International Electronic Journal of Algebra
Volume 3 (2008) 103-116

EXTENSIONS OF GENERALIZED α-RIGID RINGS

Lunqun Ouyang

Received: 10 October 2007; Revised: 12 December 2007

Communicated by Sait Halıcıoğlu

Abstract. For a ring endomorphism α, we introduce weak α-rigid and weak α-skew Armendariz rings which are a generalization of α-rigid rings, and investigated their properties. Moreover, we prove that a ring R is weak α-rigid if and only if for any n, the n by n upper triangular matrix ring $T_n(R)$ is weak α-rigid. If R is semicommutative and weak α-rigid, it is proven that the ring $R[x]$ and the ring $(R[x]/(x^n))$ where (x^n) is the ideal generated by x^n and n is a positive integer, are weak α-skew Armendariz.

Mathematics Subject Classification (2000): 16S36, 16W20, 16U99
Keywords: weak α-rigid, skew Armendariz ring, extension.

1. Introduction

Throughout this paper R denotes an associative ring with identity. $\alpha : R \longrightarrow R$ is an endomorphism of a ring R, we denote $R[x;\alpha]$ the Öre extension whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. A ring R is called *weak Armendariz* if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $pq = 0$, then $a_i b_j$ is a nilpotent element of R for each i, j. Recall that a ring R is reduced if R has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central).

Let α be an endomorphism of a ring R. According to Hong et al. [5], R is called *a α-skew Armendariz ring* if whenever $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$, $pq = 0$ implies $a_i\alpha^i(b_j) = 0$. As a generalization of the α-skew Armendariz rings, in this paper, we introduce the notion of weak α-skew Armendariz rings. We call a ring R a weak α-skew Armendariz ring if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ satisfy $pq = 0$, then $a_i\alpha^i(b_j)$ is a nilpotent element

This research is supported by the Scientific Research Fund of Hunan Provincial Education Department (07c268) and National natural science foundation of China(10771058) and Hunan Provincial Natural Science Foundation of China (06jj20053) and Scientific Research Fund of Hunan Province Education Department (06A017).

of R for each i, j. It can be easily checked that if R is a weak Armendariz ring, then it is a weak I_R- skew Armendariz ring, where I_R is an identity endomorphism of R, and all α-skew Armendariz rings are weak α-skew Armendariz. So the weak α-skew Armendariz rings are a generalization of weak Armendariz rings and α-skew Armendariz rings.

According to Krempa [10], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. We call a ring R *α-rigid* if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and α-rigid rings are reduced rings by Hong et al. [6]. Properties of α-rigid rings have been studied in Krempa [10], Hong [6], and Hirano [4]. Motivated by results in Krempa [10], Hong et al. [6] and Z. K. Liu [12], we introduce the weak α-rigid rings which are a generalization of α-rigid rings. Let α be an endomorphism of a ring R, R is said to be *weak α-rigid* if $a\alpha(a) \in nil(R) \Leftrightarrow a \in nil(R)$, where $nil(R)$ is the set of nilpotent elements of R. We will show that R is α-rigid if and only if R is weak α-rigid and reduced. So the weak α-rigid ring R is a generalization of α-rigid ring to the more general case where R is not assumed to be reduced.

For a ring R, we denote by $nil(R)$ the set of all nilpotent elements of R and by $T_n(R)$ the n by n upper triangular matrix ring over R.

2. Weak α-rigid rings

Our focus in this section is to introduce the concept of a weak α-rigid ring and study its properties. We say a ring R with an endomorphism α to be *weak α-rigid* if $a\alpha(a) \in nil(R) \Leftrightarrow a \in nil(R)$ for $a \in R$. It is easy to see that the notion of a weak α-rigid ring generalizes that of an α-rigid ring. Clearly every subring of a weak α-rigid ring is a weak α-rigid ring. The following example shows that there exists a weak α-rigid ring R such that R is not α-rigid.

Example 2.1. *Let α be an endomorphism of R and R be an α-rigid ring. Let*

$$R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a, b, c, d \in R \right\}$$

be a subring of $T_3(R)$. The endomorphism α of R is extended to the endomorphism $\overline{\alpha} : R_3 \longrightarrow R_3$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. We show that, (1) *$R_3$ is a weak $\overline{\alpha}$-rigid ring,* (2) *R_3 is not $\overline{\alpha}$-rigid.*

(1) *Suppose that* $\left(\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \overline{\alpha} \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\right) \in nil(R)$. *Then there is some positive integer n such that*

$$\left(\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\right)\right)^n = \begin{pmatrix} a\alpha(a) & * & * \\ 0 & a\alpha(a) & * \\ 0 & 0 & a\alpha(a) \end{pmatrix}^n = 0.$$

Thus $a\alpha(a) \in nil(R)$. Since R is reduced, we have $a\alpha(a) = 0$, and so $a = 0$ since R is α-rigid. Hence $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \in nil(R_3)$.

Conversely, assume that $\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \in nil(R_3)$. *Then there is some positive integer n such that*

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}^n = \begin{pmatrix} a^n & * & * \\ 0 & a^n & * \\ 0 & 0 & a^n \end{pmatrix} = 0.$$

Thus we get $a = 0$ because R is reduced. So

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix}\right) = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \in nil(R_3).$$

Therefore, R_3 is weak $\overline{\alpha}$-rigid.

(2) *Since R_3 is not reduced, R_3 is not $\overline{\alpha}$-rigid.*

Proposition 2.2. *Let α be an endomorphism of a ring R. Then R is α-rigid if and only if R is weak α-rigid and reduced.*

Proof. Assume that R is α-rigid, then R is reduced. Now we show that R is weak α-rigid. Suppose $a \in nil(R)$, then $a = 0$ since R is reduced, and so $a\alpha(a) = 0 \in nil(R)$. If $a\alpha(a) \in nil(R)$ for $a \in R$, then $a\alpha(a) = 0$, and so $a = 0 \in nil(R)$ since R is α-rigid. Therefore R is weak α-rigid and reduced. Conversely, suppose that R is weak α-rigid and reduced. Let $a\alpha(a) = 0$ for $a \in R$, then $a \in nil(R)$ since R is weak α-rigid. Thus $a = 0$ because R is reduced. Hence R is α-rigid. □

Proposition 2.3. *Let R be a weak α-rigid ring and $nil(R)$ be an ideal of R. Then we have the following:*

(1) If $ab \in nil(R)$, then $a\alpha^m(b) \in nil(R)$, $\alpha^n(a)b \in nil(R)$ for positive integers m and n.

(2) If $\alpha^k(a)b \in nil(R)$ for some positive integer k, then $ab \in nil(R)$, and $ba \in nil(R)$.

(3) If $a\alpha^t(b) \in nil(R)$ for some positive integer t, then $ab \in nil(R)$, and $ba \in nil(R)$.

Proof. (1) If $ab \in nil(R)$, then $\alpha(ab) = \alpha(a)\alpha(b) \in nil(R)$. Since $nil(R)$ is an ideal of R, we get $b\alpha(a)\alpha(b)\alpha^2(a) = b\alpha(a)\alpha(b\alpha(a)) \in nil(R)$. So $b\alpha(a) \in nil(R)$, then $\alpha(a)b \in nil(R)$. Continuing this procedure, we get $\alpha^m(a)b \in nil(R)$ for any positive integer m. Similarly, by $ab \in nil(R)$, we have $ba \in nil(R)$. Then $\alpha^n(b)a \in nil(R)$ and so $a\alpha^n(b) \in nil(R)$ for any positive integer n.

(2) If $\alpha^k(a)b \in nil(R)$ for some positive integer k, then $\alpha^k(a)\alpha^k(b) = \alpha^k(ab) = \alpha(\alpha^{k-1}(ab)) \in nil(R)$ by Proposition 2.3 (1). Since $nil(R)$ is an ideal of R, $(\alpha^{k-1}(ab))\alpha(\alpha^{k-1}(ab)) \in nil(R)$, and so $\alpha^{k-1}(ab) \in nil(R)$ by definition. Continuing this procedure, we obtain $ab \in nil(R)$.

(3) Employing the same method in the proof of (2), we get the result. □

Proposition 2.4. *Let R be weak α-rigid and nil(R) be an ideal of R. Then $\alpha(e) = e$ for any central idempotent $e \in R$.*

Proof. Let e be a central idempotent of R, then $e(1-e) = 0$ implies $\alpha(e)(1-e) \in nil(R)$ by Proposition 2.3. Thus there is some positive integer k such that $0 = (\alpha(e)(1-e))^k = \alpha(e)(1-e)$. Hence $\alpha(e) = \alpha(e)e$. Similarly, $(1-e)e = 0$ implies $\alpha(1-e)e = 0$, thus $e = \alpha(e)e$ and hence $\alpha(e) = e$. □

Let α be an endomorphism of a ring R, an ideal I of R is said to be *weak α-rigid* if $a\alpha(a) \in nil(R) \Leftrightarrow a \in nil(R)$ for $a \in I$.

Proposition 2.5. *Let R be an abelian ring with $\alpha(e) = e$ for any $e^2 = e \in R$. Then the following statements are equivalent:*

(1) R is a weak α- rigid ring.

(2) eR and $(1-e)R$ are weak α-rigid ideals.

Proof. (1) $\Longrightarrow$ (2) It is trivial.

(2) $\Longrightarrow$ (1) Let $a \in R$ be such that $a \in nil(R)$. Then $ea \in nil(R)$ and $(1-e)a \in nil(R)$. Since eR and $(1-e)R$ are weak α- rigid, there exist some positive integers m and n such that $(ea\alpha(ea))^m = e(a\alpha(a))^m = 0$, and $((1-e)a\alpha((1-e)a))^n = (1-e)(a\alpha(a))^n = 0$. Let $k = Max\{m,n\}$, then $e(a\alpha(a))^k = 0$ and $(1-e)(a\alpha(a))^k = 0$. Thus $(a\alpha(a))^k = 0$, hence $a\alpha(a) \in nil(R)$.

Conversely, assume that $a\alpha(a) \in nil(R)$. Then $ea\alpha(ea) \in nil(R)$ and $(1-e)a\alpha((1-e)a) \in nil(R)$. Thus $ea \in nil(R)$ and $(1-e)a \in nil(R)$ since eR and $(1-e)R$ are weak α-rigid ideals. So $a \in nil(R)$. Therefore R is weak α-rigid. □

3. Extensions of weak α-rigid rings

Let α be an endomorphism of a ring R. Let $T_n(R)$ denote the n by n upper triangular matrix ring over R. Then the endomorphism α of R is extended to the endomorphism $\overline{\alpha} : T_n(R) \to T_n(R)$ defined by

$$\overline{\alpha}\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right) = \begin{pmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \cdots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \cdots & \alpha(a_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha(a_{nn}) \end{pmatrix}$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$. Then we have the following:

Theorem 3.1. *Let α be an endomorphism of a ring R. Then the following statements are equivalent:*

(1) R is weak α-rigid.

(2) $T_n(R)$ is weak $\overline{\alpha}$- rigid for any positive integer n.

Proof. (1) $\Longrightarrow$ (2) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A \in nil(T_n(R))$. Then there exists some positive integer n such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^n = \begin{pmatrix} a_{11}^n & * & \cdots & * \\ 0 & a_{22}^n & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & a_{nn}^n \end{pmatrix} = 0.$$

Thus $a_{ii} \in nil(R)(i = 1, 2, \cdots, n)$. Since R is weak α- rigid, we get that $a_{ii}\alpha(a_{ii}) \in nil(R)$. So there is some positive integer t_i such that $(a_{ii}\alpha(a_{ii}))^{t_i} = 0$, $(i =$

$1, 2, \cdots, n)$. Let $t = Max\{t_i\}, (i = 1, 2, \cdots, n)$, then

$$\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right)\right)^{tn}$$

$$= \begin{pmatrix} a_{11}\alpha(a_{11}) & * & \cdots & * \\ 0 & a_{22}\alpha(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & a_{nn}\alpha(a_{nn}) \end{pmatrix}^{tn} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \overline{\alpha} \left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right) \in nil(T_n(R))$.

Now, Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A\overline{\alpha}(A) \in nil(T_n(R))$.
There is some positive integer n such that

$$\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \overline{\alpha} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}\right)^{n}$$

$= \begin{pmatrix} a_{11}\alpha(a_{11}) & * & \cdots & * \\ 0 & a_{22}\alpha(a_{22}) & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & a_{nn}\alpha(a_{nn}) \end{pmatrix}^{n} = 0.$ Thus $a_{ii}\alpha(a_{ii}) \in nil(R)$, $(i = 1, 2, \cdots n)$. Hence $a_{ii} \in nil(R)$ since R is weak α-rigid, and so there is some positive integer t_i such that $(a_{ii})^{t_i} = 0$. Let $t = Max\{t_i\}$, $(i = 1, 2, \cdots, n)$, then

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^{tn} = \begin{pmatrix} a_{11}^t & * & \cdots & * \\ 0 & a_{22}^t & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & a_{nn}^t \end{pmatrix}^{n} = \begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \cdots & \cdots & \cdots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix}^{n} = 0.$$

Hence $A \in nil(T_n(R))$. Therefore $T_n(R)$ is weak α-rigid.

(2) $\Longrightarrow$ (1) Let $a \in nil(R)$, then $a^n = 0$ for some positive integer n. Let $A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in T_n(R)$, then $A \in nil(T_n(R))$. Since $T_n(R)$ is weak $\overline{\alpha}-$ rigid,

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \overline{\alpha}\left(\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) = \begin{pmatrix} a\alpha(a) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\in nil(T_n(R))$. Thus $a\alpha(a) \in nil(R)$. Now let $a\alpha(a) \in nil(R)$, then

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \overline{\alpha}\left(\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\right) \in nil(T_n(R)).$$

Since $T_n(R))$ is weak $\overline{\alpha}-$ rigid, we have $\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in nil(T_n(R))$, and so $a \in nil(R)$. Hence R is weak α-rigid. □

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Let α be an endomorphism of a ring R, then α is extended to the endomorphism $\overline{\alpha} : T(R, R) \longrightarrow T(R, R)$ defined by $\overline{\alpha}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \alpha(r) & \alpha(m) \\ 0 & \alpha(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R, R)$.

Corollary 3.2. *Let α be an endomorphism of a ring R. Then the trivial extension $T(R, R)$ of R by R is weak $\overline{\alpha}$-rigid if and only if R is weak α-rigid.*

Proof. Since $T(R,R)$ is isomorphic to the subring $\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} | r, m \in R\right\}$ of a ring $T_2(R)$ in Theorem 3.1. And each subring of a weak α-rigid ring is also weak α-rigid. So it is easy to see that $T(R,R)$ is a weak $\overline{\alpha}$ rigid ring if and only if R is weak α-rigid. □

We say a ring R is *a weak α-skew Armendariz ring* if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x;\alpha]$ satisfy $pq = 0$, then $a_i \alpha^i(b_j)$ is a nilpotent element of R for each i, j.

Theorem 3.3. *Let R be a weak α-rigid ring with nil(R) an ideal of R. Then R is a weak α-skew Armendariz ring.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x;\alpha]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$ be such that $f(x)g(x) = 0$. Then $\sum\limits_{k=0}^{m+n} (\sum\limits_{i+j=k} a_i \alpha^i(b_j))x^k = 0$. Then we have the following equations:

$$\sum_{i+j=k} a_i \alpha^i(b_j) = 0, \ k = 0, 1, \cdots, m+n.$$

We will show that $a_i \alpha^i(b_j) \in nil(R)$ by induction on $i + j$.

If $i + j = 0$, then $a_0 b_0 = 0 \in nil(R)$ and so $b_0 a_0 \in nil(R)$.

Now suppose that k is a positive integer such that $a_i \alpha^i(b_j) \in nil(R)$ when $i + j < k$. We will show that $a_i \alpha^i(b_j) \in nil(R)$ when $i + j = k$.

Consider equation:

$$a_0 b_k + a_1 \alpha(b_{k-1}) + a_2 \alpha^2(b_{k-2}) + \cdots + a_k \alpha^k(b_0) = 0. \tag{1}$$

Multiplying (1) by b_0 from left, we have $b_0 a_k \alpha^k(b_0) = -(b_0 a_0 b_k + b_0 a_1 \alpha(b_{k-1}) + b_0 a_2 \alpha^2(b_{k-2}) + \cdots + b_0 a_{k-1} \alpha^{k-1}(b_1))$. By the induction hypothesis, $a_i \alpha^i(b_0) \in nil(R)$ for $0 \le i < k$. Thus $a_i b_0 \in nil(R)$ by Proposition 2.3, and so $b_0 a_i \in nil(R)$ for $0 \le i < k$. Hence $b_0 a_k \alpha^k(b_0) \in nil(R)$ since $nil(R)$ is an ideal of R. Thus $b_0 a_k \alpha^k(b_0) \alpha^k(a_k) = b_0 a_k \alpha^k(b_0 a_k) \in nil(R)$, this means that $b_0 a_k \in nil(R)$ and so $a_k b_0 \in nil(R)$. Thus $a_k \alpha^k(b_0) \in nil(R)$ by Proposition 2.3. Multiplying (1) by b_1 from the left, similarly, we have $a_{k-1} \alpha^{k-1}(b_1) \in nil(R)$.

Continuing this procedure yields that $a_i \alpha^i(b_j) \in nil(R)$ for $i + j = k$. Therefore by induction, we have $a_i \alpha^i(b_j) \in nil(R)$ for each i, j, and so R is a weak α-skew Armendariz ring. □

Now we can give an example of a weak α-rigid ring which is not weak α-skew Armendariz.

Example 3.4. *Let R be a ring and $M_2(R)$ the 2 by 2 matrix ring over R. Let $S = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} | A, B, C \in M_2(R) \right\}$. With usual matrix operations, S is a ring. The endomorphism α of $S \longrightarrow S$ is defined by*

$$\alpha\left(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right) = \begin{pmatrix} A & -B \\ 0 & C \end{pmatrix} \text{ for any } \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in S.$$

It is easy to see that S is weak α-rigid. Now we show that S is not weak α-skew Armendariz. Let

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix} x,$$

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} x \in S[x; \alpha].$$

Then we have $f(x)g(x) = 0$, but

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

is not a nilpotent element of S. Thus S is not weak α-skew Armendariz.

Corollary 3.5. *Let α be an endomorphism of a ring R. Then R is a weak α-skew Armendariz ring if and only if for any n, $T_n(R)$ is a weak $\overline{\alpha}$-skew Armendariz ring.*

Proof. Suppose that $T_n(R)$ is a weak $\overline{\alpha}$- skew Armendariz ring, then it is easy to see that R is a weak α-skew Armendariz ring. Conversely, Let $f(x) = A_0 + A_1x + \cdots + A_px^p$, and $g(x) = B_0 + B_1x + \cdots + B_qx^q$ be elements of $T_n(R)[x; \overline{\alpha}]$. Assume that $f(x)g(x) = 0$. Let

$$A_i = \begin{pmatrix} a^i_{11} & a^i_{12} & \cdots & a^i_{1n} \\ 0 & a^i_{22} & \cdots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i_{nn} \end{pmatrix}, \qquad B_j = \begin{pmatrix} b^j_{11} & b^j_{12} & \cdots & b^j_{1n} \\ 0 & b^j_{22} & \cdots & b^j_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^j_{nn} \end{pmatrix}.$$

Then from $f(x)g(x)=0$, it follows that

$$\left(\sum_{i=1}^{p} a_{ss}^{i}x^{i}\right)\left(\sum_{j=1}^{q} b_{ss}^{j}x^{i}\right)=0, s=1,2,\cdots,n.$$

Since R is a weak α-skew Armendariz ring, there exists some positive integer m_{ijs} such that $(a_{ss}^{i}\alpha^{i}(b_{ss}^{j})^{m_{ijs}}=0$ for any s and any i,j. Let $m_{ij}=Max\{m_{ij1},m_{ij2},\cdots,m_{ijn}\}$. Then

$$(A_i\overline{\alpha}^{i}(B_j)^{m_{ij}}=\begin{pmatrix} a_{11}^{i}\alpha^{i}(b_{11}^{j}) & * & \cdots & * \\ 0 & a_{22}^{i}\alpha^{i}(b_{22}^{j}) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^{i}\alpha^{i}(b_{nn}^{j}) \end{pmatrix}^{m_{ij}}=\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $(A_i\overline{\alpha}^{i}(B_j)\in nil(T_n(R))$. This shows that $T_n(R))$ is a weak $\overline{\alpha}$- skew Armendariz ring. □

Corollary 3.6. *Let α be an endomorphism of a ring R. Then R is a weak α-skew Armendariz ring if and only if the trivial extension $T(R,R)$ is a weak $\overline{\alpha}$- skew Armendariz ring.*

Proof. It follows from Corollary 3.5. □

Recall that if α is an endomorphism of a ring R, then the map α can be extended to an endomorphism of the polynomial ring $R[x]$ defined by $\sum_{i=0}^{m} a_ix^i \longrightarrow \sum_{i=0}^{m} \alpha(a_i)x^i$. We shall also denote the extended map $R[x]\longrightarrow R[x]$ by α and the image of $f\in R[x]$ by $\alpha(f)$. By [12.Theorem 3.8], if R is semicommutative, then $R[x]$ is weak Armendariz. We have the following Theorem which is a generalization of [12.Theorem 3.8]

Lemma 3.7. [12] *Let R be a semicommutative ring. Then nil(R) is an ideal of R.*

Lemma 3.8. [12] *Let R be a semicommutative ring. If $a_0,a_1,\cdots,a_n\in nil(R)$, then $a_0+a_1x+\cdots+a_nx\in R[x]$ is a nilpotent element.*

Theorem 3.9. *Let R be a weak α-rigid and semicommutative ring. Then $R[x]$ is a weak α-skew Armendariz ring.*

Proof. Let $f=f_0+f_1y+\cdots+f_py^p\in(R[x])[y;\alpha]$, and $g=g_0+g_1y+\cdots+g_qy^q\in(R[x])[y;\alpha]$ be such that $fg=0$. Suppose that $f_i=\sum_{s=0}^{m_i} a_s^ix^s$. Let

$m = Max\{m_i\}, (i = 0, 1, \cdots, p.)$ Then each f_i can be written in the form of $f_i = \sum_{s=0}^{m} a_s^i x^s$. Note that $xy = yx$ since $\alpha(1) = 1$, and $xa = ax$ for any $a \in R$. Thus

$$f = \sum_{i=0}^{p} \left(\sum_{s=0}^{m} a_s^i x^s \right) y^i = \sum_{s=0}^{m} \left(\sum_{i=0}^{p} a_s^i y^i \right) x^s.$$

Similarly, each g_j can be written in the form of $g_j = \sum_{t=0}^{n} b_t^j x^t$, and thus

$$g = \sum_{j=0}^{q} \left(\sum_{t=0}^{n} b_t^j x^t \right) y^j = \sum_{t=0}^{n} \left(\sum_{j=0}^{q} b_t^j y^j \right) x^t.$$

From $fg = 0$, we have the following equations:

$$\sum_{s+t=k} \left(\sum_{i=0}^{p} a_s^i y^i \right) \left(\sum_{j=0}^{q} b_t^j y^j \right) = 0. \qquad k = 0, 1, \cdots, m+n. \tag{2}$$

We will show by induction on $s+t$ that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any $0 \leq i \leq p$, and $0 \leq j \leq q$ and any s, t with $s + t = 0, 1, \cdots m + n$.

If $s + t = 0$, then $s = t = 0$. Thus $(\sum_{i=0}^{p} a_0^i y^i)(\sum_{j=0}^{q} b_0^j y^j) = 0$. Since R is semicommutative, nil(R) is an ideal of R by Lemma 3.7. Then R is weak α-skew Armendariz by Theorem 3.3. Thus $a_0^i \alpha^i(b_0^j) \in nil(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$.

Now suppose that $k \leq m + n$ is such that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$, and any s, t with $s + t < k$, we will show that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$, and any s, t with $s + t = k$. From (2), we have

$$\begin{aligned} 0 &= \sum_{s+t=k} \left(\sum_{i=0}^{p} a_s^i y^i \right) \left(\sum_{j=0}^{q} b_t^j y^j \right) = \sum_{s+t=k} \sum_{l=0}^{p+q} \left(\sum_{i+j=l} a_s^i \alpha^i(b_t^j) \right) y^l \\ &= \sum_{l=0}^{p+q} \left(\sum_{s+t=k} \sum_{i+j=l} a_s^i \alpha^i(b_t^j) \right) y^l = \sum_{l=0}^{p+q} \left(\sum_{i+j=l} \sum_{s+t=k} a_s^i \alpha^i(b_t^j) \right) y^l. \end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{s+t=k} a_s^0 b_t^0 = 0,\\
&\sum_{s+t=k} a_s^0 b_t^1 + \sum_{s+t=k} a_s^1 \alpha(b_t^0) = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sum_{s+t=k} a_s^0 b_t^l + \sum_{s+t=k} a_s^1 \alpha(b_t^{l-1}) + \cdots + \sum_{s+t=k} a_s^l \alpha^l(b_t^0) = 0,\\
&\sum_{s+t=k} a_s^p \alpha^p(b_t^q) = 0.
\end{aligned}$$

If $s < k$. Then by the induction hypothesis, $a_s^0 b_0^0 \in nil(R)$ and so $b_0^0 a_s^0 \in nil(R)$ for $s < k$. Hence $b_0^0 a_0^0 b_k^0 + b_0^0 a_1^0 b_{k-1}^0 + \cdots + b_0^0 a_{k-1}^0 b_1^0 \in nil(R)$ since R is semicommutative. Therefore if we multiply $\sum\limits_{s+t=k} a_s^0 b_t^0 = 0$ on the left side by b_0^0, then it follows that $b_0^0 a_k^0 b_0^0 \in nil(R)$, and so $b_0^0 a_k^0 \in nil(R)$ and $a_k^0 b_0^0 \in nil(R)$. If we multiply $\sum\limits_{s+t=k} a_s^0 b_t^0 = 0$ on the left side by b_1^0, then $b_1^0 a_{k-1}^0 b_1^0 = -(b_1^0 a_0^0 b_k^0 + b_1^0 a_1^0 b_{k-1}^0 + \cdots + b_1^0 a_{k-2}^0 b_2^0) - b_1^0 a_k^0 b_0^0 = -(b_1^0 a_0^0) b_k^0 - (b_1^0 a_1^0) b_{k-1}^0 - \cdots - (b_1^0 a_{k-2}^0) b_2^0 - b_1^0 (a_k^0 b_0^0) \in nil(R)$ since R is semicommutative. Thus $a_{k-1}^0 b_1^0 \in nil(R)$. Similarly, we can show that $a_{k-2}^0 b_2^0 \in nil(R), \cdots, a_0^0 b_k^0 \in nil(R)$. So we show that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=0$.

Suppose that $l \leq p+q$ is such that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any s, t with $s+t=k$ and any i, j with $i+j<l$, we will show that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=l$.

If $s < k$, then by the induction hypothesis, $a_s^i \alpha^i(b_0^0) \in nil(R)$. Thus $a_s^i b_0^0 \in nil(R)$ by Proposition 2.3, and so $b_0^0 a_s^i \in nil(R)$. If $i < l$, then by the induction hypothesis on l, $a_k^i \alpha^i(b_0^0) \in nil(R)$ for any $i < l$, which implies $a_k^i b_0^0 \in nil(R)$ and so $b_0^0 a_k^i \in nil(R)$ for any $i < l$. Multiplying $\sum\limits_{s+t=k} a_s^0 b_t^l + \sum\limits_{s+t=k} a_s^1 \alpha(b_t^{l-1}) + \cdots + \sum\limits_{s+t=k} a_s^l \alpha^l(b_t^0) = 0$ on the left side by b_0^0, we have $b_0^0 a_k^l \alpha^l(b_0^0) \in nil(R)$ since nil(R) is an ideal of R by Lemma 3.7. Thus $b_0^0 a_k^l \alpha^l(b_0^0) \alpha^l(a_k^l) = b_0^0 a_k^l \alpha^l(b_0^0 a_k^l) \in nil(R)$. Thus $b_0^0 a_k^l \in nil(R)$ which implies $a_k^l b_0^0 \in nil(R)$ and so $a_k^l \alpha^l(b_0^0) \in nil(R)$ by Proposition 2.3. Similarly, we can show that $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any s, t with $s+t=k$ and any i, j with $i+j=l$.

Therefore, by induction, we have $a_s^i \alpha^i(b_t^j) \in nil(R)$ for any $0 \leq i \leq p$, and $0 \leq j \leq q$ and any $s\ \ t$ with $s+t = 0, 1, \cdots m+n$. Now,

$$\begin{aligned} f_i\alpha^i(g_j) &= \left(\sum_{s=0}^{m} a_s^i x^s\right)\alpha^i\left(\sum_{t=0}^{n} b_t^j x^t\right) = \left(\sum_{s=0}^{m} a_s^i x^s\right)\left(\sum_{t=0}^{n} \alpha^i(b_t^j) x^t\right) \\ &= \sum_{k=0}^{m+n}\left(\sum_{s+t=k} a_s^i \alpha^i(b_t^j)\right) x^k. \end{aligned}$$

Since R is semicommutative, by Lemma 3.7, $\sum\limits_{s+t=k} a_s^i\alpha^i(b_t^j) \in nil(R)$. Thus by Lemma 3.8, $f_i\alpha^i(g_j) \in nil(R[x])$. Therefore $R[x]$ is weak α-skew Armendariz. □

Corollary 3.10. *Let R be a weak α-rigid semicommutative ring. Then $R[x]/(x^n)$ is a weak α-skew Armendariz ring where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. Denote $\overline{x}$ in $R[x]/(x^n)$ by u, so $R[x]/(x^n) = R[u] = R + Ru + \cdots + Ru^{n-1}$, where u is commutative with elements of R and $u^n = 0$. Let $f, g \in R[u][y]$ be such that $fg = 0$. Then we can suppose that $f = \sum_{i=0}^{n-1} f_i y^i$ and $g = \sum_{j=0}^{n-1} g_j y^j$, and $f_i = \sum_{s=0}^{n-1} a_s^i u^s$, and $g_j = \sum_{t=0}^{n-1} b_t^j u^t$. Then a proof similar to Theorem 3.9, we get the result. □

Acknowledgements. The author would like to thank the referee for his (her) helpful comments which improved this paper.

References

[1] D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra, 26(1998), 2265-2272.

[2] V. Dlab and C.M. Ringel, *A class of balanced non-uniserial rings*, Math. Ann. 195 (1972), 279-291.

[3] K.R. Goodearl, Ring Theory, *Marcel Dekker, New York-Basel*, 1976.

[4] Y. Hirano, *On the uniqueness of rings of coefficients in skew polynomial rings*, Publ. Math. Debrecen, 54(1999), 489-495.

[5] C.Y. Hong, N.K. Kim and T.K. Kwak, *On skew Armendariz rings*, Comm. Algebra, 31(2003), 103-122.

[6] C.Y. Hong, N.K. Kim and T.K. Kwak, *Ore extensions of Baer and p.p.-rings*, J. Pure Appl. Algebra, 151(2000), 215-226.

[7] C.Y. Hong, T.K. Kwak and S.T. Rizvi, *Extensions of generalized Armendariz rings*, Algebra Colloq. 13(2)(2006), 253-266.

[8] E. Hashemi and A. Moussavi, *Polynomial extensions of quasi-Baer rings*, Acta Math. Hungar. 107(3)(2005), 207-224.

[9] N.K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra, 223(2000), 477-488.

[10] J. Krempa, *Some examples of reduced rings*, Algebra Colloq., 3(1996), 289-300.

[11] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan. Ser.A Math. Sci, 73(1997), 14-17.

[12] Z.K. Liu and R.Y. Zhao, *On weak Armendariz rings*, Comm. Algebra, 34(2006), 2607-2616.

[13] J.M. Zelmanowitz, A class of modules with semisimple behavior, *Abelian Groups and Modules (Padova, 1994)*, *Kluwer Acad. Publ. (Dordrecht, 1995)*, pp.491-500.

Lunqun Ouyang
1. Department of Mathematics,
Hunan Science and Technology University,
Xiangtan 411201, P. R. China
2. Department of Mathematics,
Hunan Normal University,
Changsha 410081, P. R. China
E-mail: Ouyanglqtxy@163.com