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FULL HEAPS AND REPRESENTATIONS OF AFFINE WEYL GROUPS

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Abstract. We use the author's combinatorial theory of full heaps to categorify the action of a large class of Weyl groups on their root systems, and thus to give an elementary and uniform construction of a family of faithful permutation representations of Weyl groups. Examples include the standard representations of affine Weyl groups as permutations of $\mathbb{Z}$ and geometrical examples such as the realization of the Weyl group of type E_6 as permutations of 27 lines on a cubic surface; in the latter case, we also show how to recover the incidence relations between the lines from the structure of the heap. Another class of examples involves the action of certain Weyl groups on sets of pairs (t, f), where $t \in \mathbb{Z}$ and f is a function from a suitably chosen set to the two-element set $\{+, -\}$. Each of the permutation representations corresponds to a module for a Kac–Moody algebra, and gives an explicit basis for it.

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CONTENTS

	Introduction	2
1.	Heaps over Dynkin diagrams	5
2.	Full heaps	7
3.	The Weyl group and skew ideals	10
4.	Main results	15
5.	Permutations of $\mathbb{Z}$	19
6.	Geometrical examples	26
7.	The binary path representation in type affine B	30
8.	Other binary path representations	34
9.	Some related constructions	38

Acknowledgements 41

References 41

Introduction

In [7], we introduced the notion of full heaps, which are remarkable locally finite labelled posets that are closely related to Kac–Moody algebras, distributive lattices, crystal bases and Weyl groups. The main result of [7] is a construction of almost all affine Kac–Moody algebras modulo their one-dimensional centres in terms of raising and lowering operators on the space spanned by the so-called proper ideals of a suitable full heap. We mentioned briefly in [7, §8] that there is a natural action of the Weyl group W on the set $\mathcal{B}$ of proper ideals of a full heap E, and one purpose of this paper is to understand this action.

We concentrate on the case where the full heap corresponds to an affine Kac–Moody algebra. Our strategy here is to use the distributive lattice $\mathcal{B}$ to categorify the root system associated to W, in a way that is compatible with the action of W; this is based on the categorification of positive roots implicit in [7]. More precisely, we define a "decategorification" map χ from $\mathcal{B} \times \mathcal{B}$ taking values in the root lattice of W, and we concentrate on those pairs $(F, F') \in \mathcal{B} \times \mathcal{B}$ whose character is a real root; in such a situation we say that F is "skew" to F'. We prove in Theorem 4.1 that the diagonal action of W on $\mathcal{B} \times \mathcal{B}$ induces, via χ, the usual action of W on the root system. The standard fact that W acts faithfully on the (real) roots then implies that the action of W on $\mathcal{B}$ is faithful.

The permutation representations of Weyl groups arising from the action of W on $\mathcal{B}$ are very interesting, and we give explicit descriptions of some of the most important ones in this paper. These constructions may at first glance appear ad hoc, but in fact the structure of full heaps over graphs is tightly constrained: it turns out that with the exception of the Dynkin diagram $A_l^{(1)}$ (see Figure 1 below), there are no full heaps over Dynkin diagrams of finite or affine type other than the ones discussed in this paper. (We hope to give full details of these constructions elsewhere.) It should also be emphasized that the full heaps approach to these permutation representations equips them with extra structure that would not otherwise be obvious *a priori*. The reason for this is that the set $\mathcal{B}$ is naturally equipped with the structure of a partially ordered set, and also with a W-invariant irreflexive and symmetric relation, namely the aforementioned relation of skewness. The skewness relation leads to some interesting connections with geometry, graph theory, and the theory of two-graphs, some of which we mention in §6. The partial order should allow the permutation representations studied here to be deformed

in a natural way into representations of the associated affine Hecke algebra. The action of an arbitrary reflection of the Weyl group on the set $\mathcal{B}$ is easy to describe, and by using the partial order on $\mathcal{B}$, one can go on to describe the weak and strong Bruhat orders in terms of full heaps. (This phenomenon has been widely studied in the special cases that we discuss in §5; see [3, §8] for details.) We do not have space to discuss all these applications in this paper, but we hope to publish details separately.

Loosely speaking, the permutation representations arising from full heaps seem to be of three types, depending on the structure of the full heap E that gives rise to them, although the definitions themselves are completely uniform.

The first type of permutation representation arises from full heaps all of whose antichains are short (i.e., usually of size 1, and otherwise of size 2), and the permutation representations arising in this way include the standard realizations of the affine Weyl groups of types A, B, C and D as permutations of the integers. In this case, most pairs (F, F') of elements of $\mathcal{B}$ will be skew, and it is this close relationship between the lattice $\mathcal{B}$ and the root system that makes these representations so useful for understanding the Coxeter group structure of W. A comprehensive guide to these representations (and their applications) may be found in [3, §8], and we give a brief history of them at the beginning of §5. These representations also arise (together with a representation of the affine Weyl group of type G_2) in recent work of Cellini, Möseneder Frajria and Papi [4, §4], as a by-product of their work on a combinatorial interpretation of Kostant's formula for powers of the Euler product.

The second type of permutation representation arises from full heaps all of whose antichains are long, i.e., of size approaching $n/2$, where n is the number of elements in the Dynkin diagram. In these cases, the heap E is as far from being totally ordered as possible, so labelling the elements of $\mathcal{B}$ by integers is not convenient and depends heavily on choice. However, we will show how to parametrize the elements of $\mathcal{B}$ by certain pairs (t, f), where t is an integer and f is a function from some set (depending on E) to the set $\{+, -\}$ with two elements. For permutation representations of this type, most pairs (F, F') of elements of $\mathcal{B}$ will not be skew. These examples, which are described in §7 and §8, are new from the Weyl group point of view to the best of our knowledge, although they are reminiscent of the wreath product constructions of the finite Weyl groups of types B and D [9, §2.10].

The third type of permutation representation includes all other cases, meaning that the antichains of the heap are of intermediate length. The relation of skewness on $\mathcal{B}$ is the most interesting in these cases, but some ingenuity may be required

to obtain an appropriate parametrization. We look at one such example in detail in §6, namely the case of type E_6, which we obtain from type $E_6^{(1)}$ by restriction. The full heap construction realizes the finite Weyl group as a permutation group on 27 objects. These objects can be identified in a natural way with the 27 lines on a cubic surface, and remarkably, the combinatorial notion of skewness coincides with the geometric notion of skewness on the 27 lines. The approach here also makes it obvious how to lift the action of E_6 on the 27 lines to the affine Weyl group of type $E_6^{(1)}$ in an explicit way (Proposition 6.1). It is already known that the 27 lines are in correspondence with the weights of a minuscule representation of the Lie algebra E_6 and that two lines are incident if and only if the corresponding weights are not orthogonal with respect to a certain inner product (see [13, §1, §3]). However, the full heaps approach is more elementary in that one need only use the theory of Coxeter groups and their root systems, and the representation theory of Lie algebras is not required to describe the construction. There should also be a similar geometric construction for type E_7. There are other examples of this third type of representation, including some associated to the Coxeter system of type $A_l^{(1)}$, but since they are not yet well understood, we will not consider them here.

The injective homomorphisms from affine Weyl groups to permutations of $\mathbb{Z}$ used by Cellini et al [4] and by Eriksson [6] are defined in terms of the action of the affine Weyl group on a carefully chosen vector. An advantage of our combinatorial point of view using full heaps over these two approaches is that it is extremely elementary: the only Lie theory needed for our construction is the definition of a Dynkin diagram, or generalized Cartan matrix, and we do not even need any linear algebra. However, the proof that the map we define is indeed an injective homomorphism does use Lie theoretic concepts, such as the theory of Coxeter groups.

As we explained in [7], a full heap over the Dynkin diagram of an affine Kac–Moody algebra determines on the one hand a representation of the Kac–Moody algebra and on the other hand a representation of the corresponding affine Weyl group. The permutation representations we study in this paper are therefore closely related to the representation theory of Lie algebras. In particular, the parametrizations of the sets $\mathcal{B}$ given in this paper have immediate applications to Lie algebras. For example, it follows from Proposition 7.8 (respectively, Proposition 8.2) that the finite Weyl group of type B_n (respectively, D_n) has a natural faithful action on the set of all strings of length n from the alphabet $\{+,-\}$ (respectively, the set of all strings of length n from the alphabet $\{+,-\}$ that contain an even number of occurrences of $-$). From the Lie algebra point of view described in [7], these sets

of strings parametrize crystal bases of spin representations of the corresponding simple Lie algebras over $\mathbb{C}$, and the action of a Chevalley basis on them may be explicitly calculated. Note that we do not need Clifford algebras to do this, and the heaps approach makes it obvious why the modules have dimensions 2^n and 2^{n-1}, respectively. Although this result could also be achieved directly using the theory of crystal bases, the full heap may be a much simpler object than the corresponding crystal (see remarks 8.8 and 9.3).

1. Heaps over Dynkin diagrams

We first review from [7, §1] some of the basic properties of heaps over Dynkin diagrams. The definitions relating to generalized Cartan matrices come from [10], and the heap definitions are based on [19].

Let A be an n by n matrix with integer entries. We call A a *generalized Cartan matrix* if it satisfies the conditions (a) $a_{ii} = 2$ for all $1 \leq i \leq n$, (b) $a_{ij} \leq 0$ for $i \neq j$ and (c) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. In this paper, we will only consider generalized Cartan matrices with entries in the set $\{2, 0, -1, -2\}$; such matrices are sometimes called *doubly laced*. If, furthermore, A has no entries equal to -2, we will call A *simply laced*.

The Dynkin diagram $\Gamma = \Gamma(A)$ associated to a generalized Cartan matrix is a directed graph, possibly with multiple edges, and vertices indexed (for now) by the integers 1 up to n. If $i \neq j$ and $|a_{ij}| \geq |a_{ji}|$, we connect the vertices corresponding to i and j by $|a_{ij}|$ lines; this set of lines is equipped with an arrow pointing towards i if $|a_{ij}| > 1$. For example, if $a_{ij} = a_{ji} = -2$, this will result in a double edge between i and j equipped with an arrow pointing in each direction. There are further rules if $a_{ij}a_{ji} > 4$, but we do not need these for our purposes.

The Dynkin diagram (together with the enumeration of its vertices) and the generalized Cartan matrix determine each other, so we may write $A = A(\Gamma)$. If Γ is connected, we call A *indecomposable*.

Let Γ be a Dynkin diagram with vertex set P. Let C be the relation on P such that $x\ C\ y$ if and only if x and y are distinct unadjacent vertices in Γ, and let $\mathcal{C}$ be the complementary relation.

Definition 1.1. A *labelled heap* over Γ is a triple $(E, \leq, \varepsilon)$ where $(E, \leq)$ is a locally finite partially ordered set (in other words, a poset all of whose intervals are finite) with order relation denoted by $\leq$ and where ε is a map $\varepsilon : E \to P$ satisfying the following two axioms.

1. For every $\alpha, \beta \in E$ such that $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$, α and β are comparable in the order $\leq$.
2. The order relation $\leq$ is the transitive closure of the relation $\leq_{\mathcal{C}}$ such that for all $\alpha, \beta \in E$, $\alpha \ \leq_{\mathcal{C}}\ \beta$ if and only if both $\alpha \leq \beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$.

We call $\varepsilon(\alpha)$ the *label* of α. In the sequel, we will sometimes appeal to the fact that the partial order is the reflexive, transitive closure of the covering relations, because of the local finiteness condition.

Definition 1.2. Let $(E, \leq, \varepsilon)$ and $(E', \leq', \varepsilon')$ be two labelled heaps over Γ. We say that E and E' are isomorphic (as labelled posets) if there is a poset isomorphism $\phi : E \to E'$ such that $\varepsilon = \varepsilon' \circ \phi$.

A *heap* over Γ is an isomorphism class of labelled heaps. We denote the heap corresponding to the labelled heap $(E, \leq, \varepsilon)$ by $[E, \leq, \varepsilon]$.

We will sometimes abuse language and speak of the underlying set of a heap, when what is meant is the underlying set of one of its representatives.

Definition 1.3. Let $(E, \leq, \varepsilon)$ be a labelled heap over Γ, and let F a subset of E. Let ε' be the restriction of ε to F. Let $\mathcal{R}$ be the relation defined on F by $\alpha\ \mathcal{R}\ \beta$ if and only if $\alpha \leq \beta$ and $\varepsilon(\alpha)\ \mathcal{C}\ \varepsilon(\beta)$. Let $\leq'$ be the transitive closure of $\mathcal{R}$. Then $(F, \leq', \varepsilon')$ is a labelled heap over Γ. The heap $[F, \leq', \varepsilon']$ is called a *subheap* of $[E, \leq, \varepsilon]$.

If $E = (E, \leq, \varepsilon)$ is a labelled heap over Γ, then we define the *dual labelled heap*, E^* of E, to be the labelled heap $(E, \geq, \varepsilon)$. (The notion of "dual heap" is defined analogously.) There is a natural anti-isomorphism of labelled posets from E to E^*, which we will denote by $*$.

Recall that if $(E, \leq)$ is a partially ordered set, a function $\rho : E \to \mathbb{Z}$ is said to be a *rank function* for $(E, \leq)$ if whenever $a, b \in E$ are such that $a < b$ is a covering relation, we have $\rho(b) = \rho(a) + 1$. If a rank function for $(E, \leq)$ exists, we say $(E, \leq)$ is *ranked*, and we say that the heap $(E, \leq, \varepsilon)$ is ranked to mean that $(E, \leq)$ is ranked as a partially ordered set.

If F is convex as a subset of E (in other words, if $\alpha \leq \beta \leq \gamma$ with $\alpha, \gamma \in F$, then $\beta \in F$) then we call F a *convex subheap* of E. If, whenever $\alpha \leq \beta$ and $\beta \in F$ we have $\alpha \in F$, then we call F an *ideal* of E. If F is an ideal of E with $\emptyset \subsetneq F \subsetneq E$ such that for each vertex p of Γ we have $\emptyset \subsetneq F \cap \varepsilon^{-1}(p) \subsetneq \varepsilon^{-1}(p)$, then we call F a *proper ideal* of E. If $\gamma_1, \gamma_2, \ldots, \gamma_k$ are elements of E, then we define the ideal

$$\langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle = \{\alpha \in E : \alpha \leq \gamma_i \text{ for some } 1 \leq i \leq k\}.$$

If $F = \langle \gamma \rangle$ for some $\gamma \in E$, then we call F a *principal* ideal.

We will often implicitly use the fact that a subheap is determined by its set of vertices and the heap it comes from.

Definition 1.4. Let $(E, \leq, \varepsilon)$ be a labelled heap over Γ. We say that $(E, \leq, \varepsilon)$ and $[E, \leq, \varepsilon]$ are *fibred* if

(a) for each vertex p in Γ, the subheap $\varepsilon^{-1}(p)$ is unbounded above and unbounded below,
(b) for every pair p, p' of adjacent vertices in Γ and every element $\alpha \in E$ with $\varepsilon(\alpha) = p$, there exists $\beta \in E$ with $\varepsilon(\beta) = p'$ such that either α covers β or β covers α in E.

Remark 1.5.

(i) It is easily checked that these are sound definitions, because they are invariant under isomorphism of labelled heaps.
(ii) Condition (a) provides a way to name the elements of E, which we shall need in the sequel. Choose a vertex p of Γ. Since E is locally finite, $\varepsilon^{-1}(p)$ is a chain of E isomorphic as a partially ordered set to the integers, so one can label each element of this chain as $E(p, z)$ for some $z \in \mathbb{Z}$. Adopting the convention that $E(p, x) < E(p, y)$ if $x < y$, this labelling is unique once a distinguished vertex $E(p, 0) \in \varepsilon^{-1}(p)$ has been chosen for each p. When E is understood, we will use the shorthand $p(y)$ for $E(p, y)$.

2. Full heaps

We are now ready to recall the definition of our main object of study from [7], which builds on work of Stembridge [18] and Wildberger [20].

Definition 2.1. Let E be a fibred heap over a Dynkin diagram Γ with generalized Cartan matrix A. If every open interval (α, β) of E such that $\varepsilon(\alpha) = \varepsilon(\beta) = p$ and $(\alpha, \beta) \cap \varepsilon^{-1}(p) = \emptyset$ satisfies $\sum_{\gamma \in (\alpha, \beta)} a_{p, \varepsilon(\gamma)} = -2$, we call E a *full* heap.

The definition says that either (a) (α, β) contains precisely two elements with labels (q_1, q_2, say) adjacent to p and such that there is no arrow from q_1 (or q_2) to p in the Dynkin diagram, or that (b) (α, β) contains precisely one element with label (q, say) adjacent to p such that there is an arrow from q to p in the Dynkin diagram.

Definition 2.2. Let R^+ be the set of all functions $P \to \mathbb{Z}^{\geq 0}$. Let E be a labelled heap over Γ and let F be a finite subheap of E. We define the *character*, $\chi(F)$ of F to be the element of R^+ such that $\chi(F)(p)$ is the number of elements of F with ε-value p. If $\alpha \in R^+$, we write $\mathcal{L}_\alpha(E)$ to be the set of all convex subheaps F of E with $\chi(F) = \alpha$. If F consists of a single element α with $\varepsilon(\alpha) = p$, we will write $\chi(F) = p$ for short, so that $\mathcal{L}_p(E)$ is identified with the elements of E labelled by p. Since the function χ is an invariant of labelled heaps, we can extend the definition to apply to finite heaps of Γ.

We call E *periodic* if there exists a nonidentity automorphism $\phi : E \to E$ of labelled posets such that $\phi(x) \geq x$ for all $x \in E$. By [7, Remark 7.2], ϕ restricts to an automorphism of the chains $\varepsilon^{-1}(p)$ for p a vertex of Γ of the form $\phi(E(p,x)) = E(p, x+t_p)$ for some nonnegative integer t_p depending on p but not on the labelling chosen for E. If $\alpha \in R^+$ is such that $\alpha(p) = t_p$, we will say that ϕ is *periodic with period* α. If there is no automorphism ϕ' of E with period α' such that $\alpha = n\alpha'$ with $n > 1$, then we also say that E is periodic with period α and fundamental automorphism ϕ.

Note that in a periodic heap E, the automorphism ϕ induces an inclusion-preserving permutation of the proper ideals of E.

Definition 2.3. Let E be a full heap over a graph Γ and let k be a field. Let $\mathcal{B}$ be the set of proper ideals of E, so that $\mathcal{B}$ has the structure of a distributive lattice with meet and join operations $I \wedge J = I \cap J$ and $I \vee J = I \cup J$; these operations are defined by [7, Lemma 2.1 (ii)]. Let V_E be the k-span of the set $\{v_I : I \in \mathcal{B}\}$. For any proper ideal I and any finite convex subheap $L \leq E$, we write $L \succ I$ to mean that both $I \cup L$ is an ideal and $I \cap L = \emptyset$, and we write $L \prec I$ to mean that both $L \leq I$ and $I \backslash L$ is an ideal. We define linear operators X_L, Y_L and H_L on V_E as follows:

$$X_L(v_I) = \begin{cases} v_{I \cup L} & \text{if } L \succ I, \\ 0 & \text{otherwise,} \end{cases}$$

$$Y_L(v_I) = \begin{cases} v_{I \backslash L} & \text{if } L \prec I, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_L(v_I) = \begin{cases} v_I & \text{if } L \prec I \text{ and } L \not\succ I, \\ -v_I & \text{if } L \succ I \text{ and } L \not\prec I, \\ 0 & \text{otherwise.} \end{cases}$$

If p is a vertex of Γ, we write X_p for the linear operator on V_E given by

$$\sum_{L \in \mathcal{L}_p(E)} X_L,$$

and we define Y_p and H_p similarly. Note that although these sums are infinite, it follows from the definitions of fibred and full heaps that at most one of the terms in each case may act in a nonzero way on any given v_I. In this situation, we also write $p \succ I$ to mean that $L \succ I$ for some (necessarily unique) $L \in \mathcal{L}_p(E)$, and analogously we write $p \prec I$ with the obvious meaning. Note that it is not possible for both $p \prec I$ and $p \succ I$, because I cannot contain a convex chain $\alpha < \beta$ with $\varepsilon(\alpha) = \varepsilon(\beta) = p$.

By [7, Lemma 2.1], the operators X_L, Y_L and H_L are nonzero and well defined (see also [7, Definition 2.6]).

Example 2.4. The operators X_p, Y_p and H_p are of key importance in the application of full heaps to affine Kac–Moody algebras: they respectively represent the action of the Chevalley generators e_p, f_p and h_p of the derived algebra $\mathfrak{g}'(A)$.

For an example of these operators, consider, the heap E shown in Figure 15 in §7, and let F be the ideal $\langle 2(1), 5(1) \rangle$ of E. Then F consists of the elements

$$\{0(y), 1(y), 4(y) : y \leq 0\} \cup \{2(y), 3(y), 5(y) : y \leq 1\}.$$

In this case, we have $X_4(v_F) = v_{F \cup \{4(1)\}}$, $X_0(v_F) = v_{F \cup \{0(1)\}}$, and $X_p(v_F) = 0$ for $p \in \{1, 2, 3, 5\}$. We also have $Y_2(v_F) = v_{F \setminus \{2(1)\}}$, $Y_5(v_F) = v_{F \setminus \{5(1)\}}$, and $Y_p(v_F) = 0$ for $p \in \{0, 1, 3, 4\}$. We have

$$H_p(v_F) = \begin{cases} v_F & \text{if } p \in \{2, 5\}, \\ -v_F & \text{if } p \in \{0, 4\}, \\ 0 & \text{if } p \in \{1, 3\}. \end{cases}$$

Definition 2.5. Let A be a simply laced generalized Cartan matrix of affine type and let Γ be the corresponding Dynkin diagram, and suppose that μ is a nonidentity graph automorphism of Γ. Assume furthermore that (a) μ has order precisely 2 and (b) for any vertex p, $\mu(p)$ and p are not distinct adjacent vertices. The group $\{1, \mu\}$ acts on the Dynkin diagram Γ, and we denote the orbit containing the vertex p by $f(p) = \bar{p}$.

The Dynkin diagram $\overline{\Gamma}$ for $\bar{\mathfrak{g}}$ has vertices labelled by the orbits $\bar{p}$, and is such that if p and q are distinct vertices of Γ, then p and q are adjacent in Γ if and only if the (distinct) vertices $\bar{p}$ and $\bar{q}$ are adjacent in $\overline{\Gamma}$. We join $\bar{p}$ and $\bar{q}$ by a single edge,

unless Γ contains three vertices p, $\mu(p)$ and q such that q is adjacent to both p and $\mu(p)$, in which case we join $\bar{p}$ and $\bar{q}$ in $\overline{\Gamma}$ by a double edge with an arrow pointing towards $\bar{p}$. (Note that in the case where Γ is a square, it is possible that (a) p is adjacent to both q and $\mu(q)$ and (b) q is adjacent to both p and $\mu(p)$. In this case, the procedure just described results in a double edge from p to q with arrows in each direction.)

In the above situation, we will say that A (respectively, Γ) *folds* to $\overline{A}$ (respectively, $\overline{\Gamma}$) via μ.

Definition 2.6. Let A be a simply laced generalized Cartan matrix of affine type with Dynkin diagram Γ, and let $E = (E, \leq, \varepsilon)$ be a full heap over Γ.

(i) The heap E is by definition a *simply folded full heap* over Γ.

(ii) Suppose there exists a diagram automorphism μ as in Definition 2.5 such that A and Γ fold to $\overline{A}$ and $\overline{\Gamma}$ respectively via μ. Suppose furthermore that whenever we have vertices p, q of Γ satisfying (a) $\mu(p)\ \mathcal{C}\ q$, (b) $\alpha \in \varepsilon^{-1}(p)$ and (c) $\beta \in \varepsilon^{-1}(q)$, then α and β are comparable in E. Then we say that $\overline{E} := (E, \leq, f \circ \varepsilon)$ is a *simply folded full heap* over $\overline{\Gamma}$.

It is not immediate that the heap $\overline{E}$ is well defined, but this follows from [7, Proposition 6.1].

Remark 2.7. All examples of full heaps in this paper will be simply folded.

3. The Weyl group and skew ideals

We define the Weyl group, $W(\Gamma)$, associated to Γ to be the group with generators $S(\Gamma) = \{s_i \in I\}$ indexed by the vertices of Γ and defining relations

$$s_i^2 = 1 \text{ for all } i \in I,$$

$$s_i s_j = s_j s_i \text{ if } a_{ij} = 0,$$

$$s_i s_j s_i = s_j s_i s_j \text{ if } a_{ij} < 0 \text{ and } a_{ij}a_{ji} = 1,$$

$$s_i s_j s_i s_j = s_j s_i s_j s_i \text{ if } a_{ij} < 0 \text{ and } a_{ij}a_{ji} = 2.$$

Note that no relation is added in the case where $a_{ij} < 0$ and $a_{ij}a_{ji} = 4$.

Example 3.1. Define two generalized Cartan matrices

$$A_1 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Then the Weyl group corresponding to A_1 is

$$\langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^4 = 1 \rangle,$$

isomorphic to the dihedral group of order 8, and the Weyl group corresponding to A_2 is the infinite group

$$\langle s_1, s_2 : s_1^2 = s_2^2 = 1\rangle.$$

Definition 3.2. ([7, Definition 8.6])

Let A be a (not necessarily simply laced) generalized Cartan matrix with Dynkin diagram Γ, and let E be a simply folded full heap over Γ. For each vertex i of Γ, we define a linear operator S_i on V_E by requiring that

$$S_i(v_I) = \begin{cases} Y_i(v_I) & \text{if } Y_i(v_I) \neq 0, \\ X_i(v_I) & \text{if } X_i(v_I) \neq 0, \\ v_I & \text{otherwise.} \end{cases}$$

It follows that $S_i(v_I) = v_{I'}$ for some proper ideal I' of E, so we also write $S_i(I) = I'$.

The next result, whose proof is immediate from the definitions, will be useful in the sequel.

Lemma 3.3. *Maintain the notation of Definition 3.2, and define the integer c by $H_i(v_I) = cv_I$. Then we have*

$$S_i(v_I) = \begin{cases} Y_i(v_I) & \textit{if } c = 1, \\ X_i(v_I) & \textit{if } c = -1, \\ v_I & \textit{if } c = 0. \quad \square \end{cases}$$

The following results show how the Weyl group acts naturally on the lattice $\mathcal{B}$. The main purpose of this paper is to understand this action, which remarkably turns out to be faithful in all the well-understood cases.

Proposition 3.4. *Let A be a generalized Cartan matrix with Dynkin diagram Γ, and let E be a simply folded full heap over Γ, and let E^* be the dual heap. Let $\mathcal{B}$ and $\mathcal{B}^*$ be the lattices of ideals of E and E^*, respectively.*

(i) *The Weyl group acts transitively on $\mathcal{B}$ via $s_i.b = S_i(b)$ for each generator s_i of W.*

(ii) *The map $* : E \to E^*$ induces a map $*_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^*$ given by*

$$*_{\mathcal{B}}(I) = *(E \backslash I),$$

and this is an isomorphism of W-sets.

Proof. In [7, Proposition 8.7], it is shown that the assignment $s_i \mapsto -S_i$ defines a unique cyclic kW-module structure on V_E. The well-definedness of the action

follows from this by twisting by sign, together with the fact that S_i induces a function from $\mathcal{B}$ to $\mathcal{B}$. Transitivity now follows from the cyclic module structure, completing the proof of (i).

Using the definition of proper ideal, it is routine to check that the map $*_{\mathcal{B}}$ of (ii) is defined and bijective. Given $I \in \mathcal{B}$ and $I^* = *_{\mathcal{B}}(I) \in \mathcal{B}^*$ and a vertex i of Γ, we find that $L \succ I$ if and only if $L^* \prec^* I^*$, and $L \prec I$ if and only if $L^* \succ^* I^*$ (where $L^* = *_{\mathcal{B}}(L)$, and $\prec^*$ and $\succ^*$ have the obvious meanings). From the symmetry between Y_i and X_i in the definition of S_i (Definition 3.2), we can now deduce that the map $*_{\mathcal{B}}$ intertwines the two W-actions, completing the proof of (ii). $\square$

The key to understanding the Weyl group action will turn out to be the root system associated to W and the generalized Cartan matrix, which we now introduce.

Let $\Pi = \{\alpha_i : i \in I\}$ and let $\Pi^\vee = \{\alpha_i^\vee : i \in I\}$. We have a $\mathbb{Z}$-bilinear pairing $\mathbb{Z}\Pi \times \mathbb{Z}\Pi^\vee \to \mathbb{Z}$ defined by

$$\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij},$$

where (a_{ij}) is the generalized Cartan matrix. If $v = \sum_{i \in I} l_i \alpha_i \in \mathbb{Z}\Pi$, we write $v \geq 0$ to mean that $l_i \geq 0$ for all i, and we write $v > 0$ to mean that $l_i > 0$ for all i. We view $V = \mathbb{Z}\Pi$ as the integral form of a reflection representation of W, determined by the equalities $s_i(v) = v - \langle v, \alpha_i^\vee \rangle \alpha_i$ for all $i \in I$.

Indecomposable generalized Cartan matrices come in three mutually exclusive types (defined in [10, Theorem 4.3]) called *finite*, *affine* and *indefinite*. This paper is mostly concerned with finite and affine generalized Cartan matrices; the classification of these matrices may be found in [10, §4.8].

Following [10, §5], we define a *real root* to be a vector of the form $w(\alpha_i)$, where $w \in W$ and $\alpha_i \in \Pi$. If A is of finite type, all roots are real. If A is of affine type, there is a unique vector $\delta = \sum a_i \alpha_i$ such that $A\delta = 0$ and the a_i are relatively prime positive integers. Although the notion of *imaginary root* can be defined in general, in the affine type case the imaginary roots are easily characterized as precisely those vectors of the form $n\delta$ where n is a nonzero integer.

A *root* is by definition a real or imaginary root. We denote the set of roots by Δ, as in [10]. We say a root α is positive (respectively, negative) if $\alpha > 0$ (respectively, $\alpha < 0$). If α is a root, then so is $-\alpha$, and every root is either positive or negative. Following [10], we use the symbols Δ, Δ^{re} and Δ^{im} for the roots, real roots, and imaginary roots respectively. We denote the positive and negative roots by Δ_+ and Δ_- respectively, and we use the notation $\Delta_\pm^{re}$ and $\Delta_\pm^{im}$ with the obvious meanings. We will also identify the positive (real and imaginary) roots with elements of R^+ as in Definition 2.2 so that $\sum a_i \alpha_i$ corresponds to the function sending each i to a_i.

Lemma 3.5. *Let E be a simply folded full heap over a finite graph Γ, and let F and F' be proper ideals of E. Then $F\backslash F'$ and $F'\backslash F$ are finite heaps.*

Proof. By [7, Lemma 2.1 (ii)], $F \cap F'$ is a proper ideal of E. Since Γ is finite, [7, Lemma 2.1 (vi)] applied to the fact that $F \cap F' \subseteq F$ shows that $F\backslash(F \cap F')$ is finite, so that $F\backslash F'$ is finite, as required. The other assertion follows similarly. □

Lemma 3.5 ensures that the following definition makes sense.

Definition 3.6. Let E be a simply folded full heap over a finite graph Γ, and let F and F' be proper ideals of E. We define the character, $\chi(F, F') = \chi((F, F'))$, of the ordered pair (F, F') to be the function $P \to \mathbb{Z}$ given by

$$\chi(F, F')(p) = \chi(F'\backslash F)(p) - \chi(F\backslash F')(p).$$

If $\chi(F, F') \in \Delta^{re}$, then the proper ideals F and F' are said to be *skew*. We define $\Sigma(E)$ to be the set of pairs (F, F') of skew proper ideals of E.

Remark 3.7. Note that the relation "is skew to" is irreflexive and symmetric. The reason for the term "skew" will become clear when we study type E_6 in §6.

Lemma 3.8. *Let E be a simply folded full heap over a finite graph Γ, and let F and F' be proper ideals of E.*

(i) *We have $F \subseteq F'$ if and only if $\chi(F, F')(p) \geq 0$ for all vertices p of Γ.*
(ii) *We have $F' \subseteq F$ if and only if $\chi(F, F')(p) \leq 0$ for all vertices p of Γ.*
(iii) *If F, F', F'' are proper ideals of E, then $\chi(F, F'') = \chi(F, F') + \chi(F', F'')$.*

Proof. To prove (i), first note that if $F \subseteq F'$, then we have $\chi(F\backslash F')(p) = 0$ for all vertices p of Γ, and the assertion follows from the definition of $\chi(F, F')$.

Conversely, suppose that $\chi(F, F')(p) \geq 0$ for all vertices p of Γ, and let $F_p = F \cap \varepsilon^{-1}(p)$ and $F'_p = F' \cap \varepsilon^{-1}(p)$. We observe that

$$\chi(F, F')(p) = \chi(F'_p\backslash F_p)(p) - \chi(F_p\backslash F'_p)(p).$$

Since F and F' are proper ideals, it must be the case that one of the sets F_p and F'_p is included in the other, which means that at least one of $\chi(F'_p\backslash F_p)(p)$ and $\chi(F_p\backslash F'_p)(p)$ is zero. The hypothesis $\chi(F, F')(p) \geq 0$ forces $\chi(F_p\backslash F'_p)(p) = 0$ and $F_p \subseteq F'_p$. Since this is true for all p, we have $F \subseteq F'$, completing the proof of (i). The proof of (ii) follows by a similar argument.

Part (iii) also follows by using the techniques of the above paragraph. The chains F_p, F'_p, F''_p are totally ordered by inclusion and we obtain

$$\chi(F, F'')(p) = \chi(F, F')(p) + \chi(F', F'')(p),$$

from which the assertion follows. □

Lemma 3.9. *Let E be a full heap over a finite graph Γ, let $\mathcal{B}$ be the associated distributive lattice and let p and q be vertices of Γ (allowing the possibility $p = q$).*

(i) *We have $H_p X_q - X_q H_p = a_{pq} X_q$.*

(ii) *Let $F, F' \in \mathcal{B}$. Then there exists a finite sequence $F = F_0, F_1, \ldots, F_r = F'$ of elements of $\mathcal{B}$ such that for each $0 \leq i < r$, we have either $F_{i+1} = F_i \cup \{\beta_i\}$ or $F_i = F_{i+1} \cup \{\beta_i\}$, for some element $\beta_i \in E$. Furthermore, we have*

$$F' = S_{m_1} S_{m_2} \cdots S_{m_r}(F),$$

where each m_i is such that α_{m_i} appears with nonzero coefficient in $\chi(F, F')$.

(iii) *If $F, F' \in \mathcal{B}$ are such that $\chi(F, F') = \pm\alpha_q$, then $\chi(s_p(F), s_p(F')) = s_p(\pm\alpha_q)$.*

Proof. Part (i) is [7, Lemma 2.7 (2)].

For part (ii), we let $F'' = F \wedge F'$. By Lemma 3.5, $F \backslash F''$ is finite, so there is a finite sequence of operators Y_i such that

$$v_{F''} = Y_{i_1} Y_{i_2} \cdots Y_{i_k} v_F,$$

where each α_{i_l} appears with nonzero coefficient in $\chi(F'', F)$. Similarly, there is a sequence of X_i such that

$$v_{F'} = X_{j_1} X_{j_2} \cdots X_{j_{k'}} v_{F''},$$

where each α_{j_l} appears with nonzero coefficient in $\chi(F', F'')$. It follows from the definition of S_i that

$$F' = S_{j_1} S_{j_2} \cdots S_{j_{k'}} S_{i_1} S_{i_2} \cdots S_{i_k}(F).$$

Since $F'' = F \wedge F'$, an argument like that used to prove Lemma 3.8 (i) and (ii) shows that the supports of $\chi(F', F'')$ and $\chi(F'', F)$ are disjoint (in the notation of Lemma 3.8, we have $F''_p = F_p$ if $F_p \subseteq F'_p$, and $F''_p = F'_p$ otherwise). It then follows that each α_{i_l} and α_{j_l} mentioned above must appear with nonzero coefficient in $\chi(F', F'') + \chi(F'', F)$, which is equal to $\chi(F', F)$ by Lemma 3.8 (iii). This completes the proof of (ii).

To prove (iii), suppose that F, F' are as in the statement. We only need to consider the case $\chi(F, F') = \alpha_q$, because the other case follows by exchanging the roles of F and F'. We then need to prove that

$$\chi(s_p(F), s_p(F')) = \chi(F, F') - \left\langle \alpha_q, \alpha_p^\vee \right\rangle \alpha_p.$$

By Lemma 3.9 (i), we have

$$H_p \circ X_q - X_q \circ H_p = \left\langle \alpha_q, \alpha_p^\vee \right\rangle X_q.$$

Now $X_q.v_F = v_{F'}$, and by definition of H_p, there exist integers $c_1, c_2 \in \{-1, 0, 1\}$ such that $H_p \circ X_q.v_F = c_1 v_{F'}$ and $X_q \circ H_p.v_F = c_2 v_{F'}$ and $c_1 - c_2 = \left\langle \alpha_q, \alpha_p^\vee \right\rangle$.

Let us first consider the case where $\left\langle \alpha_q, \alpha_p^\vee \right\rangle = 2$, which implies that $c_1 = 1$ and $c_2 = -1$. The fact that $H_p \circ X_q.v_F = v_{F'}$ means that $H_p.v_{F'} = v_{F'}$, and, by Lemma 3.3, $S_p.v_F = Y_p.v_F$, so that S_p removes an element $\gamma_1 \in E$ (with $\varepsilon(\gamma_1) = p$) from F'. A similar argument shows that $S_p.v_F = X_p.v_F$, so that S_p adds an element $\gamma_2 \in E$ (with $\varepsilon(\gamma_2) = p$) to F. If $\gamma_1 \neq \gamma_2$, it follows that $s_p(F) \subset s_p(F')$ and

$$s_p(F') \backslash s_p(F) = F' \backslash (F \cup \{\gamma_1\} \cup \{\gamma_2\}),$$

meaning that $\chi(s_p(F), s_p(F')) = \chi(F, F') - 2\alpha_p$, as required. The other possibility is that $\gamma_1 = \gamma_2$ is the unique element of $F' \backslash F$. It follows that s_p exchanges F' and F, so that $\chi(F, F') = \alpha_p$ and $\chi(s_i(F), s_i(F')) = -\alpha_p = s_p(\alpha_p)$, as required.

The cases where $\left\langle \alpha_q, \alpha_p^\vee \right\rangle \in \{-2, -1, -0, 1\}$ follow very similar lines, but they are simpler in the sense that situations analogous to the $\gamma_1 = \gamma_2$ case above do not occur, and we always have $s_p(F) \subset s_p(F')$. The only real change needed to the above argument is that the value of $\left\langle \alpha_q, \alpha_p^\vee \right\rangle$ may lead to more than one possibility for c_1 (and c_2). □

4. Main results

In §4, we develop the main theoretical results of the paper; the remaining sections will be devoted to the study of specific examples.

Theorem 4.1. *Let E be a simply folded full heap over a (finite) Dynkin diagram Γ for an affine Kac–Moody algebra.*

(i) *Let $(F, F') \in \mathcal{B} \times \mathcal{B}$, and let $w \in W$. Then we have*

$$\chi(w(F), w(F')) = w(\chi(F, F')).$$

(ii) *Let $F, F' \in \mathcal{B}$, and let $w \in W$. Then $(F, F') \in \Sigma(E)$ if and only if $(w(F), w(F')) \in \Sigma(E)$.*

(iii) *The action of W on $\mathcal{B}$ is faithful.*

Note. It is necessary to use pairs of heaps rather than single heaps to express (i). Indeed, even if it is given that $F \subset F'$ and $w(F) \subset w(F')$, it is not generally the case that the finite subheap $w(F') \backslash w(F)$ is a function of $F' \backslash F$ and w.

Proof. To prove (i), we first deal with the case where $w = s_p$. Let F, F' be as in the statement, and let r be the integer defined in Lemma 3.9 (ii). The proof is by induction on r; the case $r = 0$ is trivial, and the case $r = 1$ is Lemma 3.9 (iii). For the inductive step, let $F = F_0, F_1, \ldots, F_r = F'$ be the sequence defined in Lemma 3.9 (ii). By Lemma 3.8 (iii), we have

$$\chi(F, F') = \sum_{i=0}^{r-1} \chi(F_i, F_{i+1}).$$

Using the additivity of the action of s_p, the case $r = 1$, and Lemma 3.8 (iii) again, we then obtain

$$\begin{aligned} s_p(\chi(F, F')) &= \sum_{i=0}^{r-1} s_p(\chi(F_i, F_{i+1})) \\ &= \sum_{i=0}^{r-1} \chi(s_p(F_i), s_p(F_{i+1})) \\ &= \chi(s_p(F), s_p(F')), \end{aligned}$$

as required. This completes the proof in the case $w = s_p$.

The proof of (i) for general w then follows by a straightforward induction on the length, m, of $w = s_{i_1} s_{i_2} \cdots s_{i_m}$.

The "only if" direction of part (ii) follows from the fact [10, §5.1] that w permutes Δ^{re}, and the "if" direction holds because w is invertible. To prove (iii), let $w \in W$ be such that $w \neq 1$; we will be done if we can show the existence of a proper ideal F'' such that $w.F'' \neq F''$. Since $w \neq 1$, it follows from standard properties [10, Lemma 3.11 (b)] that $w(\alpha_i) = \alpha < 0$ for some simple root $\alpha_i \in \Delta_+^{re}$. Let F, F' be proper ideals such that $\chi(F, F') = \alpha_i$; such ideals exist by [7, Lemma 2.1 (vii)] (which is not hard to check directly in this case). By (i), we have $\chi(w(F), w(F')) = \alpha \neq \chi(F, F')$, so we cannot have both $w(F) = F$ and $w(F') = F'$, establishing the existence of F'' as above. □

Remark 4.2. It is possible to extend the above theorem to deal with imaginary roots, but this is not interesting from the point of view of the Weyl group, because in the situations covered by the theorem, the action of W on the imaginary roots is trivial.

The Weyl groups associated to affine Kac–Moody algebras, which are the main examples of Weyl groups of interest to us, are equipped with a distinguished generator, s_0.

Definition 4.3. Let A be a generalized Cartan matrix of affine type with Dynkin diagram Γ and distinguished vertex 0, and let E be a full heap over Γ. Let W be

the Weyl group associated to Γ, and let W_0 be the (finite) subgroup generated by $S(\Gamma)\backslash\{s_0\}$. If F is a proper ideal of E, we define the *height*, $h(F)$, of F to be the maximal integer t such that $E(0,t) \in F$.

Lemma 4.4. *Maintain the notation of Definition 4.3, and fix $t \in \mathbb{Z}$.*

(i) *If F, F' are proper ideals of E, then $h(F \wedge F') = \min(h(F), h(F'))$ and $h(F \vee F') = \max(h(F), h(F'))$.*

(ii) *The subset $\mathcal{B}_t := \{F \in \mathcal{B} : h(F) = t\}$ is a sublattice of $\mathcal{B}$.*

(iii) *If I and I' are proper ideals of E and $X_i.v_I = v_{I'}$, then $h(I') = h(I) + \delta_{0i}$ (the Kronecker delta).*

(iv) *If I and I' are proper ideals of E and $Y_i.v_I = v_{I'}$, then $h(I') = h(I) - \delta_{0i}$.*

(v) *If E is periodic and ϕ is a labelled poset automorphism of E, then*

$$h(\phi(I)) = h(I) + k$$

for all proper ideals I, where $k \in \mathbb{Z}$ is such that $\phi(E(0,0)) = E(0,k)$. If $k = 1$ and $t \in \mathbb{Z}$, then $\mathcal{B}_t$ is a fundamental domain for the action of $\langle\phi\rangle$ on $\mathcal{B}$.

(vi) *The group W_0 acts transitively on $\mathcal{B}_t$, and on the $\langle\phi\rangle$-orbits of $\mathcal{B}$.*

(vii) *The map $f : F \mapsto F\backslash\langle E(0,0)\rangle$ defines a bijection between $\mathcal{B}_0$ and the set of all ideals (i.e., including $\emptyset$ and E_0) of the convex subheap E_0 of E given by*

$$E_0 = \{x \in E : x \not\geq E(0,1) \text{ and } x \not\leq E(0,0)\}.$$

Note. Some examples of the finite heaps E_0 may be found in [20], with some of the corresponding lattices of ideals in [14, §8.2].

Proof. Part (i) follows by considering the intersections of the chain $\varepsilon^{-1}(0)$ of E with F, F', $F \wedge F'$ and $F \vee F'$, and (ii) is immediate from (i). Parts (iii) and (iv) follow from the relevant definitions.

The second assertion of (v) is a consequence of the first, and the first assertion follows from the fact that $\phi(E(0,t)) = E(0,t+k)$ for any $t \in \mathbb{Z}$, by periodicity.

To prove (vi), we first note that W_0 acts on $\mathcal{B}_t$ by (iii) and (iv), because none of the generators s_i with $i \neq 0$ can change the height of $F \in \mathcal{B}_t$. To prove transitivity, choose $F, F' \in \mathcal{B}_t$. We need to prove the existence of $w \in W_0$ such that $w(F) = F'$. Since $h(F) = h(F')$, it follows that α_0 appears with zero coefficient in $\chi(F, F')$. Lemma 3.9 (ii) then completes the proof by producing the required w.

To prove that the heap E_0 in (vii) is convex, let $a, b \in E_0$ with $a < c < b$. Since $b \not\geq E(1,0)$, we have $c \not\geq E(1,0)$, and since $a \not\leq E(0,0)$, we have $c \not\leq E(0,0)$. This shows that $c \in E_0$ and it follows that E_0 is convex.

Now let I be the proper ideal $\langle E(0,0)\rangle$ of E. Since I is contained in every ideal of E of height 0, it follows that the map $f : F \mapsto F\backslash I$ is a function from $\mathcal{B}_0$ to the ideals of the heap E_0 as in the statement. The inverse of f is the map $g : g(G) = G \cup I$. To complete the proof of (vii), it remains to show that if G is an ideal of E_0, then $G \cup I$ is an ideal of E of height zero. Suppose that $x \in G \cup I$ and that $y \in E$ is such that $y < x$. Since every $x \in G \cup I$ satisfies $x \not\geq E(1,0)$ (whether $x \in G$ or $x \in I$), we have $y \not\geq E(0,1)$, and $G \cup I$ will have height 0 if it is an ideal. If $y \not\leq E(0,0)$ then $y \in E_0$ and thus $y \in G$, because G is an ideal of E_0. On the other hand, if $y \leq E(0,0)$, then $y \in I$ by definition. In either case, $y \in G \cup I$, and we conclude that $G \cup I$ is an ideal of E (of height 0). □

Theorem 4.5. *Maintain the notation of Definition 4.3, and suppose that the heap E is periodic with fundamental automorphism ϕ and period δ, where δ is the lowest positive imaginary root (see Remark 4.6 below).*

(i) *For any proper ideal F of $\mathcal{B}$ and any $k \in \mathbb{Z}$, we have $\chi(F, \phi^k(F)) = k\delta$.*

(ii) *The group generated by ϕ is isomorphic to $\mathbb{Z}$ and acts naturally on $\mathcal{B}$; if F is a proper ideal of E, then we denote by $[F]$ the $\langle\phi\rangle$-orbit containing F.*

(iii) *If $(F, F') \in \Sigma(E)$, then every ideal in $[F]$ is skew to every ideal in $[F']$; in this case we will say that the orbits $[F]$ and $[F']$ are skew.*

(iv) *There is a bijection $\omega : \mathcal{B} \to \mathbb{Z} \times \mathcal{B}_0$ given by $\omega(I) = (h(I), \phi^{-h(I)}(I))$. The induced action of W on $\mathbb{Z} \times \mathcal{B}_0$ is*

$$w.(t, I) = (t + k, \phi^{-k}(w.I)),$$

where h is the height function on ideals and $k = h(w.I)$. Applying the natural projection $\mathbb{Z} \times \mathcal{B}_0 \to \mathcal{B}_0$ and identifying $\mathcal{B}_0$ with the $\langle\phi\rangle$-orbits of $\mathcal{B}$ as in Lemma 4.4 (v), we recover the action of W on the $\langle\phi\rangle$-orbits.

Proof. Part (i) follows directly from the hypotheses.

The definition of fibred heap ensures that ϕ has infinite order. If $I \in \mathcal{B}$, we define $\phi(I)$ to be $\{\phi(i) : i \in I\}$; since this is an invertible map sending proper ideals to proper ideals, (ii) follows.

Let F and F' be as in (iii), and let $\phi^a(F)$ and $\phi^b(F')$ be typical elements of $[F]$ and $[F']$ respectively. By part (i), Lemma 3.8 (iii), and the fact that $\chi(G, G') = -\chi(G', G)$, we have

$$\chi(\phi^a(F), \phi^b(F')) = \chi(\phi^a(F), F) + \chi(F, F') + \chi(F', \phi^b(F')) = \chi(F, F') + (b - a)\delta.$$

By [10, Proposition 6.3 (d)], the sum of a real root and an integer multiple of δ is again a real root, so $\phi^a(F)$ and $\phi^b(F')$ are skew, as required.

The inverse of the map given in (iv) is $\omega^{-1}(t, I) = \phi^t(I)$. The formula for the induced action follows by a direct check. The last assertion is a consequence of Lemma 4.4 (v) and the fact that $\phi^{-k}(w.I)$ and $w.I$ are in the same orbit. □

Remark 4.6. The hypotheses about the period δ of E used in Theorem 4.5 are true for all the full heaps appearing in this paper (although not for the examples mentioned in §9). If the underlying Dynkin diagram of E corresponds to an untwisted affine Kac–Moody algebra, this was proved in [7, Lemma 7.4 (ii)]. All the examples of full heaps in this paper are of this type, with two exceptions, namely examples 5.4 and 8.7, which correspond to twisted affine Kac–Moody algebras. For these two cases, the data in [10, §4.8] can be used to verify the hypothesis directly.

5. Permutations of $\mathbb{Z}$

The first examples of Weyl group representations that we will consider realize the affine Weyl groups as permutations of the integers, which Björner and Brenti [3, p293] consider to be part of the folklore of the subject. In type affine A, the permutation representation of Example 5.1 first appeared in work of Lusztig [12], although without a proof of faithfulness, and the representation was further developed by Shi [15] and Björner and Brenti [2]. In type affine C, the permutation representation of Example 5.3 first appeared in work of Bédard [1], again without a proof of faithfulness, and was further studied by Shi [16]. These two examples are the simplest considered in this paper, in the sense that they are the only ones for which the heap is a totally ordered set. A unified treatment (with proofs) of affine Weyl groups of types A, B, C and D as permutations of $\mathbb{Z}$ is given in Eriksson's thesis [6]. These explicit constructions of affine Weyl groups may easily be reformulated so as to express the affine Weyl groups in terms of semidirect products of familiar groups. Conversely, the representations of these groups as permutations of $\mathbb{Z}$ can easily be produced from their realizations as semidirect products, although neither construction is obvious from the usual definition of affine Weyl groups via generators and relations.

All the examples of full heaps in §5 come from [7, Appendix]. All these heaps are periodic, and the dashed boxes in the diagrams indicate the repeating motif. We use the $p(y)$ notation mentioned in Remark 1.5 (ii) to name individual elements in a corresponding labelled heap. With this notation, it is the p-component of the vertex $p(y)$ that is repeating, while the y-component is determined by the labelled poset automorphism ϕ as in the definition of periodic heap.

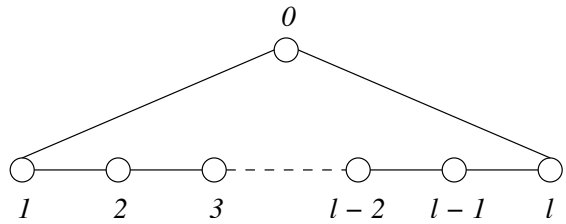

FIGURE 1. The Dynkin diagram of type $A_l^{(1)} (l > 1)$

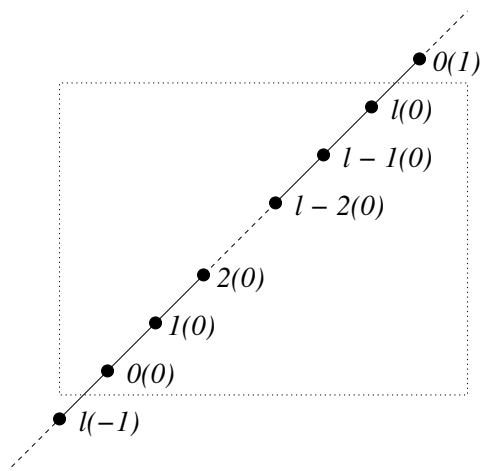

FIGURE 2. A full heap, E, over the Dynkin diagram of type $A_l^{(1)} (l > 1)$

Example 5.1. Consider the full heap shown in Figure 2 over the affine Dynkin diagram shown in Figure 1. In this case, every proper ideal is principal, so the set of proper ideals is precisely the set

$$\{\langle p(y)\rangle : 0 \leq p \leq l,\ y \in \mathbb{Z}\}.$$

Because the proper ideals are totally ordered, the map $\zeta : \mathcal{B} \to \mathbb{Z}$ defined by $\zeta(\langle p(y)\rangle) = (l+1)y + p + 1$ is an isomorphism of totally ordered sets (where $\mathbb{Z}$ is ordered in the usual way). In this way, the action of W on $\mathcal{B}$ induces an action of w on $\mathbb{Z}$, which is faithful by Theorem 4.1 (iii).

With these identifications, the action of s_i on $\mathbb{Z}$ is as follows:

$$s_i(z) = \begin{cases} z+1 & \text{if } z \equiv i \mod (l+1), \\ z-1 & \text{if } z \equiv i+1 \mod (l+1), \\ z & \text{otherwise.} \end{cases}$$

Thus, we recover the familiar realization of the affine Weyl group of type A as permutations of the integers, as described by Lusztig [12], together with a proof that this representation is faithful.

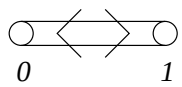

FIGURE 3. The Dynkin diagram of type $A_1^{(1)}$

Remark 5.2. The case $l = 1$ of Example 5.1 can also be checked to give a faithful representation, and the analogue of the heap in Figure 2 is indeed a full heap over the Dynkin diagram shown in Figure 3.

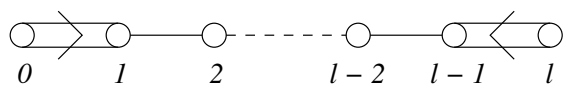

FIGURE 4. The Dynkin diagram of type $C_l^{(1)} (l > 1)$

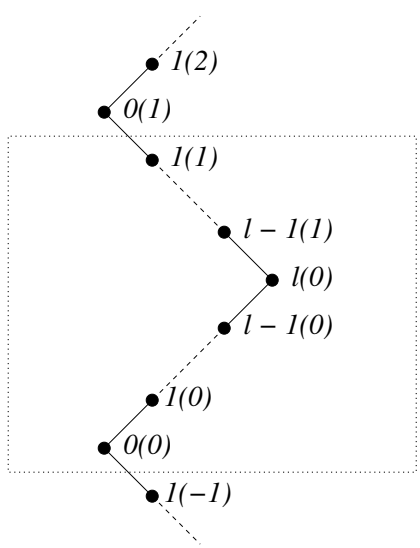

FIGURE 5. A full heap, E, over the Dynkin diagram of type $C_l^{(1)} (l > 1)$

Example 5.3. Consider the self-dual full heap shown in Figure 5 over the affine Dynkin diagram shown in Figure 4. As in Example 5.1, every proper ideal is principal, so the set of proper ideals is precisely the set

$$\{\langle p(y)\rangle : 0 \leq p \leq l,\ y \in \mathbb{Z}\}.$$

Because the proper ideals are totally ordered, we can again define an isomorphism $\zeta : \mathcal{B} \to \mathbb{Z}$ of totally ordered sets, as follows:

$$\zeta(\langle p(y)\rangle) = \begin{cases} 2ly+1 & \text{if } p=0, \\ 2ly+l+1 & \text{if } p=l, \\ ly+p+1 & \text{if } p \notin \{0,l\} \text{ and } y \text{ is even}, \\ l(y+1)-p+1 & \text{if } p \notin \{0,l\} \text{ and } y \text{ is odd}. \end{cases}$$

With these identifications, the action of s_j on $\mathbb{Z}$ is

$$s_j(z) = \begin{cases} z+1 & \text{if } z \equiv \pm j \mod 2l, \\ z-1 & \text{if } z \equiv (\pm j)+1 \mod 2l, \\ z & \text{otherwise.} \end{cases}$$

Thus, we recover the familiar realization of the affine Weyl group of type C as permutations of the integers, as described by Bédard [1], together with a proof that this representation is faithful.

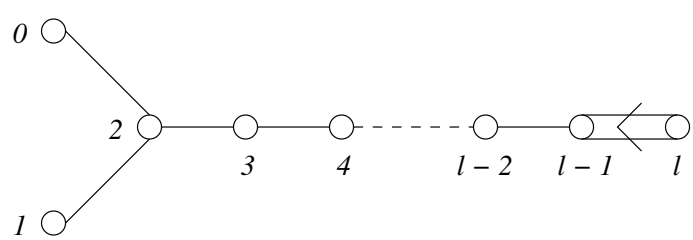

FIGURE 6. The Dynkin diagram of type $A^{(2)}_{2l-1}$

Example 5.4. Consider the self-dual full heap E of Figure 7, over the Dynkin diagram of type $A^{(2)}_{2l-1}$ shown in Figure 6. All proper ideals of E are principal, except those of the form $\langle 0(y), 1(y+1)\rangle$. We refine the order on the ideals to a total one by stipulating that, for all $y \in \mathbb{Z}$, $\langle 0(y)\rangle < \langle 1(y+1)\rangle$.

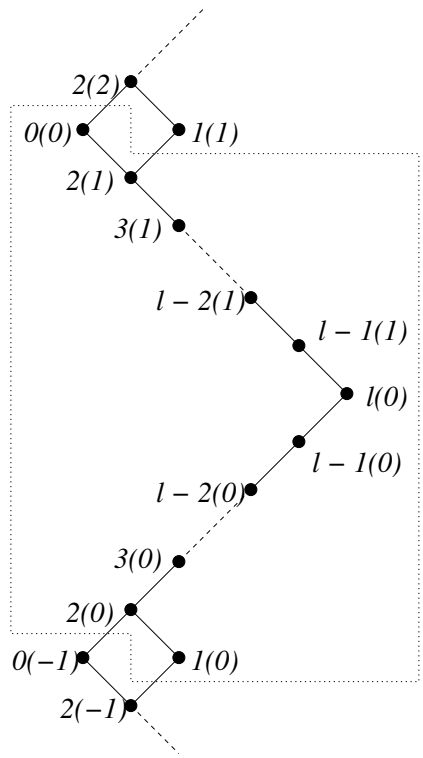

FIGURE 7. A full heap, E, over the Dynkin diagram of type $A^{(2)}_{2l-1}$

With this refinement, we can define an isomorphism $\zeta : \mathcal{B} \to \mathbb{Z}$ of totally ordered sets, as follows:

$$\zeta(\langle 0(y-1), 1(y)\rangle) = 2ly + 2,$$

$$\zeta(\langle p(y)\rangle) = \begin{cases} 2l(y+1) & \text{if } p = 0, \\ 2ly + 1 & \text{if } p = 1, \\ ly + p + 1 & \text{if } 1 < p < l \text{ and } y \text{ is even}, \\ l(y+1) - p + 1 & \text{if } 1 < p < l \text{ and } y \text{ is odd}, \\ 2ly + l + 1 & \text{if } p = l. \end{cases}$$

With these identifications, the action of s_j on $\mathbb{Z}$ (if $j \notin \{0, 1\}$) is

$$s_j(z) = \begin{cases} z + 1 & \text{if } z \equiv \pm j \mod 2l, \\ z - 1 & \text{if } z \equiv (\pm j) + 1 \mod 2l, \\ z & \text{otherwise.} \end{cases}$$

We have

$$s_0(z) = \begin{cases} z+1 & \text{if } z \equiv \pm 1 \mod 2l, \\ z-1 & \text{if } z \equiv 0 \text{ or } z \equiv 2 \mod 2l, \\ z & \text{otherwise,} \end{cases}$$

and

$$s_1(z) = \begin{cases} z+2 & \text{if } z \equiv -1 \text{ or } z \equiv 0 \mod 2l, \\ z-2 & \text{if } z \equiv 1 \text{ or } z \equiv 2 \mod 2l, \\ z & \text{otherwise.} \end{cases}$$

Thus, we recover the familiar realization of the affine Weyl group of type B as permutations of the integers, together with a proof that this representation is faithful.

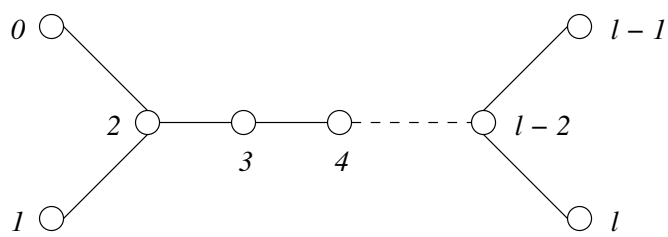

FIGURE 8. The Dynkin diagram of type $D_l^{(1)}$

Example 5.5. Consider the self-dual full heap shown in Figure 9 over the affine Dynkin diagram shown in Figure 8. In this case, it is not true that every proper ideal is principal. The only non-principal proper ideals are those of the form $\langle 0(y), 1(y+1)\rangle$ or $\langle l-1(y), l(y)\rangle$, where $y \in \mathbb{Z}$. The set of proper ideals is not totally ordered by inclusion, but we may refine the order to a total one by stipulating that, for all $y \in \mathbb{Z}$, $\langle l-1(y)\rangle < \langle l(y)\rangle$ and $\langle 0(y)\rangle < \langle 1(y+1)\rangle$.

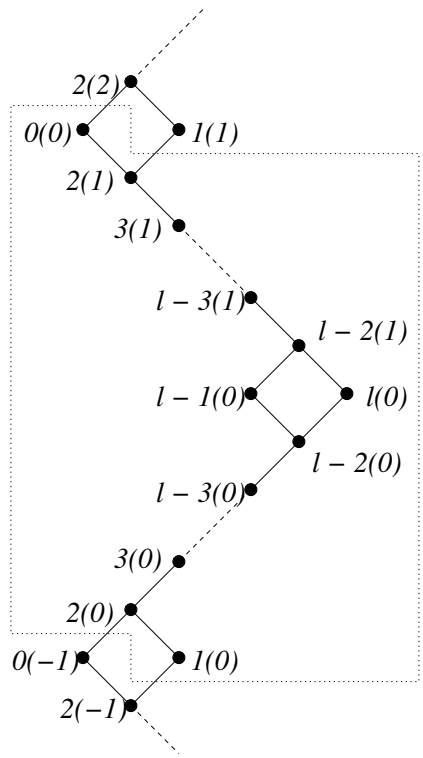

FIGURE 9. A full heap, E, over the Dynkin diagram of type $D_l^{(1)}$

With this refinement, we can define an isomorphism $\zeta : \mathcal{B} \to \mathbb{Z}$ of totally ordered sets, as follows:

$$\zeta(\langle 0(y-1), 1(y)\rangle) = 2ly + 2,$$
$$\zeta(\langle l-1(y), l(y)\rangle) = 2ly + l + 2,$$
$$\zeta(\langle p(y)\rangle) = \begin{cases} 2l(y+1) & \text{if } p = 0, \\ 2ly + p & \text{if } p = 1, \\ 2ly + p + 1 & \text{if } p \in \{l-1, l\}, \\ ly + p + 1 & \text{if } 1 < p < l-1 \text{ and } y \text{ is even}, \\ l(y+1) - p + 1 & \text{if } 1 < p < l-1 \text{ and } y \text{ is odd}. \end{cases}$$

With these identifications, the action of s_j on $\mathbb{Z}$ (if $j \notin \{0, 1, l-1, l\}$) is

$$s_j(z) = \begin{cases} z+1 & \text{if } z \equiv \pm j \mod 2l, \\ z-1 & \text{if } z \equiv (\pm j)+1 \mod 2l, \\ z & \text{otherwise.} \end{cases}$$

If $j \in \{0, l-1\}$, then we have

$$s_j(z) = \begin{cases} z+1 & \text{if } z \equiv j-1+c(j) \text{ or } z \equiv j+1+c(j) \mod 2l, \\ z-1 & \text{if } z \equiv j+c(j) \text{ or } z \equiv j+2+c(j) \mod 2l, \\ z & \text{otherwise,} \end{cases}$$

where we define $c(0) = 0$ and $c(l-1) = 1$. Finally, if $j \in \{1, l\}$, we have

$$s_j(z) = \begin{cases} z+2 & \text{if } z \equiv j-2+c(j) \text{ or } z \equiv j-1+c(j) \mod 2l, \\ z-2 & \text{if } z \equiv j+c(j) \text{ or } z \equiv j+1+c(j) \mod 2l, \\ z & \text{otherwise,} \end{cases}$$

where we define $c(1) = 0$ and $c(l) = 1$.

Thus, we recover the familiar realization of the affine Weyl group of type D as permutations of the integers, together with a proof that this representation is faithful.

By comparing the four examples above, the reader may correctly suspect that the further the proper ideals are from being totally ordered by inclusion, the less helpful it is to think of the action of W on $\mathcal{B}$ as a periodic permutation of the integers, even though this can be done in principle.

6. Geometrical examples

The application of Theorem 4.1 to type E_6 has some interesting connections with geometry, as we shall now show by considering the full heap E in Figure 11 over the Dynkin diagram of Figure 10. (The heap E is not self-dual, and by Proposition 3.4 (ii), we could equally well have started with the dual heap, E^*.)

The heap E is periodic with fundamental automorphism ϕ and period $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_0$. By explicit enumeration, we find that there are 27 orbits of proper ideals of E under the action of $\langle \phi \rangle$.

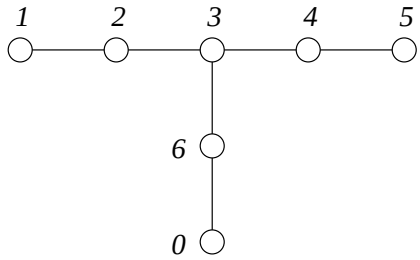

FIGURE 10. The Dynkin diagram of type $E_6^{(1)}$

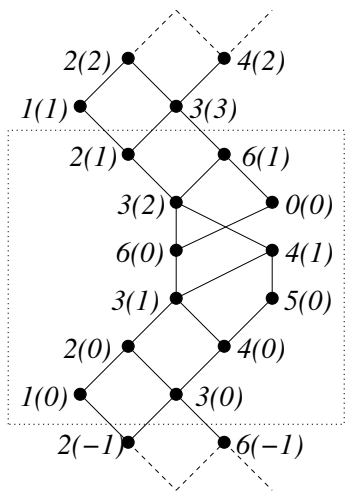

FIGURE 11. A full heap, E, over the Dynkin diagram of type $E_6^{(1)}$

A complete set of representatives, together with labels, is as follows:

$$
\begin{array}{rclrclrcl}
a_1 & = & \langle 1(0)\rangle, & a_2 & = & \langle 2(1)\rangle, & a_3 & = & \langle 3(2)\rangle, \\
a_4 & = & \langle 4(1), 6(0)\rangle, & a_5 & = & \langle 6(0), 5(0)\rangle, & a_6 & = & \langle 6(0)\rangle, \\
b_1 & = & \langle 1(1), 0(0)\rangle, & b_2 & = & \langle 2(1), 0(0)\rangle, & b_3 & = & \langle 3(2), 0(0)\rangle, \\
b_4 & = & \langle 4(1), 0(0)\rangle, & b_5 & = & \langle 5(0), 0(0)\rangle, & b_6 & = & \langle 0(0)\rangle, \\
c_{12} & = & \langle 6(1)\rangle, & c_{13} & = & \langle 2(1), 6(1)\rangle, & c_{14} & = & \langle 3(0)\rangle, \\
c_{15} & = & \langle 4(0)\rangle, & c_{16} & = & \langle 5(0)\rangle, & c_{23} & = & \langle 6(1), 1(1)\rangle, \\
c_{24} & = & \langle 1(0), 3(0)\rangle, & c_{25} & = & \langle 4(0), 1(0)\rangle, & c_{26} & = & \langle 5(0), 1(0)\rangle, \\
c_{34} & = & \langle 2(0)\rangle, & c_{35} & = & \langle 4(0), 2(0)\rangle, & c_{36} & = & \langle 5(0), 2(0)\rangle, \\
c_{45} & = & \langle 3(1)\rangle, & c_{46} & = & \langle 3(1), 5(0)\rangle, & c_{56} & = & \langle 4(1)\rangle.
\end{array}
$$

Proposition 6.1. *In the action of W on the $\langle\phi\rangle$-orbits of $\mathcal{B}$, the Coxeter generators are represented by the following products of six transpositions:*

$$s_1 \mapsto (a_1a_2)(b_1b_2)(c_{13}c_{23})(c_{14}c_{24})(c_{15}c_{25})(c_{16}c_{26}),$$
$$s_2 \mapsto (c_{12}c_{13})(a_2a_3)(b_2b_3)(c_{24}c_{34})(c_{25}c_{35})(c_{26}c_{36}),$$
$$s_3 \mapsto (c_{13}c_{14})(c_{23}c_{24})(a_3a_4)(b_3b_4)(c_{35}c_{45})(c_{36}c_{46}),$$
$$s_4 \mapsto (c_{14}c_{15})(c_{24}c_{25})(c_{34}c_{35})(a_4a_5)(b_4b_5)(c_{46}c_{56}),$$
$$s_5 \mapsto (c_{15}c_{16})(c_{25}c_{26})(c_{35}c_{36})(c_{45}c_{46})(a_5a_6)(b_5b_6),$$
$$s_6 \mapsto (c_{23}b_1)(c_{13}b_2)(c_{12}b_3)(a_4c_{56})(a_5c_{46})(a_6c_{45}),$$
$$s_0 \mapsto (a_1b_1)(a_2b_2)(a_3b_3)(a_4b_4)(a_5b_5)(a_6b_6).$$

Proof. From the definition of S_i, it is clear that S_i commutes with ϕ, and thus that the action of W on $\mathcal{B}$ commutes with the action of $\langle\phi\rangle$ on $\mathcal{B}$. The formulae for the generators may be checked by a routine but rather lengthy calculation. □

The significance of the above result is that the Coxeter group of type E_6 (which is the subgroup $W(E_6)$ generated by $s_1, s_2, \ldots, s_6$) is well known to be the automorphism group of a famous configuration of 27 lines on a cubic surface (see [8, Theorem V.4.9] and [8, Exercise V.4.11 (b)]). We now explain how the representation of $W(E_6)$ on the orbits of $\mathcal{B}$ is isomorphic, as a permutation group, to the action of $W(E_6)$ on the 27 lines. Coxeter [5, §1] gave the permutations representing the generators explicitly. In [5], the symbols a_i, b_i, c_{ij} are the names of the 27 lines, and the correspondence between our notation for the group generators and the notation of [5] is

$$s_1 \mapsto (1\ 2), \quad s_2 \mapsto (2\ 3), \quad s_3 \mapsto (3\ 4), \quad s_4 \mapsto (4\ 5), \quad s_5 \mapsto (5\ 6), \quad s_6 \mapsto Q.$$

Under these identifications, the permutations representing the action of $W(E_6)$ in [5, §1] agree with those given by Proposition 6.1. It follows that Proposition 6.1 gives an explicit action of the affine Weyl group of type E_6 on the 27 lines.

We now summarise the geometric relationship between the 27 lines. (More details may be found in [5, §1] or [8, §V.4]; note that Hartshorne uses the notation E_i, F_{ij}, G_i for a_i, c_{ij}, b_i respectively.) Any two distinct lines that do not intersect are skew. The lines $a_1, \ldots, a_6$ are mutually skew, as are the lines $b_1, \ldots, b_6$. The five skew lines $a_2, \ldots, a_6$ have a common transversal, namely b_1, and so on. The line c_{ij} intersects a_k (respectively, b_l) if and only if $k \in \{i, j\}$ (respectively, $l \in \{i, j\}$). The lines c_{ij} and c_{kl} intersect if $\{i, j\} \cap \{k, l\} = \emptyset$; otherwise they are skew. (Note that the action of s_0 in Proposition 6.1 also preserves this relationship.)

The generators $s_1, \ldots, s_5$ act on the subscripts of the lines by transpositions, as suggested in the correspondence of the previous paragraph. For example, c_{24} is moved by s_1 to c_{14}, by s_2 to c_{34}, by s_3 to c_{23} and by s_4 to c_{25}. The only orbit of $\mathcal{B}$ moved to a different $\langle\phi\rangle$-orbit by all of s_1, s_2, s_3 and s_4 is the one containing $\langle 1(0), 3(0)\rangle$. Since the action on the 27 lines is transitive (as can be checked from the formulae in [5, §1]) and the action of W_0 on the $\langle\phi\rangle$-orbits is transitive (by Lemma 4.4 (vi)), we see that the dictionary between the $\langle\phi\rangle$-orbits of $\mathcal{B}$ and the 27 lines is unique.

Remarkably, there is a concise description of the incidence relations between the 27 lines in terms of $\mathcal{B}$ and the root system alone; this is the reason for the term "skew" introduced in Definition 3.6. The proof we give below is not conceptual, and it would be nice to know a reason why this should be true.

Proposition 6.2. *Let F, F' be two proper ideals of the heap E of Figure 11 over the Dynkin diagram of type $E_6^{(1)}$ in Figure 10. Let $l([F])$ and $l([F'])$ be the corresponding lines on the cubic surface. Then the orbits $[F]$ and $[F']$ are skew (in the sense of Theorem 4.5 (iii)) if and only if $l([F])$ is skew to $l([F'])$ (in the geometric sense).*

Proof. The affine Weyl group W preserves the heap-theoretic notion of skewness by Theorem 4.1 (ii). It also preserves the geometric notion of skewness: the subgroup W_0 is known to preserve geometric skewness, and the generator s_0 also preserves it, as can be checked directly from the incidence relations and the formula in Proposition 6.1. By transitivity of the actions, it suffices to check the assertion for a fixed F.

Let us choose $F = \langle 1(0)\rangle$, so that $l(F) = a_1$. In this case, a_1 is skew to all other lines except (a) those of the form b_i for $2 \leq i \leq 6$ and (b) those of the form c_{1i} for $2 \leq i \leq 6$. Comparing F with representatives of the orbits corresponding to the other lines, we find that $[F]$ is skew to all orbits except those corresponding to the aforementioned lines. □

Proposition 6.2 may be restated in the language of algebraic geometry as follows, where $l_1.l_2 \in \mathbb{Z}$ denotes the intersection number of the lines l_1 and l_2 as described in [8, Theorem V.1.1].

Corollary 6.3. *Let F, F' be two proper ideals of the heap E of Figure 11 over the Dynkin diagram of type $E_6^{(1)}$ in Figure 10, and let $l([F])$ and $l([F'])$ be the corresponding lines on the cubic surface. Then the intersection number $l([F]).l([F'])$*

is given by

$$l([F]).l([F']) = \begin{cases} -1 & \textit{if } \chi(F,F') = t\delta \textit{ for some } t \in \mathbb{Z}, \\ 0 & \textit{if } \chi(F,F') \in \Delta^{re}, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. The conditions in the statement are easily checked (using the definitions and [8, Theorem V.1.1, Theorem V.4.9]) to be equivalent to the respective conditions (a) $[F]$ and $[F']$ are equal, (b) $[F]$ and $[F']$ are skew and (c) $[F]$ and $[F']$ are neither skew nor equal. □

Note that an explicit description of the (faithful) action of the affine Weyl group W on $\mathcal{B}$ itself may be obtained by using Theorem 4.5 (iv).

An analogous construction to the above one for type E_6 can also be performed for type E_7. The Dynkin diagram of type $E_7^{(1)}$ is shown in Figure 12, and the unique (and, therefore, self-dual) full heap over this graph is shown in Figure 13.

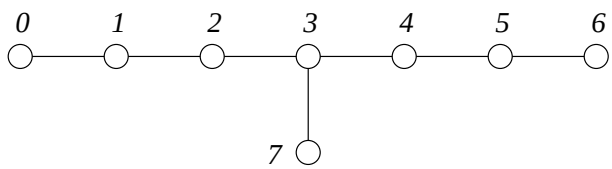

FIGURE 12. The Dynkin diagram of type $E_7^{(1)}$

Remark 6.4. It is known (see for example [13, §4]) that the unique normal subgroup of index 2 in the finite Weyl group of type E_7 is the automorphism group of a certain configuration of lines, namely 56 lines on the Del Pezzo surface of degree two defined as the double cover of the projective plane branched over the quartic. Since there are 56 orbits of proper ideals of the heap E in Figure 13 under the action of $\langle\phi\rangle$, we expect that the E_6 construction presented here in detail, when imitated for type E_7, should produce a permutation representation that is closely related to this geometrical situation.

7. The binary path representation in type affine B

In this section, we will apply our theory to the full heaps corresponding to the spin representations of the simple Lie algebra of type B.

Consider the self-dual full heap shown on the left of Figure 15 over the affine Dynkin diagram shown in Figure 14. The right hand side of Figure 15 shows the convex subheap E_0 of Lemma 4.4 (vii).

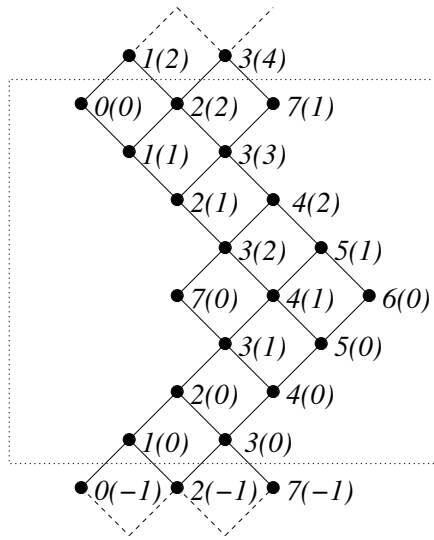

FIGURE 13. A full heap over the Dynkin diagram of type $E_7^{(1)}$

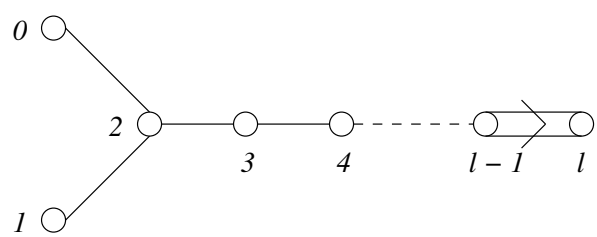

FIGURE 14. The Dynkin diagram of type $B_l^{(1)}$

In this example, it is convenient to regard the vertices of E_0 as occupying positions in a $\mathbb{Z} \times \mathbb{Z}$ grid. More precisely, in the heap E_0 arising from type $B_l^{(1)}$, the set of occupied positions is

$$\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : 0 \leq x < l \text{ and } -x \leq y \leq x \text{ and } x - y \in 2\mathbb{Z}\}.$$

Note that the smallest y such that $(i, y) \in E_0$ is $y = -i$.

Definition 7.1. Let E_0 be as above. To each ideal G of E_0 and integer i with $0 \leq i < l$, we define

$$g(G, i) = \max\{y : (i, y) \in G\} + 1,$$

with the convention that $g(G, i) = -i - 1$ if $\{y : (i, y) \in G\}$ is empty.

Definition 7.2. A *binary path* of type $B_l^{(1)}$ is by definition a function

$$f : \{x \in \mathbb{Z} : -1 \leq x < l\} \to \mathbb{Z}$$

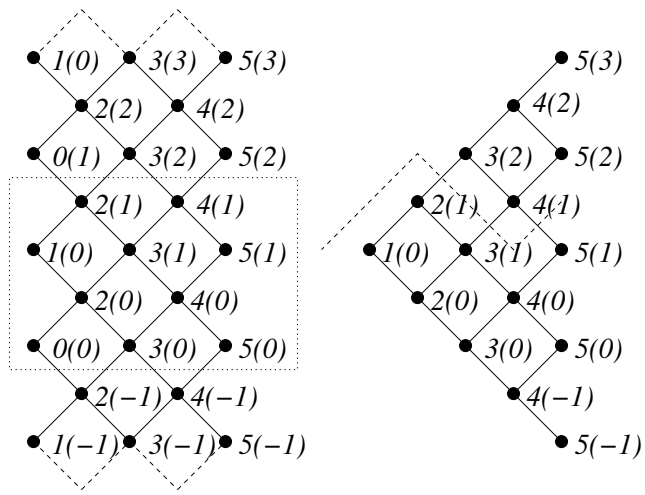

FIGURE 15. A full heap, E, over the Dynkin diagram of type $B_l^{(1)}$ for $l = 5$, and the convex subheap E_0

such that for all $0 \leq i < l$ we have $f(-1) = 0$ and $f(i) = f(i-1) \pm 1$. We denote the set of all binary paths of type $B_l^{(1)}$ by $P(B_l^{(1)})$.

Lemma 7.3. *There is a bijection π from the set of binary paths of type $B_l^{(1)}$ to the set of ideals of the heap E_0 arising from type $B_l^{(1)}$, where we regard an ideal of E_0 as a subset of $\mathbb{Z} \times \mathbb{Z}$ as above. Explicitly, we have*

$$\pi(f) = \{(x, y) \in E_0 : y < f(x)\}.$$

Proof. For $-1 \leq j < l$, define

$$(\iota(G))(j) = \begin{cases} 0 & \text{if } j = -1, \\ g(G, j) & \text{otherwise,} \end{cases}$$

where the function g is as in Definition 7.1. Note that if $(x, y) \in G$, then both of $(x-1, y-1)$ and $(x+1, y-1)$ must lie in G because G is an ideal; this is clear from the nature of the covering relations in the Hasse diagram of E_0, shown in Figure 15. We also have $g(G, 0) = \pm 1$. Using these observations and the definitions, we find that $\iota(G)$ is a binary path of type $B_l^{(1)}$. The same techniques show that $\pi(f)$ (as in the statement) will be an ideal of E_0. A routine check shows that $\pi(\iota(G)) = \iota(\pi(G)) = G$, and the assertions now follow. □

Example 7.4. The dashed line in the depiction of the heap E_0 in Figure 15 connects the points in the binary path f for which $f(-1) = f(3) = 0$, $f(0) = f(2) = f(4) = 1$ and $f(1) = 2$. The corresponding ideal of E_0 is $G = \langle 2(1), 5(1) \rangle$, which

contains 10 elements of E_0. The corresponding ideal of E is $\langle 2(1), 5(1), 0(0)\rangle = \langle 2(1), 5(1)\rangle$.

Corollary 7.5. *The heap E_0 arising from type $B_l^{(1)}$ has 2^l ideals (including $\emptyset$ and E_0).*

Proof. This follows from Lemma 7.3 and the observation that there are 2^l binary paths of type $B_l^{(1)}$. □

It is convenient to represent binary paths of type $B_l^{(1)}$ by the set $\mathbf{2^l}$ of functions from $\{1, 2, \ldots, l\}$ to $\{+, -\}$, which we will denote by strings of length l from the alphabet $\{+, -\}$. These also index the ideals of E_0 by Lemma 7.3.

Definition 7.6. Given a binary path f, we define the string $\sigma(f)$ to have symbol $+$ as the i-th letter if $f(i-1) > f(i-2)$, and symbol $-$ as the i-th letter if $f(i-1) < f(i-2)$.

Example 7.7. If f is the path in Example 7.4, then $\sigma(f) = (++--+)$.

Proposition 7.8. *The action of $W(B_l^{(1)})$ on $\mathcal{B}$ induces a transitive and faithful action of the finite group W_0 of type B_l on $\mathbf{2^l}$. For $1 \le i < l$, the generator s_i acts by exchanging the symbols at positions i and $i+1$. The generator s_l acts by altering the symbol at position l.*

This action extends by Theorem 4.5 (iv) to a faithful action of W on $\mathbb{Z} \times \mathbf{2^l}$, in which s_i acts as (id, s_i) and s_0 acts as follows, where $\mathbf{s}'$ is the substring obtained from $\mathbf{s}$ by deleting the first two symbols:

$$s_0((t,\ ++\mathbf{s}')) = (t+1,\ --\mathbf{s}'),$$
$$s_0((t,\ --\mathbf{s}')) = (t-1,\ ++\mathbf{s}'),$$
$$s_0((t,\ +-\mathbf{s}')) = (t,\ +-\mathbf{s}'),$$
$$s_0((t,\ -+\mathbf{s}')) = (t,\ -+\mathbf{s}').$$

Proof. The action of W_0 on the strings is induced by combining the identifications of Lemma 4.4 (vi) and (vii), Lemma 7.3 and Definition 7.6. It is transitive by Lemma 4.4 (vi) and faithful by Theorem 4.1 (iii). The other assertions now follow from the definitions. □

Example 7.9. Consider the string $(++--+)$ in Example 7.7. The action of s_3 on the string exchanges two minus signs, so is the identity; similarly, the action of s_3 on the ideal G of Example 7.4 is the identity. We have $s_4.(++--+) = (++-+-)$: the symbols at positions 4 and 5 are exchanged. The action of s_4

on G adds the element 4(1) produce the ideal corresponding to $(++-+-)$. We have $s_5.(++--+) = (++---)$: in this case the last symbol is changed. This corresponds to removing the element 5(1) from the ideal G.

8. Other binary path representations

The approach of §7 can be imitated for Weyl groups of types $D_l^{(1)}$, $A_{2l-1}^{(1)}$, and $C_l^{(1)}$. Since only minor modifications to the arguments and definitions are required, we give only a summary of the results.

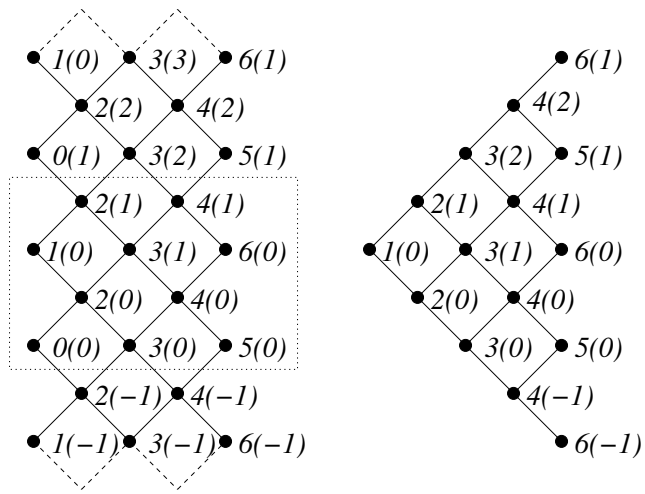

FIGURE 16. A full heap, E, over the Dynkin diagram of type $D_l^{(1)}$ for $l = 6$, corresponding to a spin representation, and the convex subheap E_0

The self-dual full heap E in Figure 16 over the Dynkin diagram of type $D_l^{(1)}$ (see Figure 8) corresponds to a spin representation of the simple Lie algebra of type D_l. The heap E is ranked (see Definition 1.3); we will call the subheap of E given by the elements of rank k the "k-th layer" of E. The even numbered labels in the set $X = \{2, 3, 4, \ldots, l-2\}$ occur in the k-th layer if and only if k is even, and the odd numbered labels in X occur in the k-th layer if and only if k is odd. Label 0 (respectively, 1, l, $l-1$) occurs in the k-th layer if and only if k is congruent to 1 (respectively, 3, $l+1$, $l+3$) modulo 4. (This condition ensures that the unique maximal element of the convex subheap E_0 is labelled l.) Another isomorphism class of heaps may be obtained in each case by twisting by the graph automorphism exchanging vertices $l-1$ and l.

The ideals of the subheap E_0 in Figure 16 are indexed by $\mathbf{2^{l-1}}$, similarly to the case of type B. There is also an analogous notion of binary paths of type $D_l^{(1)}$.

With minor modifications to take account of the alternating pattern of $l-1$ and l in the column $x = l-2$ of the heap E_0, the argument used to prove Proposition 7.8 proves the following

Lemma 8.1. *The action of $W(D_l^{(1)})$ on $\mathcal{B}$ induces a transitive and faithful action of the finite group W_0 of type D_l on $\mathbf{2^{l-1}}$. For $1 \le i < l-1$, the generator s_i acts by exchanging the symbols at positions i and $i+1$. Let G be the ideal of E_0 corresponding to a string $\mathbf{s}$ of length $l-1$, and define $k = g(G, l-2)$, where g is as in Definition 7.1. Call G "even" if $k = l-1 \mod 4$, and "odd" otherwise (i.e., if $k = l+1 \mod 4$). Then s_{l-1} and s_l act on the $(l-1)$-st symbol of $\mathbf{s}$ according to the following rules:*

$$s_{l-1}(-) = \begin{cases} + & \text{if } G \text{ is even,} \\ - & \text{otherwise;} \end{cases}$$

$$s_{l-1}(+) = \begin{cases} - & \text{if } G \text{ is odd,} \\ + & \text{otherwise,} \end{cases}$$

$$s_l(-) = \begin{cases} + & \text{if } G \text{ is odd,} \\ - & \text{otherwise;} \end{cases}$$

$$s_l(+) = \begin{cases} - & \text{if } G \text{ is even,} \\ + & \text{otherwise.} \end{cases} \quad \square$$

Lemma 8.1 can be summarized more concisely by appending a symbol "+" to $\mathbf{s}$ if $\mathbf{s}$ corresponds to an even ideal, and a symbol "−" if it corresponds to an odd ideal. Note that this will produce elements of $\mathbf{2^l}$ containing an even number of − signs.

Proposition 8.2. *The action of $W(D_l^{(1)})$ on $\mathcal{B}$ induces a transitive and faithful action of the finite group W_0 of type D_l on the subset of $\mathbf{2^l}$ consisting of strings that contain an even number of − signs. For $1 \le i < l$, the generator s_i acts by exchanging the symbols at positions i and $i+1$. The generator s_l acts by exchanging the symbols at positions $l-1$ and l, and then altering each of the symbols at these positions. This action extends by Theorem 4.5 (iv) to a faithful action of W on $\mathbb{Z} \times \mathbf{2^l}$, in which s_i acts as (id, s_i) for $i \neq 0$, and the action of s_0 is as described in Proposition 7.8.* $\square$

We now turn to type $A^{(1)}_{2l-1}$, whose Dynkin diagram is shown in Figure 1; note that the diagram has an even number, $2l$, of vertices. The relevant full heap is the ranked heap E for which the k-th layer consists of all odd (respectively, even) numbered vertices if k is odd (respectively, even). It is convenient to regard the elements of E as occupying places in a $\mathbb{Z}_{2l} \times \mathbb{Z}$ grid on a cylinder, in which vertex p in layer k corresponds to the position (p, k). (Note that precisely half the possible positions are occupied by elements of E.)

Definition 8.3. To each proper ideal F of the heap E defined above, we define

$$g(F, i) = \max\{y : (i, y) \in F\} + 1.$$

We also define a binary path of type $A^{(1)}_{2l-1}$ to be a function

$$f : \mathbb{Z}_{2l} \to \mathbb{Z}$$

such that for all $i \in \mathbb{Z}$ we have $f(i+1) = f(i) \pm 1$, where indices are read modulo $2l$. We denote the set of all binary paths of type $A^{(1)}_{2l-1}$ by $P(A^{(1)}_{2l-1})$.

We omit the proof of the following lemma, because it is very similar to the proof of Lemma 7.3.

Lemma 8.4. *There is a bijection π from the set of binary paths of type $A^{(1)}_{2l-1}$ to the set of proper ideals of the heap E defined above, identifying a proper ideal with the corresponding subset of $\mathbb{Z}_{2l} \times \mathbb{Z}$ as above. Explicitly, we have*

$$\pi(f) = \{(x, y) \in E : y < f(x)\}. \quad \square$$

The corresponding notion of strings in this case is the set $\mathbf{2}_+^{\mathbb{Z}_{2l}}$, which consists of those functions $f : \mathbb{Z}_{2l} \to \{+, -\}$ such that $|f^{-1}(+)| = |f^{-1}(-)|$.

Definition 8.5. Given a binary path f, we define the string $\sigma(f) \in \mathbf{2}_+^{\mathbb{Z}_{2l}}$ to have symbol $+$ as the i-th letter if $f(i) > f(i-1)$, and symbol $-$ as the i-th letter if $f(i) < f(i-1)$, where indices are read modulo $2l$. (Note that $\sigma(f)$ will have an equal number of $+$ and $-$ signs.)

The analogue of propositions 7.8 and 8.2 for type $A^{(1)}_{2l-1}$ is as follows; the proof follows the same lines as Proposition 7.8. In order to state the result, it is convenient to represent an element $\mathbf{s} \in \mathbf{2}_+^{\mathbb{Z}_{2l}}$ by the sequence $(f(0), f(1), \ldots, f(2l-1))$, and to define $\mathbf{s}'$ to be the substring obtained from $\mathbf{s}$ by deleting the first and last elements in the sequence representation.

Proposition 8.6. *The action of $W(A^{(1)}_{2l-1})$ on $\mathcal{B}$ induces a transitive and faithful action of the finite group W_0 of type A_{2l-1} on $\mathbf{2}^{\mathbb{Z}_{2l}}_{+}$. For $i \not\equiv 0 \mod 2l$, the generator s_i acts by exchanging the symbols at positions $i-1$ and i. This action extends by Theorem 4.5 (iv) to a faithful action of W on $\mathbb{Z} \times \mathbf{2}^{\mathbb{Z}_{2l}}_{+}$, in which s_i acts as (id, s_i) for $i \not\equiv 0 \mod 2l$, and s_0 acts as follows:*

$$s_0((t,\ +\mathbf{s}'-)) = (t+1,\ -\mathbf{s}'+),$$
$$s_0((t,\ -\mathbf{s}'+)) = (t-1,\ +\mathbf{s}'-),$$
$$s_0((t,\ +\mathbf{s}'+)) = (t,\ +\mathbf{s}'+),$$
$$s_0((t,\ -\mathbf{s}'-)) = (t,\ -\mathbf{s}'-). \quad \square$$

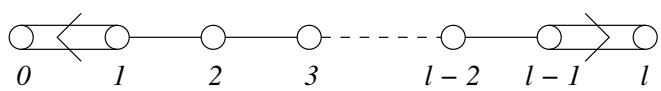

FIGURE 17. The Dynkin diagram of type $D^{(2)}_{l+1}$

There is also a binary path representation for the affine Weyl group $W(C^{(1)}_l) \cong W(D^{(2)}_{l+1})$ of type C. The corresponding full heap is similar to the one for type $B^{(1)}_l$ (see Figure 15), except that the generators corresponding to 0 and 1 in Figure 14 are identified with each other. More precisely, the relevant full heap is the ranked heap E over the Dynkin diagram of Figure 17 for which the k-th layer consists of all odd (respectively, even) numbered vertices if k is odd (respectively, even). The Weyl group action may be explicitly described as follows.

Proposition 8.7. *The action of $W(C^{(1)}_l)$ on $\mathcal{B}$ induces a transitive and faithful action of the finite group W_0 of type C_l on $\mathbf{2}^{\mathbf{l+1}}$. For $i \neq 0$, the generator s_i acts by exchanging the symbols at positions i and $i+1$. This action extends by Theorem 4.5 (iv) to a faithful action of W on $\mathbb{Z} \times \mathbf{2}^{\mathbf{l+1}}$, in which s_i acts as (id, s_i) for $i \neq 0$, and s_0 acts as follows, where $\mathbf{s}'$ is the substring obtained from $\mathbf{s}$ by deleting the first symbol:*

$$s_0((t,\ +\mathbf{s}')) = (t+1,\ -\mathbf{s}'),$$
$$s_0((t,\ -\mathbf{s}')) = (t-1,\ +\mathbf{s}'). \quad \square$$

Remark 8.8. The representations described in propositions 7.8, 8.2, 8.7 and 8.6 may, by taking a suitable limit, be extended to representations of the infinitely generated Weyl groups of types A_∞, B_∞, C_∞ and D_∞ respectively. (The Weyl groups of types B_∞ and C_∞ are isomorphic. See [10, §7.11] for the relevant definitions.) In these cases, the full heaps over the Dynkin diagrams are countably infinite as before, but

the lattice $\mathcal{B}$ is uncountable. It can be shown that the Weyl group acts faithfully in these cases, although of course the action is intransitive, for reasons of cardinality. We omit the details for reasons of space.

9. Some related constructions

In §9, we look briefly at some other examples of permutation representations of Weyl groups and related groups, in particular, examples arising from the lattices of ideals of heaps that are not full heaps. We will not give full details in the interests of space.

A well known result in the theory of lattices [17, §3.4] is that the set of ideals of a partially ordered set $(E, \leq)$ forms a distributive lattice, $J = J(E)$, where the operations of meet and join are set theoretic intersection and union, respectively. A covering relation in J consists of two ideals $F \subset F'$ such that $F' \backslash F$ is a singleton. In the case where $(E, \leq)$ is a heap, the elements of E have labels, so in the case of a covering relation, we may label the edge from F to F' in the Hasse diagram for J by the label of the single element in $F' \backslash F$.

In [7, Theorem 8.3], it is shown how the distributive lattice associated to a full heap over an (untwisted) affine Kac–Moody algebra has the structure of a crystal in the sense of Kashiwara [11]; both these structures can be regarded as edge-labelled Hasse diagrams of partially ordered sets.

Another class of examples of edge-labelled graphs in the theory of Coxeter groups are the so-called Cox Box blocks of Eriksson's thesis [6, §2.7]. Cox Box blocks are easy to construct, although there is a lot of freedom of choice in their construction; as Eriksson says in [6, §2.7], they are "perhaps more of a game than a systematic approach". Another disadvantage of Cox Box blocks is that they need not lead to faithful representations of the associated Coxeter group [6, §2.7.1]. Remarkably however, although the Cox Box blocks have no inherent partial order associated to them, many of the most natural examples do turn out to be isomorphic as edge-labelled graphs to the lattice of ideals of a heap. Moreover, in many cases, the heap that arises is a full heap, which by the theory in this paper shows that the associated Coxeter group representation will be faithful, at least if the associated Kac–Moody algebra is one of the affine types. From this point of view, the action of a Coxeter generator, s_i, on an ideal is easy to describe directly, as follows. First locate the vertex v of the edge-labelled graph corresponding to the ideal. If there is no edge labelled i emerging from v, then s_i acts as the identity. Otherwise, follow the (unique) edge labelled i to another vertex, v' say; then $s_i.v = v'$.

Some examples of Cox Box blocks are shown in [6, p62]. The type F_4 graph given there may not be the Hasse diagram of a distributive lattice, but the other ten examples shown all occur as the lattice of ideals of some heap, and in all but two cases, this heap is a full heap. These two exceptions are types $G_2^{(1)}$ and type H_3, as we now explain.

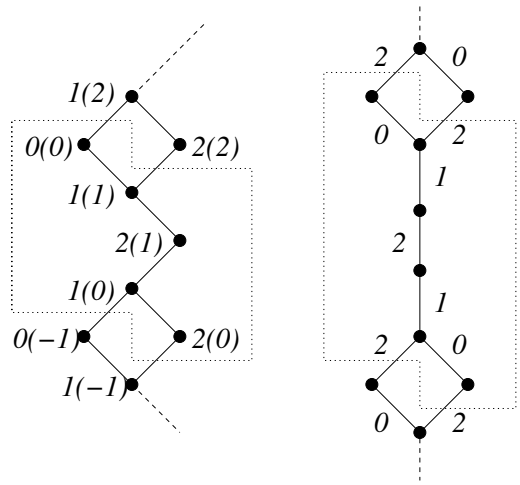

FIGURE 18. A heap of type $G_2^{(1)}$ (left), and its corresponding lattice of ideals (right)

Example 9.1. Let W be the affine Weyl group of type $G_2^{(1)}$ given by the presentation

$$W = \langle s_0, s_1, s_2 : s_0^2 = s_1^2 = s_2^2 = 1, (s_0 s_1)^3 = 1, (s_1 s_2)^6 = 1, (s_0 s_2)^2 = 1.\rangle$$

Ignoring the labels on the edges, the Dynkin diagram in this case is a graph Γ in which s_1 is connected to s_0 and s_2, and s_0 is not connected to s_2. Consider the heap E over Γ shown on the left in Figure 18. The Hasse diagram of the lattice of ideals of E, considered as a labelled graph, is shown on the right in Figure 18. This lattice is isomorphic as a labelled graph to Eriksson's Cox Box blocks model for type $G_2^{(1)}$ [6, p62].

Remarkably, our techniques can be modified to produce a categorification of the action of the Weyl group on the root system in this case. The key to this is to think of the heap elements $E(2, 2k+1)$ as "double vertices": more precisely, we redefine χ so that $\chi(\{E(2, 2k+1)\}) = 2\alpha_2$, but we let $\chi(\{E(2, 2k)\}) = \alpha_2$ as before. Our arguments can then be adapted to prove that the representation is faithful. This is not isomorphic to the representation constructed by Cellini et al in [4, Theorem 4.5],

as the latter contains points that are fixed by all Weyl group elements, although it may be the case that the representations become isomorphic after the removal of these fixed points. It may also be possible to construct a Lie algebra representation from our example by ad hoc modification of the definitions, as Wildberger does in [21] for finite type G_2.

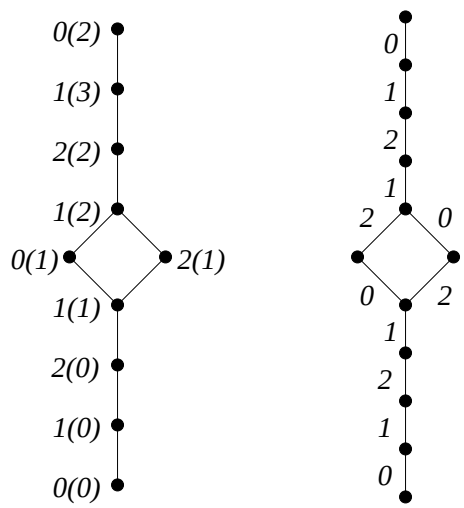

FIGURE 19. A heap of type H_3 (left), and its corresponding lattice of ideals (right)

Example 9.2. Let W be the finite Coxeter group of type H_3 given by the presentation

$$W = \langle s_0, s_1, s_2 : s_0^2 = s_1^2 = s_2^2 = 1, (s_0 s_1)^3 = 1, (s_1 s_2)^5 = 1, (s_0 s_2)^2 = 1.\rangle$$

Ignoring the labels on the edges, the Dynkin diagram in this case is a graph Γ in which s_1 is connected to s_0 and s_2, and s_0 is not connected to s_2. Consider the heap E over Γ shown on the left in Figure 19. The Hasse diagram of the lattice of ideals of E, considered as a labelled graph, is shown on the right in Figure 19. This lattice is isomorphic as a labelled graph to Eriksson's Cox Box blocks model for type H_3 [6, p62].

Eriksson [6, §2.7.2] states (without proof, but correctly!) that this representation is permutation group isomorphic to the action of the Coxeter group of type H_3 as rigid rotations and reflections on the twelve vertices of the icosahedron; it follows that the representation is faithful as both the Coxeter group and the symmetry group are known to have order 120.

Remark 9.3. Note that the edge-labelled graphs in examples 9.1 and 9.2 are not much more complicated than the heaps that give rise to them: this is because the heap in each case is close to being totally ordered. In other examples, such as those of §7 and §8, the Cox Box blocks construction is so much more complicated than the heap that it becomes difficult to write down; an extreme example of this is the situation of Remark 8.8, in which the edge-labelled graph has uncountably many vertices.

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References

[1] R. Bédard, *Cells for two Coxeter groups*, Comm. Algebra, **14** (1986), 1253–1286.

[2] A. Björner and F. Brenti, *Affine permutations of type A*, Electron. J. Combin. **3** (1996), research paper 18.

[3] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, *Springer*, New York, 2005.

[4] P. Cellini, P. Möseneder Frajria and P. Papi, *The $\widehat{W}$-orbit of ρ, Kostant's formula for powers of the Euler product and affine Weyl groups as permutations of $\mathbb{Z}$*, J. Pure Appl. Algebra, **208** (2007), 1103–1119.

[5] H.S.M. Coxeter, *The polytope 2_{21} whose twenty-seven vertices correspond to the lines to the general cubic surface*, Amer. J. Math. **62** (1940), 457–486.

[6] H. Eriksson, *Computational and combinatorial aspects of Coxeter groups*, Ph.D. thesis, KTH, Stockholm, Sweden, 1994.

[7] R.M. Green, *Full heaps and representations of affine Kac–Moody algebras*, Int. Electron. J. Algebra **2** (2007), 138–188.

[8] R. Hartshorne, Algebraic Geometry, *Springer-Verlag*, New York, 1977.

[9] J.E. Humphreys, Reflection Groups and Coxeter Groups, *Cambridge University Press*, Cambridge, 1990.

[10] V.G. Kac, Infinite dimensional Lie algebras (third edition), *Cambridge University Press*, Cambridge, UK, 1990.

[11] M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.

[12] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups*, Trans. Amer. Math. Soc. **277** (1983), 623–653.

[13] L. Manivel, *Configurations of lines and models of Lie algebras*, J. Algebra **304** (2006), 457–486.

[14] N. Perrin, *Small resolutions of minuscule Schubert varieties*, preprint; `math.AG/0601117`.

[15] J.Y. Shi, The Kazhdan–Lusztig cells in certain affine Weyl groups, *Lecture Notes in Mathematics*, **1179**, Springer, Berlin, 1986.

[16] J.Y. Shi, *The verification of a conjecture on left cells of certain Coxeter groups*, Hiroshima Math. J. **24** (1994), 627–646.

[17] R.P. Stanley, Enumerative Combinatorics, Volume 1, *Cambridge University Press*, Cambridge, UK, 1997.

[18] J.R. Stembridge, *Minuscule elements of Weyl groups*, J. Algebra, **235** (2001), 722–743.

[19] G.X. Viennot, Heaps of pieces, I: basic definitions and combinatorial lemmas, Combinatoire Énumérative, ed. G. Labelle and P. Leroux, *Springer-Verlag*, Berlin, 1986, pp. 321–350.

[20] N.J. Wildberger, *A combinatorial construction for simply-laced Lie algebras*, Adv. Appl. Math. **30** (2003), 385–396.

[21] N.J. Wildberger, *A combinatorial construction of G_2*, J. Lie Theory **13** (2003), 155–165.

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