International Electronic Journal of Algebra
Volume 3 (2008) 75-82

STRONGLY FI-LIFTING MODULES

Y. Talebi and T. Amoozegar

Received: 3 April 2007; Revised: 31 July 2007
Communicated by Derya Keskin Tütüncü

Abstract. A module M is called *lifting* if every submodule A of M contains a direct summand B of M such that $B \overset{ce}{\hookrightarrow} A$ in M. We call M is *(strongly) FI-lifting* if every fully invariant submodule A of M contains a (fully invariant) direct summand B of M such that $B \overset{ce}{\hookrightarrow} A$ in M. The class of FI-lifting modules properly contains the class of lifting modules and the class of strongly FI-lifting modules. But a strongly FI-lifting module need not be a lifting module and vice versa. In this paper we investigate whether the class of (strongly) FI-lifting modules are closed under particular class of submodules, direct summands and direct sums.

Mathematics Subject Classification (2000): 16D90, 16D99
Keywords: supplemented modules, lifting modules, FI-lifting modules

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unital right R-modules. Suppose that $0 \subseteq A \subseteq B \subseteq M$. By $A \ll B$ we mean A is a small submodule of B. We say that A is a *coessential submodule of B in M* (denoted by $A \overset{ce}{\hookrightarrow} B$ in M) if $B/A \ll M/A$. We recall the definition of an amply supplemented module. If N and L are submodules of the module M, then N is called a *supplement* (*weak supplement*) of L, if $N + L = M$ and $N \cap L \ll N$ ($N \cap L \ll M$). M is called *supplemented* (*weakly supplemented*) if each of its submodules has a supplement (weak supplement) in M. M is called *amply supplemented*, if for all submodules N and L of M with $N + L = M$, N contains a supplement of L in M. A submodule A of M is said to be *coclosed in M* if it has no proper coessential submodule in M. A module M is called *lifting* if every submodule A of M contains a direct summand B of M such that $B \overset{ce}{\hookrightarrow} A$ in M. A module M is called *hollow* if every proper submodule of M is small. Recall that a submodule K of M is called *fully invariant* (denoted by $K \trianglelefteq M$) if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$. We denote by $\mathrm{Rad}\,(M)$ the radical of a module M.

Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. The injective hull of N in $\sigma[M]$ is denoted by $\widehat{N}$. A module $N \in \sigma[M]$ is said to be M*-small* if $N \ll \widehat{N}$. It is easy to see that N is M-small if and only if $N \ll L$ for some $L \in \sigma[M]$. Recall that a module $N \in \sigma[M]$ is called M-singular if $N \simeq A/B$ for some $A \in \sigma[M]$ and B essential in A. Hence M-singular modules can be considered as the dual of M-small modules. The M-singular submodule of $N \in \sigma[M]$, denoted by $Z_M(N)$, is defined as $\mathrm{Tr}\,(\mathcal{U}, \mathrm{N})$, where $\mathcal{U}$ denote the class of all M-singular modules. Dually, in [9], $\overline{Z}_M(N)$ is defined as follows:

$$\overline{Z}_M(N) = \mathrm{Re}\,(\mathrm{N}, \mathcal{S}) = \bigcap \{\mathrm{Ker}\,(\mathrm{g}) \mid \mathrm{g} \in \mathrm{Hom}\,(\mathrm{N}, \mathrm{L}), \mathrm{L} \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the class of all M-small modules. We call N an M*-cosingular* (*non-*M*-cosingular*) module if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). It is easy to see that a module $N \in \sigma[M]$ is non-M-cosingular if and only if every nonzero factor module of N is non-M-small.

2. FI-Lifting Modules

In this section we define FI-lifting modules. We show that this class of modules contain properly the class of lifting modules and is closed under fully invariant coclosed submodules and finite direct sums. We prove that ring R is FI-lifting as an R-module if and only if R/I has a projective cover for every two sided ideal I of R.

The following has been proved by Keskin [6, Lemma 1.1] and Lomp [7, 1.2.1].

2.1. Lemma. *Let M be an R-module and $N \subseteq M$. Consider the following conditions :*

(1) *N is a supplement submodule of M;*

(2) *N is coclosed in M;*

(3) *for all $X \subseteq N, X \ll M$ implies $X \ll N$.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *hold. If M is a weakly supplemented module, then* (3) $\Rightarrow$ (1) *holds.*

2.2. Lemma. *Let X be a supplement submodule of M and $K \subseteq X$. Then X/K is a supplement submodule of M/K.*

Proof. See [3, 20.5(1)]. □

2.3. Lemma. *Let M be a module. Then:*

(1) *Any sum or intersection of fully invariant submodules of M is again a fully invariant submodule of M (in fact the fully invariant submodules form a complete modular sublattice of the lattice of submodules of M).*

(2) *If $X \subseteq Y \subseteq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

(3) *If $M = \bigoplus_{i \in I} X_i$ and S is a fully invariant submodule of M, then $S = \bigoplus_{i \in I} \pi_i(S) = \bigoplus_{i \in I}(X_i \cap S)$, where π_i is the i-th projection homomorphism of M.*

Proof. See [2, Lemma 1.1]. □

We note that if $M = \bigoplus_{i=1}^{n} M_i$ and N is a fully invariant submodule of M, then $N = \bigoplus_{i=1}^{n}(N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i.

2.4. Definition. Let M be an R-module. We say that M is an *FI-lifting module* if every fully invariant submodule A of M contains a direct summand B of M such that $B \overset{ce}{\hookrightarrow} A$ in M.

By [8, 4.8], M is lifting if and only if every submodule N of M can be written in the form $N = A \oplus S$ where A is a direct summand of M and $S \ll M$. We prove a similar result for FI-lifting modules.

2.5. Proposition. *The following are equivalent for an R-module M:*

(a) *M is an FI-lifting module;*

(b) *every fully invariant submodule A of M can be written as $A = B \oplus S$, where B is a direct summand of M and $S \ll M$;*

(c) *every fully invariant submodule A of M can be written as $A = B + S$, where B is a direct summand of M and $S \ll M$.*

Proof. (a)⇒(b) is trivial and (b)⇒(c) is obvious.

(c)⇒(a). Suppose $A \trianglelefteq M$. By hypothesis $A = B + S$, where B is a direct summand of M and $S \ll M$. Suppose $M = B \oplus C$. Then C is a supplement of B and since $S \ll M$, C is a supplement of A also (see [3, 20.4(4)]). Hence $C \cap A \ll C$ and so $A/B \ll M/B$. □

2.6. Theorem. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a direct sum of FI-lifting modules. Then M is FI-lifting.*

Proof. Let $N \trianglelefteq M$. Then $N = \bigoplus_{i=1}^{n}(N \cap M_i)$ and $N \cap M_i$ is a fully invariant submodule of M_i. As each M_i is FI-lifting we have $N \cap M_i = L_i \oplus S_i$ where L_i is a direct summand of M_i and $S_i \ll M_i$. Put $L = \bigoplus_{i=1}^{n} L_i$ and $S = \bigoplus_{i=1}^{n} S_i$. Then $N = L \oplus S$ where L is a direct summand of M and $S \ll M$. □

2.7. Corollary. *If M is a finite direct sum of lifting (e.g. hollow) modules, then M is FI-lifting.*

2.8. Example. It is obvious that every lifting module is FI-lifting. The converse is not true. For example, let p be any prime integer and consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$. Since any hollow module is FI-lifting, Corollary 2.7 implies M is FI-lifting. But M is not a lifting module (see [3, Example 23.5]).

2.9. Example. *An infinite direct sum of FI-lifting modules need not be FI-lifting.* For example, let R be a semiperfect ring which is not right perfect and F be the countably generated free R-module. Then $\operatorname{Rad}(F)$ is not small in F and is fully invariant in F. $\operatorname{Rad}(F)$ cannot contain a nonzero direct summand of N, since for any projective module P, $P \neq \operatorname{Rad}(P)$.

2.10. Proposition. *Let M be an FI-lifting R-module and X be a fully invariant submodule of M which is coclosed (direct summand) in M. Then X is FI-lifting.*

Proof. Let $A \trianglelefteq X$. Then $A \trianglelefteq M$ by Lemma 2.3. Since M is FI-lifting, A contains a direct summand B of M such that $A/B \ll M/B$. By Lemmas 2.1 and 2.2, $A/B \ll X/B$. Also B is a direct summand of X. □

By [8, 4.8], a module is lifting if and only if it is amply supplemented and its coclosed submodules are direct summands. We note that an *FI-lifting module need not be amply supplemented*. For example let K be the quotient field of a discrete valuation domain which is not complete. Then $K \oplus K$ is not amply supplemented (see [3, 23.7]), but is FI-lifting by Corollary 2.7.

2.11. Corollary. *Let M be an FI-lifting module which is amply supplemented. Then $\overline{Z}^2(M)$ is FI-lifting, where $\overline{Z}^2(M) = \overline{Z}(\overline{Z}(M))$.*

Proof. By [9, 2.1] and [9, 3.4], $\overline{Z}^2(M)$ is a coclosed fully invariant submodule of M. Now the corollary follows from Proposition 2.10. □

2.12. Theorem. *Let P be a projective module. Then P is FI-lifting if and only if P/A has a projective cover for every fully invariant submodule A of P.*

Proof. Suppose P is a projective FI-lifting module and A is a fully invariant submodule of P. Then $A = X \oplus S$ where X is a direct summand of P and $S \ll P$. Suppose $P = X \oplus Y$. As $S \ll P$, $(X + S)/X \ll P/X$. Hence the natural map $f : P/X \to P/(X + S) = P/A$ is a projective cover.

Conversely, suppose P/A has a projective cover for every fully invariant submodule A of P. Let $f : Q \to P/A$ be a projective cover of P/A. Then there exists a map $h : P \to Q$ such that $fh = \eta$ where $\eta : P \to P/A$ is the natural map. As $\operatorname{Ker} f \ll Q$ and η is an epimorphism, h is an epimorphism and hence h splits. Suppose $P = \operatorname{Ker} h \oplus B$. Then $A = \operatorname{Ker} h \oplus (A \cap B)$ and $A \cap B \ll P$. Thus P is FI-lifting. □

2.13. Corollary. *Suppose R is a ring. The module R_R is FI-lifting if and only if R/I has a projective cover for every two sided ideal I of R.*

3. Strongly FI-lifting Modules

In this section we define strongly FI-lifting modules. This class of modules is properly contained in the class of FI-lifting modules; but there is no containment relation between the class of strongly FI-lifting modules and the class of lifting modules. We show that a direct summand of a strongly FI-lifting module is strongly FI-lifting and that a finite direct sum of copies of a strongly FI-lifting module is strongly FI-lifting.

3.1. Definition. Let M be an R-module. We say that M is a *strongly FI-lifting module* if every fully invariant submodule A of M contains a fully invariant direct summand B of M such that $B \overset{ce}{\hookrightarrow} A$ in M.

As in Proposition 2.5 we can prove the following.

3.2. Proposition. *The following are equivalent for an R-module M:*

(a) M is a strongly FI-lifting module;

(b) every fully invariant submodule A of M can be written as $A = B \oplus S$, where B is a fully invariant direct summand of M and $S \ll M$;

(c) every fully invariant submodule A of M can be written as $A = B + S$, where B is a fully invariant direct summand of M and $S \ll M$.

3.3. Proposition. *Let M be an FI-lifting with* $\operatorname{Rad}(M) = 0$. *Then every fully invariant submodule (in particular M) is strongly FI-lifting module.*

Proof. Let N be a fully invariant submodule of M. Suppose A is fully invariant in N. Then A is fully invariant in M also (see Lemma 2.3). As M is FI-lifting,

$A = B \oplus S$ where B is a direct summand of M and $S \ll M$ (see Proposition 2.5). Since $\mathrm{Rad}\,(M) = 0$, $S = 0$ and so A is a direct summand of M and hence of N. Thus N is strongly FI-lifting. □

3.4. Theorem. *A direct summand of a strongly FI-lifting module is strongly FI-lifting.*

Proof. Let $M = X \oplus Y$ be a strongly FI-lifting module. Assume that $S_1 \trianglelefteq X$. Then there exists $S_2 \trianglelefteq Y$ such that $S_1 \oplus S_2 \trianglelefteq M$ [4, Lemma 1.11]. Since M is a strongly FI-lifting, $S_1 \oplus S_2 = B \oplus S$ where $S \ll M$ and B is a fully invariant direct summand of M. But $B \trianglelefteq M$ implies that $B = (X \cap B) \oplus (Y \cap B)$ and $X \cap B$ is fully invariant in X. Also $X \cap B$ is a direct summand of M. We have $S_1 = \pi_X(B) + \pi_X(S) = (X \cap B) + \pi_X(S)$ where $\pi_X : M \to X$ is the projection along Y. As $S \ll M, \pi_X(S) \ll X$. By Proposition 3.2, X is a strongly FI-lifting module. □

3.5. Proposition. *Let $M = \bigoplus_{i=1}^n M_i$ and let $M_i \trianglelefteq M$ for all $1 \leq i \leq n$. Then M is strongly FI-lifting if and only if M_i is strongly FI-lifting, for all $1 \leq i \leq n$.*

Proof. If M is strongly FI-lifting then each M_i is so, by Proposition 3.4.

Conversely, suppose $M = \bigoplus_{i=1}^n M_i$ where each M_i is strongly FI-lifting and fully invariant in M. Let $N \trianglelefteq M$. Then $N = \bigoplus_{i=1}^n (N \cap M_i)$ and $(N \cap M_i) \trianglelefteq M_i$, for all $1 \leq i \leq n$. As M_i is strongly FI-lifting, $N \cap M_i = B_i \oplus S_i$ where B_i is a fully invariant direct summand of M_i and $S_i \ll M_i$ (see Proposition 3.2). Put $B = \bigoplus_{i=1}^n B_i$ and $S = \bigoplus_{i=1}^n S_i$. Then $N = B \oplus S$ where B is a direct summand of M and $S \ll M$. As $B_i \trianglelefteq M_i$ and $M_i \trianglelefteq M$, $B_i \trianglelefteq M$ for all $1 \leq i \leq n$. Hence $B \trianglelefteq M$. Therefore M is strongly FI-lifting. □

3.6. Theorem. *Suppose N is a strongly FI-lifting module and $M = \bigoplus_{i=1}^n M_i$ where each $M_i \simeq N$. Then M is a strongly FI-lifting module.*

Proof. There exist isomorphisms $f_i : M_1 \to M_i$ for $i = 2, \cdots, n$. If A is a fully invariant submodule of M, then it is easy to see that $A = A_1 \oplus f_2(A_1) \oplus \cdots \oplus f_n(A_1)$ where $A_1 = M_1 \cap A$.

As M_1 is strongly FI-lifting and A_1 is a fully invariant submodule of M_1, we have $A_1 = L_1 \oplus S_1$ where L_1 is a fully invariant submodule of M_1 and $S_1 \ll M_1$ (see Proposition 3.2). Put $L := L_1 \oplus f_2(L_1) \oplus \cdots \oplus f_n(L_1)$ and $S := S_1 \oplus f_2(S_1) \oplus \cdots \oplus f_n(S_1)$. Then $A = L \oplus S$, L is a fully invariant direct summand of M and $S \ll M$. Hence M is strongly FI-lifting. □

From Theorem 3.4 and Theorem 3.6 we get the following.

3.7. Corollary. *Suppose R is a ring and R_R is strongly FI-lifting. Then any finitely generated projective R-module is strongly FI-lifting.*

3.8. Remarks.

(1) *A finite direct sum of strongly FI-lifting modules need not be strongly FI-lifting.*

(2) *An FI-lifting module need not be strongly FI-lifting.*

(3) *A strongly FI-lifting module need not be lifting.*

(4) *A countable direct sum of copies of a strongly FI-lifting module need not be strongly FI-lifting.*

(5) *A lifting module need not be strongly FI-lifting.*

(1) Consider the module M given in Example 2.8. As any hollow module is strongly FI-lifting, M is a direct sum of two strongly FI-lifting modules. Consider the submodule $N = \mathbb{Z}/p\mathbb{Z} \oplus p^2\mathbb{Z}/p^3\mathbb{Z}$ of M. N is not small in M and contains no nonzero fully invariant direct summand of M. Hence M is not strongly FI-lifting.

(2) The example given in (1) is FI-lifting but not strongly FI-lifting.

(3) Consider a hollow module H such that $\mathrm{End}\,(H)$ is not local (for an example see [5]). Then $M = H \oplus H$ is strongly FI-lifting by Theorem 3.6. M is not a lifting module by [3, 23.16 and 23.18].

(4) Consider the module F given in Example 2.9.

(5) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ where p is a prime integer. M is lifting by [3, 23.20]. But M is not strongly FI-lifting. For, consider the fully invariant submodule $N = \mathbb{Z}/p\mathbb{Z} \oplus p\mathbb{Z}/p^2\mathbb{Z}$ of M. N is a fully invariant submodule of M which is not small in M. But N does not contain any nonzero fully invariant direct summand of M.

3.9. Proposition. *Let M be a strongly FI-lifting module and X a supplement submodule of M such that $X \trianglelefteq M$. If any of the following conditions is satisfied, then X is strongly FI-lifting.*

(1) X is indecomposable.

(2) $\mathrm{Rad}\,(X) = 0$.

(3) M is a self-injective module.

Proof. (1) Since X is an indecomposable fully invariant supplement submodule of M, X is a direct summand of M. By Theorem 3.4, X is strongly FI-lifting.

(2) Every strongly FI-lifting module is FI-lifting . By Propositions 2.10 and 3.3, X is strongly FI-lifting.

(3) Suppose Y is a fully invariant submodule of X. Then Y is a fully invariant submodule of M and hence $Y = A \oplus S$ where A is a fully invariant and a direct summand of M and $S \ll M$ (see Proposition 3.2). Then A is a direct summand of X also and $S \ll X$ as X is a supplement in M (see Lemma 2.1). Since any map from $X \to X$ can be extended to a map $M \to M$, A is fully invariant in X also. $\square$

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments which improved the presentation of the paper, especially for the Theorem 2.12 and Remarks 3.8.

References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, *Springer-Verlag*, New York, 1992.

[2] G.F. Birkenmeier, B.J. Müller and S.T. Rizvi, *Modules in which every fully invariant submodule is essential in a direct summand*, Comm. Algebra, 30(2002), 1395-1415.

[3] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting Modules, *Frontiers in Mathematics, Birkhäuser Verlag*, 2006.

[4] S.T. Rizvi and S.R. Cosmin, *Baer and Quasi-Baer Modules*, Comm. Algebra, 32(2004), 103-123.

[5] A. Facchini, and L. Salce, *Uniserial modules: Sums and isomorphisms of subquotients*, Comm Algebra, 18(1990), 499-517.

[6] D. Keskin, *On Lifting Modules*, Comm. Algebra, 28(2000), 3427-3440.

[7] C. Lomp, On Dual Goldie Dimension, Diplomarbeit (M.Sc. Thesis), University of Düsseldorf, Germany, 1996.

[8] S.M. Mohamed and B.J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Notes Series 147, *Cambridge University Press*, 1990.

[9] Y. Talebi and N. Vanaja, *The Torsion Theory Cogenerated by M-small Modules*, Comm. Algebra, 30(2002), 1449-1460.

[10] R. Wisbauer, Foundations of module and ring theory, *Gordon and Breach*, Reading, 1991.

Y. Talebi and **T. Amoozegar**
Department of Mathematics, Faculty of Science,
University of Mazandaran, Babolsar, Iran
E-mail: talebi@umz.ac.ir