p}_iM)$ for some i $(0 \leq i \leq n-1)$. Then there exists a positive integer k and a collection of prime ideals $\mathfrak{q}_0, \mathfrak{q}_1, \cdots, \mathfrak{q}_k$ satisfying $\mathfrak{q}_0 \in AP(N + \mathfrak{p}_iM)$ and $\mathfrak{q}_0 \underset{N+\mathfrak{p}_iM}{\longrightarrow} \mathfrak{q}_1 \underset{N+\mathfrak{p}_iM}{\longrightarrow} \mathfrak{q}_2 \underset{N+\mathfrak{p}_iM}{\longrightarrow} \cdots \underset{N+\mathfrak{p}_iM}{\longrightarrow} \mathfrak{q}_k = \mathfrak{q}$. Now since $\mathfrak{q}_0 \in AP(N + \mathfrak{p}_iM)$, then we have $\mathfrak{p}_i \underset{N}{\longrightarrow} \mathfrak{q}_0$. We also have (as in the remarks preceding this result) $\mathfrak{p}_i \subseteq \mathfrak{q}_0 \subseteq \cdots \subseteq \mathfrak{q}_k$, so that for each l $(1 \leq l \leq k)$, $\mathfrak{q}_l \in AP((N + \mathfrak{p}_iM) + \mathfrak{q}_{l-1}M) = AP(N + \mathfrak{q}_{l-1}M)$. In other words, we have

$$\mathfrak{p}_0 \underset{N}{\longrightarrow} \mathfrak{p}_1 \underset{N}{\longrightarrow} \mathfrak{p}_2 \underset{N}{\longrightarrow} \cdots \underset{N}{\longrightarrow} \mathfrak{p}_i \underset{N}{\longrightarrow} \mathfrak{q}_0 \underset{N}{\longrightarrow} \mathfrak{q}_1 \underset{N}{\longrightarrow} \cdots \underset{N}{\longrightarrow} \mathfrak{q}_k = \mathfrak{q},$$

and the first part is proved.

Now let j be such that $(i \leq j \leq n)$. Observe that $(N + \mathfrak{p}_iM : M) = \mathfrak{p}_i$, so that $\mathfrak{p}_i \in AP(N + \mathfrak{p}_iM)$ (Corollary 3.2). Since $j \geq i$, the result follows immediately from Definition 3.3 and the remarks following it. $\square$

Before getting to the first of the two main results of this section, we remark that our aim in having introduced generalised associated primes has been to provide a means of determining $AP(\operatorname{rad} N)$ – or at least a means of narrowing the search for the elements of $AP(\operatorname{rad} N)$. Indeed, the second of our two main results of this section, Theorem 3.6, shows that $AP(\operatorname{rad} N)$ is in fact contained in $GAP(N)$. However, this second result would provide little benefit, if it turned out that $GAP(N)$ were infinite.

Theorem 3.5. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M. Then $GAP(N)$ is finite.*

Proof. Suppose the result is false. Because M is Noetherian there exists a submodule K which is maximal in the collection of proper submodules of M which have infinitely many generalised associated prime ideals. By Proposition 1.3 we may write $AP(K) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$, where m is some positive integer. There are two cases to consider: either $(K : M)$ is prime or it isn't. If $(K : M)$ is prime, then it belongs to $AP(K)$ (Corollary 3.2), so without loss of generality, in this case we suppose $\mathfrak{q}_1 = (K : M)$. Note that, regardless of whether $(K : M)$ is prime, for any i such that $\mathfrak{q}_i \neq (K : M)$, then $\mathfrak{q}_i M \nsubseteq K$, and thus $K \subsetneq K + \mathfrak{q}_i M$. Moreover, we have $(K + \mathfrak{q}_i M : M) = \mathfrak{q}_i$ (Lemma 2.3) and therefore $\mathfrak{q}_i \in AP(K + \mathfrak{q}_i M)$ (again, Corollary 3.2). For convenience, we let $t = 1$ if $(K : M)$ is prime; otherwise let $t = 0$.

Now let $\mathfrak{p} \in GAP(K)$. There exist a positive integer n and prime ideals $\mathfrak{p}_i$ $(0 \leq i \leq n)$ of R such that $\mathfrak{p}_0 \in AP(K)$ and

$$\mathfrak{p}_0 \underset{K}{\longrightarrow} \mathfrak{p}_1 \underset{K}{\longrightarrow} \mathfrak{p}_2 \underset{K}{\longrightarrow} \cdots \underset{K}{\longrightarrow} \mathfrak{p}_n = \mathfrak{p}.$$

Now $\mathfrak{p}_0 = \mathfrak{q}_j$ for some j $(1 \leq j \leq m)$. If $t < j \leq m$ (i.e., $\mathfrak{q}_j \neq (K : M)$) then we have $\mathfrak{p} \in GAP(K + \mathfrak{q}_j M)$ (Lemma 3.4). The only other case to consider is where $\mathfrak{p}_0 = \mathfrak{q}_1 = (K : M)$ (i.e., $t = 1 = j$). In this case, $\mathfrak{p}_0 M \subseteq K$ and thus $\mathfrak{p}_1 \in AP(K)$. If $\mathfrak{p}_1 M \subseteq K$ then $\mathfrak{p}_0 = \mathfrak{p}_1$, which implies that $\mathfrak{p}_2 \in AP(K)$. Repeating this argument, either $\mathfrak{p} = \mathfrak{p}_0 = \mathfrak{q}_1$ or $\mathfrak{p} \in GAP(K + \mathfrak{p}_k M)$ where k is the least positive integer such that $1 \leq k \leq n$ and $\mathfrak{p}_k \neq \mathfrak{p}_0$. In the latter case, $\mathfrak{p}_k \in AP(K)$ and thus $\mathfrak{p}_k = \mathfrak{q}_h$ for some h $(t < h \leq m)$.

We have proved that, for any $\mathfrak{p} \in GAP(K)$, either $\mathfrak{p} \in AP(K)$ or $\mathfrak{p} \in GAP(K + \mathfrak{q}_r M)$ for some r $(t < r \leq m)$. It follows that

$$GAP(K) \subseteq \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\} \bigcup GAP(K + \mathfrak{q}_{t+1} M) \bigcup \cdots \bigcup GAP(K + \mathfrak{q}_m M).$$

But by the choice of K, $GAP(K + \mathfrak{q}_i M)$ is finite for all i $(t < i \leq m)$. Hence $GAP(K)$ is finite, a contradiction. The result follows. $\square$

Let N be a proper submodule of M. Recall that, at least in the Noetherian case, for every prime ideal $\mathfrak{p} \in AP(\operatorname{rad} N)$, then there exists a $\mathfrak{p}$-prime submodule P of M which is a minimal prime over N (Lemma 2.1), but that the converse of this does not hold (Example 1.9). With this in mind, we let $MP(N)$ denote the set of all prime ideals $\mathfrak{p}$ such that there exists a $\mathfrak{p}$-prime submodule P of M which is a minimal prime over N.

For a proper ideal $\mathfrak{a}$ of R, we denote the *height* of $\mathfrak{a}$ by $ht(\mathfrak{a}) = \inf\{ht(\mathfrak{p}) : \mathfrak{p}$ is a minimal prime over $\mathfrak{a}\}$. In case $\mathfrak{a} = R$, we take $ht(\mathfrak{a})$ to be one greater than the dimension of R (or infinity, if R is infinite dimensional). Obviously, in the context of Theorem 3.5, there is a finite upper bound on the heights of the elements of $GAP(N)$. We shall see from Theorem 3.6 that the requirement in Theorem 2.11 for R to be finite dimensional is unnecessary; simply replace $\dim R$ with $\gamma(N) = \max\{ht(\mathfrak{p}) : \mathfrak{p} \in GAP(N)\}$.

Theorem 3.6. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M. Then $AP(\operatorname{rad} N) \subseteq GAP(N)$.*

Proof. By Lemma 2.1, it suffices to show that $MP(N) \subseteq GAP(N)$. Let $\mathfrak{p} \in MP(N)$; i.e., there exists a $\mathfrak{p}$-prime submodule P which is minimal over N. We induct on $n = ht(\mathfrak{p}) - ht(N : M)$. If $\mathfrak{p}$ is a minimal prime of $(N : M)$ (in particular, if $n = 0$) then $\mathfrak{p} \in AP(N) \subseteq GAP(N)$. Alternatively, if $\mathfrak{p}$ is not a minimal prime of $(N : M)$, then $\mathfrak{p}$ properly contains some minimal prime ideal $\mathfrak{q}$ to $(N : M)$. By the previous argument, $\mathfrak{q} \in GAP(N)$. Note that $(N + \mathfrak{q}M : M) = \mathfrak{q}$, by Lemma 2.3. It follows that $N + \mathfrak{q}M$ cannot be prime, since it is contained properly in P, which is itself a minimal prime to N. By the same token, $cl_{\mathfrak{q}}(N + \mathfrak{q}M)$ is a minimal prime to $N + \mathfrak{q}M$ (Theorem 2.5). Taking into account the obvious fact that P is likewise a minimal prime to $N + \mathfrak{q}M$, we see that $cl_{\mathfrak{q}}(N + \mathfrak{q}M) \nsubseteq P$. It follows that $N + \mathfrak{q}M$ must have a normal primary decomposition of the form $cl_{\mathfrak{q}}(N + \mathfrak{q}M) \bigcap \left(\bigcap_{j=2}^{k} Q_j\right)$, where for each j $(2 \le j \le k)$, Q_j is $\mathfrak{q}_j$-primary, and $\mathfrak{q} \subsetneq \mathfrak{q}_j$. Now since $cl_{\mathfrak{q}}(N+\mathfrak{q}M) \nsubseteq P$, but $cl_{\mathfrak{q}}(N+\mathfrak{q}M) \bigcap \left(\bigcap_{j=2}^{k} Q_j\right) = N+\mathfrak{q}M \subseteq P$, then $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some i $(2 \le i \le k)$. In particular $\mathfrak{q}_i \in AP(N + \mathfrak{q}M)$ and $\mathfrak{q} \subsetneq \mathfrak{q}_i \subseteq \mathfrak{p}$. Hence, as before, we have $\mathfrak{q}_i = (N + \mathfrak{q}_i M : M)$, and clearly P is a minimal prime to $N + \mathfrak{q}_i M$, so that $\mathfrak{p} \in MP(N + \mathfrak{q}_i M)$. Now since $ht(\mathfrak{p}) - ht(\mathfrak{q}_i) < n$, the proof follows by induction. $\square$

Corollary 3.7. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M. Then $\operatorname{rad} N = \bigcap_{\mathfrak{p}_i \in GAP(N)} cl_{\mathfrak{p}_i}(N + \mathfrak{p}_i M)$.*

Proof. Observe that by Lemma 2.3 and Theorem 2.5, $cl_{\mathfrak{p}_i}(N+\mathfrak{p}_i M)$ is a prime submodule containing N for every $\mathfrak{p}_i \in GAP(N)$, so that $\operatorname{rad} N \subseteq \bigcap_{\mathfrak{p}_i \in GAP(N)} cl_{\mathfrak{p}_i}(N+\mathfrak{p}_i M)$. The result now follows from Theorems 3.6 and 2.7. $\square$

Of course, the prime decomposition given in Corollary 3.7 is not necessarily a normal prime decomposition – some redundancies are fairly likely to exist. In the next section we shall provide a result which effectively eliminates whatever redundancies there might be.

4. Eliminating Redundant Primes

In light of Theorem 2.7, eliminating the redundant prime submodules in the (not necessarily normal) prime decomposition in Corollary 3.7 amounts to determining which elements of $GAP(N)$ do not belong to $AP(\operatorname{rad} N)$. Admittedly, at first glance this could effectively be as difficult as determining $\operatorname{rad} N$ in the first place. While it might be fairly easy, for a given submodule L of M, to compute $AP(L)$, we have already seen that this does not give us $AP(\operatorname{rad} L)$. It would be quite helpful if one would only need to consider the associated primes of a known submodule, rather than its radical. This was indeed the thinking behind the construction of $GAP(N)$.

Note that in the context of Corollary 3.7, $(N + \mathfrak{p}_i M : M) = \mathfrak{p}_i$ for each $\mathfrak{p}_i \in GAP(N)$. For this reason, we are mainly concerned at this stage with submodules L such that $(L : M)$ is a prime ideal. Recall that for any submodule L of a Noetherian module M, then L is a primary submodule if and only if $AP(L)$ consists solely of one prime ideal of R.

Lemma 4.1. *Let R be Noetherian, let N be a submodule of a finitely generated R-module M and let $\mathfrak{p}$ be a prime ideal of R such that $(N : M) \subseteq \mathfrak{p}$. Then $N + \mathfrak{p}M$ is prime if and only if $AP(N + \mathfrak{p}M) = \{\mathfrak{p}\}$.*

Proof. This follows immediately from the preceding remarks, from Lemma 2.3 and from the remarks following Proposition 1.1. □

We saw in Theorem 3.5 that $GAP(N)$ is finite for any submodule N of a Noetherian module M, and in the subsequent remarks, we let $\gamma(N) = \max\{ht(\mathfrak{p}) : \mathfrak{p} \in GAP(N)\}$.

Corollary 4.2. *Let R be Noetherian and let N be a submodule of a finitely generated R-module M such that $(N : M) = \mathfrak{p}$ is a prime ideal of R. Then N is prime if and only if $ht\left(\operatorname{ann}\operatorname{Ext}^k(M/N, R)\right) > k$ for all k such that $ht(\mathfrak{p}) < k \leq \gamma(N)$.*

Proof. The result follows from Lemma 4.1 and [3, Theorem 1.1]. □

Let R be Noetherian, let N be a proper submodule of a finitely generated R-module M, and let $\mathfrak{p} \in GAP(N)$. We already know that the minimal primes over $(N : M)$ belong to $AP(\operatorname{rad} N)$ (Corollary 3.2), so we consider the case that $\mathfrak{p}$ is not minimal over $(N : M)$. One fairly obvious strategy for determining whether $\mathfrak{p} \in AP(\operatorname{rad} N)$ is to begin with those primes in $GAP(N)$ of least height, and work one's way up. Recall that $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is minimal amongst those $\mathfrak{p}$-prime submodules of M that contain N (but it might not be a minimal prime over N).

It turns out that in following this strategy, then by the time one gets round to trying to determine whether $\mathfrak{p} \in AP(\operatorname{rad} N)$, one can already know all the elements of $AP(\operatorname{rad} N)$ that are properly contained in $\mathfrak{p}$. With this in mind, for any $\mathfrak{p} \in GAP(N)$ such that $\mathfrak{p}$ is not minimal over $(N : M)$, let $k(\mathfrak{p})$ denote the set of prime ideals $\mathfrak{q}$ belonging to $AP(\operatorname{rad} N)$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$, let $K(\mathfrak{p}) = \bigcap_{\mathfrak{q} \in k(\mathfrak{p})} cl_{\mathfrak{q}}(N + \mathfrak{q}M)$, and let $k(\mathfrak{p})^c = AP(\operatorname{rad} N) \setminus k(\mathfrak{p})$. We are now ready for the main result of this section.

Theorem 4.3. *Let R be Noetherian and let N be a proper submodule of a finitely generated R-module M. Let $\mathfrak{p} \in GAP(N)$ such that $\mathfrak{p}$ is not a minimal prime over $(N : M)$. Then the following are equivalent:*

(i) $\mathfrak{p} \in AP(\operatorname{rad} N)$;

(ii) $K(\mathfrak{p}) \nsubseteq cl_{\mathfrak{p}}(N + \mathfrak{p}M)$;

(iii) $(N + \mathfrak{p}M : K(\mathfrak{p})) = \mathfrak{p}$.

Proof. (i) $\iff$ (ii). Recall that $\operatorname{rad} N = \bigcap_{\mathfrak{q} \in AP(\operatorname{rad} N)} cl_{\mathfrak{q}}(N + \mathfrak{q}M)$ is a normal prime decomposition (Theorem 2.7), and note that $K(\mathfrak{p})$ is actually the intersection of some of the components of this decomposition.

Now if $\mathfrak{p} \in AP(\operatorname{rad} N)$, then it is clear from Theorem 2.7 that $K(\mathfrak{p}) \nsubseteq cl_{\mathfrak{p}}(N + \mathfrak{p}M)$. Conversely, observe that

$$cl_{\mathfrak{p}}(N + \mathfrak{p}M) \supseteq \operatorname{rad} N = K(\mathfrak{p}) \bigcap \Big(\bigcap_{\mathfrak{q} \in k(\mathfrak{p})^c} cl_{\mathfrak{q}}(N + \mathfrak{q}M) \Big).$$

Now if $K(\mathfrak{p}) \nsubseteq cl_{\mathfrak{p}}(N + \mathfrak{p}M)$, then

$$\mathfrak{p} \supseteq \Big(\bigcap_{\mathfrak{q} \in k(\mathfrak{p})^c} cl_{\mathfrak{q}}(N + \mathfrak{q}M) : M \Big) = \bigcap_{\mathfrak{q} \in k(\mathfrak{p})^c} \mathfrak{q},$$

and thus $\mathfrak{p} \supseteq \mathfrak{a}$ for some $\mathfrak{a} \in k(\mathfrak{p})^c$. It follows that $\mathfrak{p} = \mathfrak{a}$.

(ii) $\Rightarrow$ (iii). Since $cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ is $\mathfrak{p}$-prime and $K(\mathfrak{p}) \nsubseteq cl_{\mathfrak{p}}(N + \mathfrak{p}M)$, then $(cl_{\mathfrak{p}}(N + \mathfrak{p}M) : K(\mathfrak{p})) = \mathfrak{p}$. We also have $\mathfrak{p}M \subseteq N + \mathfrak{p}M \subseteq cl_{\mathfrak{p}}(N + \mathfrak{p}M)$ and so $\mathfrak{p} \subseteq (\mathfrak{p}M : K(\mathfrak{p})) \subseteq (N + \mathfrak{p}M : K(\mathfrak{p})) \subseteq (cl_{\mathfrak{p}}(N + \mathfrak{p}M) : K(\mathfrak{p})) = \mathfrak{p}$.

(iii) $\Rightarrow$ (ii). Observe that $((N + \mathfrak{p}M) \bigcap K(\mathfrak{p}) : K(\mathfrak{p})) = (N + \mathfrak{p}M : K(\mathfrak{p})) = \mathfrak{p}$. It is an easy exercise to show that

$$\left(\frac{(N + \mathfrak{p}M) \bigcap K(\mathfrak{p})}{N} : \frac{K(\mathfrak{p})}{N} \right) = \mathfrak{p}.$$

Now since $K(\mathfrak{p})/N$ is finitely generated, then by [13, Theorem 3.3] there exists a $\mathfrak{p}$-prime submodule P' of $K(\mathfrak{p})$ containing N such that

$$\frac{(N+\mathfrak{p}M)\bigcap K(\mathfrak{p})}{N} \subseteq \frac{P'}{N}.$$

Thus $\mathfrak{p}M\bigcap K(\mathfrak{p}) \subseteq (N+\mathfrak{p}M)\bigcap K(\mathfrak{p}) \subseteq P'$, and hence, by the Lying Over Theorem for modules (see [17, Theorem 3.1]), there exists a $\mathfrak{p}$-prime submodule P of M such that $P\bigcap K(\mathfrak{p}) = P'$. This implies that $K(\mathfrak{p}) \nsubseteq P$, whereas $cl_{\mathfrak{p}}(N+\mathfrak{p}M) \subseteq P$, and the proof is complete. $\square$

As for determining the members of $AP(\operatorname{rad} N)$, recall that the strategy outlined above is to work one's way up according to the height of the primes in $GAP(N)$, eliminating the unnecessary primes along the way. It would be quite helpful if it were true that, for any element $\mathfrak{p} \in GAP(N)$ such that $\mathfrak{p} \notin AP(\operatorname{rad} N)$, then for every $\mathfrak{q} \in GAP(N)$ such that $\mathfrak{p} \subsetneq \mathfrak{q}$, we would likewise have $\mathfrak{q} \notin AP(\operatorname{rad} N)$. Unfortunately, this is not the case, as the next example demonstrates.

Example 4.4. *Let $R = F[x,y,z]$, where F is a field, let $M = R\oplus R\oplus R$, and let N be the submodule of M generated by the elements $(x,0,0)$, $(0,y,0)$, $(0,0,x)$, $(0,0,y)$ and $(0,0,z)$. Then $GAP(N) = \{Rx,\ Ry,\ Rx+Ry,\ \mathfrak{m}\}$, where $\mathfrak{m} = Rx+Ry+Rz$.*

However, $N = \operatorname{rad} N = P_1\bigcap P_2\bigcap P_3$ is a normal prime decomposition, where P_1 is the Rx-prime submodule $Rx\oplus R\oplus R$, P_2 is the Ry-prime submodule $R\oplus Ry\oplus R$ and P_3 is the $\mathfrak{m}$-prime submodule $\mathfrak{m}M$. Thus $AP(\operatorname{rad} N) = \{Rx, Ry, \mathfrak{m}\}$ (missing out $Rx+Ry$).

We conclude with one further example, in order to demonstrate some of the results of this paper. The computations were carried out in the computer algebra system Macaulay2.

Example 4.5. *Let $R = (\mathbb{Z}/101\mathbb{Z})[w,x,y,z]$ and let $M = R^{(5)}$, with standard basis $\{e_1,\dots,e_5\}$. Let N be the submodule of M given by*

$$\begin{aligned} N = {} & R(x^2y^3e_1 + w^2xe_2 - x^3y^2e_3) + R(x^2ye_4 - z^2e_5) \\ & + R(wz^2e_2 + x^3ye_3 + x^2y^2e_4) + R(xy^2e_4 + w^3e_5). \end{aligned}$$

Since N is generated by 4 elements, then $(N:M) = 0$. Let $\mathfrak{p} = R(w^3x + yz^2)$ and note that $\mathfrak{p}$ is a prime ideal of R. We have

$$\begin{aligned} GAP(N) = \{ & 0, Rx, Ry, \mathfrak{p}, Rw+Rx, Rw+Ry, Rx+Ry, Rx+Rz, \\ & Rw+Rx+Ry, Rw+Rx+Rz, Rw+Ry+Rz, \\ & Rx+Ry+Rz, Rw+Rx+Ry+Rz\}, \end{aligned}$$

and working our way along this list of prime ideals, from least to greatest height, we get the following prime submodules, which, as it turns out, will yield a normal prime decomposition of $\operatorname{rad} N$*:*

$$P_0 = (N :_M r), \text{ where } r = w^3x^5y^2 + x^4y^3z^2$$
$$P_1 = (N + RxM :_M wz^4) = Rx \oplus R \oplus Rx \oplus Rx \oplus R$$
$$P_2 = (N + RyM :_M wz^2) = Ry \oplus R \oplus Ry \oplus Ry \oplus R$$
$$P_3 = (N + \mathfrak{p}M :_M x^2y^3z^2)$$
$$P_4 = (N + (Rx + Rz)M :_M w^3) = (Rx + Rz)^{(4)} \oplus R.$$

(We remark that neither P_0 *nor* P_3 *has a terribly simple description – the latter being generated by 16 elements, which we decline to list here.) Note that for all the prime ideals* $\mathfrak{q}$ *such that* $\mathfrak{q} \in GAP(N) \setminus \{0, Rx, Ry, \mathfrak{p}, Rx + Rz\}$*, we have* $(N + \mathfrak{q}M : K(\mathfrak{q})) \neq \mathfrak{q}$*. We find then that* $\operatorname{rad} N$ *is given by*

$$\operatorname{rad} N = P_0 \bigcap P_1 \bigcap P_2 \bigcap P_3 \bigcap P_4,$$

and that

$$AP(\operatorname{rad} N) = \{0, Rx, Ry, \mathfrak{p}, Rx + Rz\}.$$

As a final remark, we are able to compute (based on ideas found in [14]*) the uniform dimension of* $M/\operatorname{rad} N$*, and we find that* $u(M/\operatorname{rad} N) = 7$*.*

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