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ON REDUCED MODULES AND RINGS

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Abstract. In this paper we extend several results known for reduced rings to reduced modules. We prove that for a semiprime module or a module with zero Jacobson radical, the concepts of reduced, symmetric, ps-Armendariz and ZI modules coincide. New examples of reduced modules are furnished: flat modules over reduced rings and modules with zero Jacobson radical over left quo rings are reduced. Rings over which all modules are reduced/symmetric are characterized.

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1. INTRODUCTION

A ring is *reduced* if it has no nonzero nilpotent elements. Reduced rings have been studied for over forty years (see [19]), and the reduced ring $R_{red} = R/Nil(R)$ associated with a commutative ring R has been of interest to commutative algebraists. Recently the reduced ring concept was extended to modules by Lee and Zhou in [15] and the relationship of reduced modules with (what we call as) ZI modules was studied by Baser and Agayev in [5]. In this paper we extend several results involving reduced rings and related rings to modules.

All our rings are associative with identity, subrings and ring homomorphisms are unitary and - unless otherwise mentioned - modules are unitary left modules. Domains need not be commutative. R denotes a ring and M denotes an R-module. Module homomorphisms are written on the side opposite that of scalars. We may not mention which letters denote elements of rings and which of modules over them, when this is clear from the context. All our left-sided results have right-sided counterparts. For unexplained concepts and results we refer to [20].

2. REDUCED MODULES AND ZI MODULES

In this section we continue the study of reduced modules, ZI modules and their relationships with other modules carried out in [15], [7], [5] and [1].(The ZI property has been called semicommutativity in [11], [7], [5] and [1] and the 'insertion of factors property (IFP)' elsewhere in the literature.)

2A. We show in this part that (Proposition 2.2) reduced implies symmetric implies ZI (for modules) and also that (Proposition 2.7) flat modules over reduced rings are reduced.

Following [14], R is *symmetric* if whenever $a, b, c \in R$ satisfy $abc = 0$, we have $bac = 0$; it is easily seen that this is a left-right symmetric concept. A module ${}_RM$ is *symmetric* ([14] and [16]), if whenever $a, b \in R$, $m \in M$ satisfy $abm = 0$, we have $bam = 0$. Extending the definition of a ZI ring (zero-insertive ring) M is *ZI* if the condition $am = 0$ implies $aRm = 0$. (We denote the annihilator of $m \in M$ by $l(m)$; thus M is ZI if and only if $l(m)$ is an ideal for each m.) A ring is *reversible* if $ab = 0$ implies $ba = 0$; symmetric rings are reversible since $ab1 = 0$ implies $ba1 = 0$; reversible rings are ZI. A module M is *reduced* (see Lemma 1.2 of [15]) if whenever $a \in R$, $m \in M$ satisfy $a^2m = 0$, then $aRm = 0$. Reduced modules are clearly ZI.

The following known/easily proved results will be used.

Remark 2.1. (1) In a ZI module the condition $abm = 0$ implies $acbdm = 0$ for all elements c, d of the ring.

(2) Suppose that whenever elements $a \in R, m \in M$ of a ZI R-module M satisfy $a^2m = 0$ we have $am = 0$. Then M is reduced.

(3) If for a left ideal B of R the R-module R/B is ZI, then B is an ideal of R.

(4) A ring R is ZI (resp., reduced, symmetric) if and only if the module ${}_RR$ is ZI (resp., reduced, symmetric).

(5) The class of ZI/reduced/symmetric modules is closed under direct products, submodules and (therefore) direct sums.

By Theorem I.3 of [3] reduced rings are symmetric. This result extends to modules:

Proposition 2.2. *Reduced modules are symmetric and symmetric modules are ZI.*

Proof. Let $a, b \in R$, $m \in M$ (a reduced R-module) be such that $abm = 0$. Then $(bab)^2m = 0 \Rightarrow babRm = 0 \Rightarrow babam = 0 \Rightarrow bam = 0$. Next let $a \in R$, $m \in M$ (a symmetric R-module) be such that $am = 0$. Then for each $b \in R$, we have $bam = 0$ which implies $abm = 0$. □

Proposition 2.2 of [5] is the 'reduced' case of Proposition 2.3 and Proposition 2.5 of [7] is the ZI case. All these 'change of rings' results will be used without explicit mention.

Proposition 2.3. *Let $\theta : R \longrightarrow A$ be a ring homomorphism, and M an A-module; then M is an R-module via $r.m = \theta(r).m$.*

(1) *If ${}_AM$ is reduced/ZI/symmetric, then so is ${}_RM$.*

(2) *If θ is onto and ${}_RM$ is reduced/ZI/symmetric, then so is ${}_AM$.*

Proof. By preceding remarks we consider only the 'symmetric' case.

(1) Suppose ${}_AM$ is symmetric and let $a, b \in R, m \in M$ such that $abm = 0$. Then, by definition, $0 = abm = \theta(ab)m = \theta(a)\theta(b)m$. Since ${}_AM$ is symmetric, we have $bam = \theta(ba)m = \theta(b)\theta(a)m = 0$ showing that ${}_RM$ is symmetric.

(2) Let $a, b \in A, m \in M$ such that $abm = 0$. Since θ is onto there exists $r, s \in R$ such that $\theta(r) = a, \theta(s) = b$. Now $0 = abm = \theta(r)\theta(s)m = rsm$. Since ${}_RM$ is symmetric, we have $bam = \theta(s)\theta(r)m = srm = 0$, and thus ${}_AM$ is symmetric. □

The R-endomorphism ring of M is denoted by $E(M)$. We denote the left $E(M)-$, right R-bimodule $Hom_R(M, R)$ by M^*. The 'generalized associativity situation' in the standard Morita context $(R, M, M^*, E(M))$ is exploited without explicit mention.

An R-module M is *torsionless* if M is a submodule of a direct product of copies of R, or, equivalently, if given $m \in M$, $m \neq 0$, there exists $q \in M^*$ such that $mq \neq 0$. If M is a faithful R-module, then R is a submodule of a direct product of copies of M. An application of Remark 2.1(4)-(5) yields the following proposition.

Proposition 2.4. *The following conditions are equivalent.*

(1) *R is a reduced (resp.,symmetric) ring.*

(2) *Every torsionless R-module is reduced (resp.,symmetric).*

(3) *Every submodule of a free R-module is reduced (resp.,symmetric).*

(4) *There exists a faithful, reduced (resp.,symmetric) R-module.*

Remark 2.5. For an R-module M, let $\overline{R}$ denote the ring $R/ann(M)$. Consider the following conditions.

(1) The left R-module M is reduced (resp.,symmetric).

(2) The left $\overline{R}$-module M is reduced (resp.,symmetric).

(3) $\overline{R}$ is a reduced (resp.,symmetric) ring.

(4) The right $E(M)$-module M is reduced (resp.,symmetric).

(5) The ring $E(M)$ is reduced (resp.,symmetric).

An application of Proposition 2.3 yields the equivalence of conditions (1) and (2); since the left R-, right $E(M)$-bimodule M is faithful as a left $\overline{R}$-module and is also faithful as a right $E(M)$-module, applying (4)⇒(1) of Proposition 2.4 we get (2)⇒(3) and (4)⇒(5).

Next we recall a well-known result.

Proposition 2.6. *Suppose that M is a flat left R-module. Then for every exact sequence*

$$0 \to K \to F \to M \to 0 \qquad (I)$$

where F is R-free , we have $(IF) \cap K = IK$ for each right ideal I of R; in particular, we have $xF \cap K = xK$ for each element x of R.

Next we prove

Proposition 2.7. *Flat modules over reduced rings are reduced.*

Proof. Let M be a flat module over the reduced ring R. Let $m \in M$ and $a \in R$ satisfy $a^2m = 0$. Suppose that for the epimorphism $\beta : F \to M$ the sequence

$$0 \to K \to F \to M \to 0 \qquad (II)$$

is exact . Now there exists $y \in F$ such that $y\beta = m$. This implies that $(a^2y)\beta = a^2m = 0$, and therefore $a^2y \in (a^2F) \cap K = a^2K$, by Proposition 2.6. Hence $a^2y = a^2k$ for some $k \in K$, yielding $a^2(y-k) = 0$. As the free R-module F is reduced we have (for each $b \in R$) $ab(y - k) = 0$ implying $abm = ab(y\beta) = ab(k\beta) = 0$. Thus M is reduced. □

The following analogue of Proposition 2.7 has a similar proof.

Proposition 2.8. *Flat modules over symmetric rings are symmetric.*

2B. The study of reduced modules in [15] was partly motivated by the relationships of reduced rings with Armendariz rings, a notion introduced in [18]. A ring R is *Armendariz* if given polynomials $f(X) = \Sigma a_iX^i$ and $g(X) = \Sigma b_jX^j$ with coefficients in R, the condition $f(X)g(X) = 0$ implies $a_ib_j = 0$ for every i and j. It was pointed out in 4.7 of [18] that this concept can be extended to modules and to the power series situation.

In [13] Kim, Lee and Lee studied the power series analogue of the 'Armendariz ring' concept; they called such rings power-serieswise Armendariz rings. Extending this concept to modules we call an R-module M *ps-Armendariz* if whenever $f(X) =$

$\sum a_i X^i \in R[[X]],\ g(X) = \sum m_j X^j \in M[[X]]$ (the power series module) satisfy $f(X)g(X) = 0$, we have $a_i m_j = 0$, $\forall i$ and $\forall j$.

The following result generalizes Lemma 2.3(2) of [13]. (Lemma 2.3(3) of [13] also extends to modules.)

Proposition 2.9. *Reduced modules are ps-Armendariz and ps-Armendariz modules are ZI.*

Proof. Reduced modules are ps-Armendariz by Lemma 1.5 of [15]. Next suppose the module M is ps-Armendariz. Let $a \in R$, $m \in M$ satisfy $am = 0$. Then for all $b \in R$, we have $(a - abX)(m + bmX + b^2mX^2 + \dots) = 0$ yielding $abm = 0$, as ${}_RM$ is ps-Armendariz. □

In the context of Proposition 2.9 the following results are of interest. The easy proof of Proposition 2.10 is omitted.

Proposition 2.10. *Let D be a commutative domain with field of fractions K. The D-module K/D is ps-Armendariz if and only if $D = K$.*

The following analogue of Theorem 12 of [2] has a similar proof.

Proposition 2.11. *Let D be a commutative domain and M a D-module. Then the idealization $D(+)M$ is ps-Armendariz if and only if M is ps-Armendariz.*

Remark 2.12 supplements some examples given in [13].

Remark 2.12. By Proposition 2.10 the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is not ps-Armendariz. It is Armendariz since (by Proposition 3.8 of [7]) all $\mathbb{Z}$-modules are Armendariz. Similar assertions can be made about the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ (the p-Prüfer group for a prime p). It follows that $\mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$ and $\mathbb{Z}(+)\mathbb{Z}_{p^\infty}$ are commutative (therefore ZI) rings which are Armendariz but not ps-Armendariz.

Remark 2.13. In view of Theorems 2.15 and 2.16 of [7] and Propositions 2.7 and 2.8 we can ask whether flat modules over ps-Armendariz rings are ps-Armendariz.

Remark 2.14. By Propositions 2.2 and 2.9 the classes of symmetric modules and of ps-Armendariz modules (over a given ring) lie between the classes of reduced modules and ZI modules. A similar remark also holds for rings. By Proposition 2 of [12], if K is a field, the ring $R_0 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K \right\}$ is Armendariz.

The same proof actually shows that the ring R_0 is ps-Armendariz. It is not symmetric since $E_{23}E_{12}I = 0$ but $E_{12}E_{23}I \neq 0$. By Remark 2.12, the ring $\mathbb{Z}(+)(\mathbb{Q}/\mathbb{Z})$ is symmetric but not ps-Armendariz.

2C. In this part we characterize rings over which all modules are reduced / symmetric. We begin with a lemma.

Lemma 2.15. *The following conditions are equivalent for a left R-module M.*

(1) *M is reduced*
(2) *All cyclic submodules of M are reduced.*

Proof. We have only to prove (2) $\Rightarrow$ (1). Let $a \in R$, $m \in M$ such that $a^2m = 0$.. Since the R-module Rm is reduced it follows that $aRm = 0$ proving that M is reduced. □

A ring R is *(von Neumann) regular* (resp., *strongly regular*) if for every $a \in R$, there exists $b \in R$ such that $a = aba$ (resp., $a = ba^2$). A ring R is *left duo* (resp.,*left quo*) if every left ideal (resp., maximal left ideal) of R is an ideal. R is a strongly regular ring if and only if R is regular and reduced if and only if R is regular and left duo if and only if R is regular and left quo; in particular, strong regularity is a left-right symmetric property.

Theorem 2.16. *For a ring R, the following conditions are equivalent.*

(1) *Every left R-module is reduced.*
(2) *Every cyclic left R-module is reduced.*
(3) *R is strongly regular.*

Proof. (1) $\Leftrightarrow$ (2) by Lemma 2.15.

(2) $\Rightarrow$ (3) Let $a \in R$. Now the cyclic R-module R/Ra^2 is reduced. Denoting residue classes in R/Ra^2 by bars, $a^2\overline{1} = 0$ which implies $\overline{a} = a\overline{1} = 0$ that is $a = ba^2$ for some $b \in R$.

(3) $\Rightarrow$ (2) Let M be cyclic over R. Then $M \cong R/B$ for some left ideal B of R. But R is strongly regular, so B is actually an ideal which implies that R/B is a strongly regular and hence a reduced ring. Hence by Proposition 2.3(1) the left R-module $M(\cong R/B)$ is reduced. □

It follows from Proposition 2.2 and Theorem 2.16 that a ring R is strongly regular if and only if every R-module is flat and symmetric, a result due to Raphael [16]. In Theorem 2.18 we characterize rings over which all modules are symmetric. We begin with a lemma.

Lemma 2.17. *The following conditions are equivalent for a ring R.*

(1) *Whenever A, B are left ideals of R, we have $AB = BA$.*
(2) *Whenever $a, b \in R$ we have $RaRb = RbRa$.*
(3) *Whenever $a, b \in R$ there exists $r \in R$ such that $ba = rab$.*

Proof. Conditions (1) and (2) are trivially equivalent.

(2) $\Rightarrow$(3) Letting $b = 1$ in $RaRb = RbRa$, we get $RaR = Ra$, yielding R is left duo. Further, $RbRa = RaRb = Rab$ implies $ba = rab$ for some $r \in R$.

(3) $\Rightarrow$ (2) Given $x \in R$, we have $a(xb) = rxba$ for some $r \in R$ yielding $RaRb \subset RbRa$ and the result follows by symmetry. □

A ring satisfies the *condition* C_l if it satisfies the conditions of Lemma 2.17; C_r is its right-sided version.

Theorem 2.18. *The following conditions are equivalent for a ring R.*

(1) *R satisfies condition C_l.*
(2) *Every left R-module is symmetric.*
(3) *Every cyclic left R-module is symmetric.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) trivially.

(3) $\Rightarrow$ (1) Let $a, b \in R$. Then R/Rab is symmetric as a left R-module. We denote the residue class in R/Rab of an element $x \in R$ by $\overline{x}$. As $ab\overline{1} = 0$ in R/Rab, we have $\overline{ba} = ba\overline{1} = 0$ and hence there exists $r \in R$ such that $ba = rab$. □

Remark 2.19. Some results connecting the condition C_r with duo rings have been recorded in §3 of [6]. Since the conditions C_l and C_r are of independent interest we record a few (possibly folklore) remarks about them.

(1) If each (principal) left ideal is generated by a family of central elements, the ring clearly satisfies C_l. Among examples of such rings are commutative rings, strongly regular rings and the ring of power series in one indeterminate over a division ring.
(2) The class of rings satisfying the condition C_l is closed under homomorphic images.
(3) As was noted in the proof of Lemma 2.17, if R satisfies the condition C_l then R is left duo.
(4) It follows from (3) that the conditions 'R is strongly regular', 'R is regular and satisfies C_l' and 'R is regular and satisfies C_r' are equivalent.

(5) Let R be left and right duo, so that $Ra = aR$ holds for each $a \in R$. Then for $a, b \in R$, we have $baR = bRa = Rba$ implying that C_l holds if and only if C_r holds.

(6) Let L be a field, $K = L(X)$ and σ the ring endomorphism of K defined by $\sigma(X) = X^2$ and σ is the identity map on L. Consider $R = K(+)_\sigma K$, the twisted Nagata extension, where on $K \oplus K$ the product $(a,b)(c,d) =_{def} (ac, \sigma(b)d)$. Then R has exactly one right ideal A apart from R and 0; A is also a left ideal, and hence R satisfies C_r. It does not satisfy C_l; this can either be directly verified or deduced from the fact that R is not left duo.

(7) Brungs' Example in §3 of [6] is a noetherian left and right duo domain which does not satisfy either condition C_l or C_r.

2D. Let P denote one of the following five 'vanishing' conditions which apply to rings as well as modules: ZI, reduced, symmetric, ps-Armendariz, Armendariz and ZI. Trivially every vector space satisfies each of these conditions. Hence it is not the case that if a module satisfies P then its endomorphism ring also has P. However, we record below some positive results that can be proved for cyclic modules.

Proposition 2.20. *Let the cyclic R-module M be a ZI/ reduced/ symmetric/ ps-Armendariz/ Armendariz and ZI R-module. Then its endomorphism ring $E(M)$ also has the same property.*

Proof. Write $M \cong R/B$ where B is a left ideal of R. Since (in each of these situations) the R-module R/B is ZI, B is an ideal of R. Now, by change of rings arguments (using Proposition 2.3 and its analogues) the R/B-module M and so the ring R/B has the same property as ${}_RM$. As $E(M) \cong E({}_{R/B}M) \cong R/B$, it follows that the ring $E(M)$ also has the same property. □

Remark 2.21. If R/B is an Armendariz R-module then, as the following example shows, B may not be an ideal of R and the method of Proposition 2.20 will not work. Example 14 of [11] is an Armendariz ring which is not ZI. Indeed, let R be such a ring. Then for some $b \in R$ the left ideal $l(b)$ is not an ideal. Since (as left R-modules) $R/l(b) \cong Rb \leq R$, the cyclic R-module $R/l(b)$ is Armendariz.

Remark 2.22. Let B be a left ideal of a ring R. By the *idealizer* of B in R we mean the subring $I(B) := \{x \in R | Bx \leq B\}$ of R. $I(B)$ contains B as an ideal, and is the largest subring of R to do so. The rings $E({}_RR/B)$ and $I(B)/B$ are isomorphic, and indeed R/B has a natural right $I(B)/B$-module structure. We use these ideas to prove Proposition 2.23. (Clearly, the same method can be used elsewhere.)

Proposition 2.23. *If M is a cyclic, Armendariz R-module, then $E(M)$ is an Armendariz ring.*

Proof. Let $M \cong R/B$ and consider the ring $I(B)/B$ in the notation of Remark 2.22. Let $f(X) = \Sigma a_i X^i$ and $g(X) = \Sigma b_j X^j$ be polynomials with coefficients in $I(B)$. Assume that $\overline{f(X)}, \overline{g(X)} \in [I(B)/B][X]$ satisfy $\overline{f(X)g(X)} = 0$. Now $f(X), g(X) \in R[X]$, and regarding $\overline{g(X)}$ as an element of $[R/B][X]$ we have $f(X)\overline{g(X)} = 0$. By assumption the left R-module R/B is Armendariz, and so we get $a_i b_j \in B$ for all i, j. It follows that $E(M) \cong I(B)/B$ is an Armendariz ring. □

2E. Let S_0 denote the set of all nonzerodivisors of R. For an R-module M we write $T(M) = \{m \in M \mid rm = 0 \quad \text{for some} \quad r \in S_0\}$; the module M is *torsion free* if $T(M) = 0$. In Proposition 2.24 we note that $T(M)$ is a submodule of M if M is ZI and record some properties inherited by $M/T(M)$ from M - for an analogue of 2.24(3) see Lemma 1 of [14].

Proposition 2.24. (1) *Let M be a ZI module. Then $T(M)$ is a submodule of M and the factor module $M/T(M)$ is also ZI (and torsion free).*

(2) *Let M be reduced. Then $M/T(M)$ is reduced.*

(3) *Let M be symmetric. Then $M/T(M)$ is symmetric.*

(4) *Let M be Armendariz and ZI. Then $M/T(M)$ is Armendariz and ZI.*

Proof. We denote by $\overline{m}$ the residue class in $M/T(M)$ of the element $m \in M$.

(1) Let $x, y \in T(M)$ so that $rx = sy = 0$ for some $r, s \in S_0$. As M is ZI we have $rtx = 0$ for each $t \in R$ implying $tx \in T(M)$; in particular, $rsx = 0$ which yields $rs(x - y) = 0$ with $rs \in S_0$. Thus $x - y \in T(M)$ also showing that $T(M)$ is a submodule of M. Given $\overline{m} \in T(M/T(M))$ there exists $s \in S_0$ such that $s\overline{m} = 0$, yielding $sm \in T(M)$. It follows that for some $r \in S_0, rsm = 0$. Thus, $m \in T(M)$ showing that $T(M/T(M)) = 0$. Next, let $a\overline{m} = 0$ and let $b \in R$; then $am \in T(M)$ implying $ram = 0$ for some $r \in S_0$. As M is ZI, we have $rabm = 0$ yielding $ab\overline{m} = 0$. Thus $M/T(M)$ is ZI.

(2) Let for $a \in R$ and $\overline{m} \in M/T(M), a^2\overline{m} = 0$ so that $a^2 m \in T(M)$. Then for some $r \in S_0, ra^2 m = 0$ holds, yielding, as M is ZI, $(ra)^2 m = 0$. As M is actually reduced, this yields $raRm = 0$ showing $aR\overline{m} = 0$. Thus $M/T(M)$ is a reduced R-module.

(3) Let for elements $a, b \in R$ and $\overline{m} \in M/T(M), ab\overline{m} = 0$ so that for some $r \in S_0$ we have $rabm = 0$. Using M is symmetric, we deduce $br(am) = 0$ which implies

$rbam = 0$. Thus $ba\overline{m} = 0$ in $M/T(M)$ proving that the R-module $M/T(M)$ is symmetric.

(4) Since M is ZI, $M/T(M)$ is certainly ZI by (1). Let $f(X) = \Sigma a_i X^i$ and $g(X) = \Sigma m_j X^j$ be nonzero polynomials with coefficients in R and M respectively. Let $\overline{g(X)}$ denote the canonical image of $g(X)$ in $[M/T(M)][X]$ and assume that $f(X)\overline{g(X)} = 0$. Let $t = deg(f) + deg(g)$. Then $\Sigma_{i+j=k} a_i m_j \in T(M)$ for $k = 0, 1, \ldots, t$ implying (for each k) the existence of elements $r_k \in S_0$ satisfying $r_k \Sigma_{i+j=k} a_i m_j = 0$. Let r denote the product in any order of the elements r_k. Then $r \in S_0$, and as M is ZI we have $r\Sigma_{i+j=k} a_i m_j = 0$ for each k. Now write $u(X) := rf(X) \in R[X]$. Since M is an Armendariz module $u(X)g(X) = 0$ yields $ra_i m_j = 0$ for all i, j. Hence $a_i \overline{m_j} = 0$ (in $M/T(M)$) for all i, j showing that the R-module $M/T(M)$ is Armendariz. □

Remark 2.25. In the context of Proposition 2.24(4) it is natural to ask whether for every ps-Armendariz module M the module $M/T(M)$ is also ps-Armendariz.

2F. An R-module M is *semiprime* [21] if given $m \in M$, $m \neq 0$, there exists $q \in M^*$ such that $(mq)m \neq 0$. The ring R is semiprime (i.e., has no nonzero nilpotent ideals) if and only if the module ${}_RR$ is semiprime. A module M is Z$-$*regular* (' Zelmanowitz regular') if given $m \in M$, there exists $q \in M^*$ such that $(mq)m = m$. Semisimple, projective modules are Z-regular, Z-regular modules are semiprime and semiprime modules are torsionless (see the definition in **2A**) . A module M is *cyclically semiprime* if every cyclic submodule of M is semiprime. Torsion free modules over domains being cyclically semiprime, the $\mathbb{Z}$-module $\mathbb{Q}$ is cyclically semiprime but not semiprime.

It is easily seen that semiprime ZI rings are reduced. In Proposition 2.26 and Corollary 2.27 we extend this result to modules.

Proposition 2.26. *For a cyclically semiprime module M the following conditions are equivalent.*

(1) *M is reduced.*
(2) *M is symmetric.*
(3) *M is ps-Armendariz.*
(4) *M is ZI.*

Proof. Since (1)$\Rightarrow$(2)$\Rightarrow$(4) and (1)$\Rightarrow$(3)$\Rightarrow$(4) always hold (by Propositions 2.2 and 2.9) we prove (4)$\Rightarrow$(1). Let $a^2m = 0, abm \neq 0$ for $a, b \in R, m \in M$. As the R-module Rm is semiprime there exists $q \in (Rm)^*$ such that $[(abm)q]abm \neq 0$. However, $a.am = 0$ and M is ZI imply $0 = ab(mq)abm \neq 0$, a contradiction. □

Corollary 2.27. *If M is a cyclically semiprime module over the ZI ring R then M is reduced.*

Proof. Every cyclic submodule of M is certainly torsionless and hence, by Propositions 2.7 and 2.9 of [7], M is ZI; by Proposition 2.26, it must be reduced. □

3. SEMIPRIMITIVE MODULES

In this section we study the relationships of reduced modules with semiprimitive modules and other related classes of modules and rings.

3A. The *Jacobson radical* $Rad(M)$ of a module M is the intersection of all its maximal submodules. A module M is *semiprimitive* if $Rad(M) = 0$. A ring R is *semiprimitive* if $Rad({}_RR) = 0$; it is well-known that this is a left-right symmetric concept.

In Proposition 3.2 we extend the result 'semiprimitive, ZI rings are reduced' to modules. The following lemma is well-known.

Lemma 3.1. (1) *Submodules of semiprimitive modules are semiprimitive.*

(2) *Let B be an ideal of a ring R and let M be an R/B-module. Then $Rad({}_{R/B}M)$ $= Rad({}_RM)$. In particular, M is semiprimitive as an R-module if and only if M is semiprimitive as an R/B-module.*

Proposition 3.2. *For a semiprimitive module M the following conditions are equivalent.*

(1) *M is reduced.*
(2) *M is symmetric.*
(3) *M is ps-Armendariz.*
(4) *M is ZI.*

Proof. By the proof of Proposition 2.26 we have only to prove (4)⇒(1). By Lemma 2.15 it is sufficient to prove that the R-module Rm is reduced for each $m \in M$. Since M is ZI, $B := l(m)$ is an ideal of R and $R/B \cong Rm$ as left modules. Since M is semiprimitive, by Lemma 3.1, so is the R-module R/B and hence also the ring R/B. Since the semiprimitive ZI ring R/B is reduced the R-module Rm is reduced by Proposition 2.3(1). □

Remark 3.3. Example (1) below shows that neither of Propositions 2.26 and 3.2 can be deduced from the other. Example (3) shows that the semiprimitive analogue of Corollary 2.27 does not hold.

(1) Reduced rings are semiprime, and so also are semiprimitive rings but a simple $\mathbb{Z}$-module (which is reduced as well as semiprimitive as a $\mathbb{Z}$-module) is never (cyclically) semiprime. The reduced, cyclically semiprime $\mathbb{Z}$-module $\mathbb{Q}$ satisfies $Rad(\mathbb{Q}) = \mathbb{Q}$.

(2) For a prime integer p the $\mathbb{Z}$-module $(\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$ is reduced, but is neither cyclically semiprime nor semiprimitive.

(3) Let A be a semiprimitive ring which is not reduced, for example the ring of all 2×2 matrices over a field. Since every ring is the homomorphic image of some $\mathbb{Z}$-algebra of polynomials (in non-commuting indeterminates) A is a factor ring of some domain R. By 'change of rings' results the R-module A is semiprimitive but not reduced.

3B. In this part - among other things - we extend to modules some results known for quo rings. We refer to [4] and §3 of [10] for undefined concepts and basic results needed by us.

Closely related to regular rings are V-rings. A ring R is a left *V-ring* if every simple left R-module is injective. By a celebrated theorem of Kaplansky, the two concepts are equivalent for commutative rings. A module ${}_RM$ is a *V-module* (also called *co-semisimple*) if every simple R-module is M-injective (in the sense of Azumaya), equivalently, if every factor module of M is semiprimitive. A module ${}_RM$ is a *p-V-module* if every simple R-module is p-M-injective; a ring R is a left *p-V-ring* if ${}_RR$ is a p-V-module. Elliger [8] calls M *regular* if every cyclic submodule of M is a direct summand; the ring R is regular if and only if the module ${}_RR$ is so. Semisimple modules as well as Z-regular modules (defined in **2F**) are regular, regular modules as well as V-modules are p-V-modules, and p-V-modules are semiprimitive.

Modules over left duo rings being ZI, several results of §2 are applied to them. In Propositions 3.6 and 3.11 we consider modules defined over the larger class of left quo rings. We first recall Proposition 4.4 and Corollary 4.5 from [17].

Proposition 3.4. *If R is a left (or right) quo ring, then $R/Rad(R)$ is a reduced ring.*

Corollary 3.5. *If R is a semiprimitive left (or right) quo ring, then R is reduced.*

In Proposition 3.6 we extend these two results to modules.

Proposition 3.6. *The following conditions are equivalent for a ring R.*

(1) *R is left quo.*
(2) *For each R-module M, the module $M/Rad(M)$ is reduced.*
(3) *Each semiprimitive R-module is reduced.*
(4) *For each R-module M, the module $M/Rad(M)$ is ZI.*
(5) *Each semiprimitive R-module is ZI.*
(6) *Each simple left R-module is reduced.*
(7) *Each simple left R-module is ZI.*

Proof. Since $Rad(M/Rad(M)) = 0$ (always) it follows that for any property Q of modules, the conditions 'For each R-module M, the module $M/Rad(M)$ has Q' and 'Each semiprimitive R-module has Q' are equivalent. Therefore (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) hold. Clearly (3) $\Rightarrow$ (5) $\Rightarrow$ (7).

(7) $\Rightarrow$ (1) For each maximal left ideal μ of R, the simple R-module R/μ is ZI, and therefore, by Remark 2.1(3), μ is an ideal. Hence R is left quo.

(1) $\Rightarrow$ (6) If W is a simple R-module, then $W \cong R/\mu$ for some maximal left ideal μ of R. Since R is left quo, μ must be an ideal, and the ring R/μ (being a division ring) is a reduced ring. Hence, by Proposition 2.3(1), $W \cong R/\mu$ must be a reduced R-module.

(6) $\Rightarrow$ (3) If $\{M_i\}_{i \in I}$ are all the maximal submodules of the semiprimitive module M, then we have a canonical monomorphism $M \longrightarrow \Pi_{i \in I}(M/M_i)$. As each M/M_i is a simple R-module it is a reduced R-module. Hence M must be reduced. □

Proposition 3.7 is a part of Theorem 4.10 of [17]. In Proposition 3.11 and Corollary 3.12 it is (partially) extended to modules.

Proposition 3.7. *Let R be a left quo ring. Then the following conditions are equivalent.*

(1) R is regular.
(2) R is a left p-V-ring.
(3) R is a left V-ring.

Propositions 3.8 - 3.10 are easily proved results needed by us.

Proposition 3.8. *Let $\theta : R \longrightarrow A$ be an onto ring homomorphism, and M an A-module; then M is an R-module via $r.m = \theta(r).m$.*

(1) *${}_AM$ is a regular module if and only if ${}_RM$ is a regular module*
(2) *${}_AM$ is a V-module if and only if ${}_RM$ is a V-module.*
(3) *${}_AM$ is a p-V-module if and only if ${}_RM$ is a p-V-module.*

Proof. Straightforward verification. □

Proposition 3.9. *If every cyclic submodule of M is a V-module, then M is a V-module.*

Proof. This is an immediate consequence of Proposition 2.3 of [4]. □

Proposition 3.10. *Let M be a cyclic, semiprimitive module over the left quo ring R. Then for some ideal B of R we have $M \cong R/B$ as left R-modules.*

Proof. We have $M \cong R/B$ where B is a left ideal of R. As the R-module R/B is semiprimitive, the left ideal B is the intersection of all maximal left ideals of R containing B. As R is left quo each maximal left ideal is an ideal. Hence B is an ideal of R. □

Next we prove

Proposition 3.11. *Let R be a left quo ring and let M be a cyclic R-module. Then the following conditions are equivalent.*

(1) *M is a regular module.*
(2) *M is a V-module.*
(3) *M is a p-V-module.*

Proof. In all three cases M is semiprimitive, and using Proposition 3.10 we may assume that $M \cong R/B$ for some ideal B of R. By 'change of rings' Proposition 3.8 the ring R/B has the 'same' property as the R-module M. The ring R/B is certainly left quo. Hence by Propositions 3.7 and 3.8 the result follows. □

Corollary 3.12. *Left p-V-modules over left quo rings are V-modules.*

Proof. In the cyclic case this holds by Proposition 3.11. In the general case we use Proposition 3.9. □

Regular ZI rings are strongly regular and therefore reduced, left and right V-rings. We have the following extension of this result.

Proposition 3.13. *Let $_RM$ be a regular, ZI module. Then M is a reduced V-module.*

Proof. Since regular modules are semiprimitive, by Proposition 3.2, the module M is reduced. To prove that M is also a V-module assume first that M is cyclic. Since M is ZI, $M \cong R/B$ where B is an ideal of R. Applying Proposition 3.8(1) we deduce that R/B is a strongly regular ring. Hence R/B is a left V-ring and therefore by Proposition 3.8(2) ${}_RM$ is a V-module. In the general case we use Proposition 3.9. □

Remark 3.14. Our Propositions 3.11 and 3.13 are analogues of parts of Theorems 4.4 and 4.8 of [10]. However, the defining condition of the (Zelmanowitz) regular modules studied by Hirano is stronger than that of the regular modules considered here.

4. NONSINGULAR MODULES

For basic results concerning nonsingular rings and modules we refer to [9]; see also [5]. Reduced rings are nonsingular by Exercise 1D3 of [9] but reduced modules need not be nonsingular - for each prime integer p the $\mathbb{Z}$-module $\mathbb{Z}/p\mathbb{Z}$ is reduced but not nonsingular as noted in Example 2.19 of [5]. If R is an infinite product of fields and B is the direct sum of the same fields, then R/B is a flat, torsion free (as defined in **2E**), reduced and singular R-module. While torsionless modules over reduced rings are, of course, both reduced and nonsingular, we do not know an example of a torsionless, reduced module which is not nonsingular.

Nonsingular rings (modules) need not be reduced; the ring of 2×2 matrices over a field is (left and right) nonsingular. The next result, a part of Proposition 1.27 of [9], is extended in Theorem 4.2 and its corollaries.

Proposition 4.1. *Commutative nonsingular rings are reduced.*

Theorem 4.2. *Let ${}_RM$ be a nonsingular module. Then M is reduced in each of the following cases.*

(1) *R is left duo.*
(2) *M is ZI over the reversible ring R;*
(3) *M is symmetric.*

Proof. In all cases, M is (nonsingular and) ZI. To prove the result we assume that $a \in R, m \in M$ satisfy $a^2m = 0, am \neq 0$ and derive a contradiction in each case. (See Remark 2.1(2).) Since $l(am)$ is not a large left ideal of R, for some nonzero $c \in R$ we have $l(am) \cap Rc = 0$. Now $c \notin l(am) \Rightarrow cam \neq 0$. Case (1): As R is left duo, $ca \in Rc$. Now $ca^2m = 0$ implies $ca \in l(am) \cap Rc = 0 \Rightarrow 0 = cam \neq 0$, a contradiction. Cases (2) and (3): Since M is ZI, $a^2m = 0$ implies $acam = 0 \Rightarrow ac \in l(am) \cap Rc = 0$. In case (2), since R is reversible we have $ca = 0$ implying $0 = cam \neq 0$, while in case (3) M is symmetric implies $0 \neq cam = acm = 0$. □

Since symmetric rings are reversible, and reversible rings are ZI we deduce the following results; they may be known but we could not find references in the literature.

Corollary 4.3. *Left duo, left nonsingular rings are reduced.*

Corollary 4.4. *Let R be a reversible ring. Then R is left non-singular if and only if R is reduced if and only if R is right nonsingular.*

Corollary 4.5. *Let R be a symmetric ring. Then R is left non-singular if and only if R is reduced if and only if R is right nonsingular.*

Remark 4.6. In view of the fact that all left (or right) duo rings and all reversible rings are ZI, it is natural to ask whether all left nonsingular ZI rings (nonsingular ZI modules) are reduced.

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