International Electronic Journal of Algebra
Volume 3 (2008) 125-134

GENERALIZED PROJECTIVITY OF QUASI-DISCRETE MODULES

Yosuke Kuratomi

Received: 4 November 2007; Revised: 17 December 2007
Communicated by John Clark

Abstract. In 2004, S.H.Mohamed and B.J.Müller [11] defined generalized projectivity (dual ojectivity) as follows: given modules A and B, A is *generalized B-projective (B-dual ojective)* if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A' \oplus A''$, $B = B' \oplus B''$, a homomorphism $\varphi : A' \to B'$ and an epimorphism $\psi : B'' \to A''$ such that $g \circ \varphi = f|_{A'}$ and $f \circ \psi = g|_{B'}$. Generalized projectivity plays an important role in the study of direct sums of lifting modules (cf. [8, 11]). Since the structure of generalized projectivity is complicated, it is difficult to determine whether generalized projectivity is inherited by (finite) direct sums. This problem is not easy even in the case that each module is quasi-discrete. In this paper we consider this problem.

Mathematics Subject Classification (2000): 16E40; 16E70
Keywords: quasi-discrete module, lifting module, generalized projective

1. Preliminaries

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules. A submodule S of a module M is said to be a *small* submodule if $M \neq K + S$ for any proper submodule K of M and we write $S \ll M$ in this case. Let M be a module and let N and K be submodules of M with $K \subseteq N$. Then K is said to be a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M in this case. Let X be a submodule of M. Then X is called a *co-closed* submodule in M if X has no proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M. $K <_\oplus N$ means that K is a direct summand of N. Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \xrightarrow{\varphi} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \xrightarrow{\varphi} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \xrightarrow{\varphi} M_2 \rangle \oplus M_2$.

A module M has the *finite internal exchange property* if, for any finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ and any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i = 1, \cdots, n)$ such that $M = X \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_n}$.

A module M is said to be a *lifting* module if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$ in M.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. Then M is said to be a *lifting module for* the decomposition $M = \oplus_I M_i$ if, for any submodule X of M, there exist $X^* \subseteq M$ and $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $X^* \subseteq_c X$ in M and $M = X^* \oplus (\oplus_I \overline{M_i})$, that is, M is a lifting module and satisfies the internal exchange property in the direct sum $M = \oplus_I M_i$.

Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$, if and only if Y is minimal with respect to $M = X + Y$. Note that a supplement Y of X in M is co-closed in M. A module M is $(\oplus-)$*supplemented* if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is supplement of X in M. A module M is called *amply supplemented* if X contains a supplement of Y in M whenever $M = X + Y$. We note that

$$\text{lifting} \Rightarrow \text{amply supplemented} \Rightarrow \text{supplemented}.$$

In this paper, we consider the following results:

Result 1. *Let N be a quasi-discrete module and $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. If M'_i is generalized N-projective for any $M'_i <_\oplus M_i$ $(i = 1, \cdots, n)$, then M is generalized N-projective.*

Result 2. *Let M be a quasi-discrete module and $N = N_1 \oplus \cdots \oplus N_m$ be lifting for $N = N_1 \oplus \cdots \oplus N_m$. If N_i and M are relatively generalized projective $(i = 1, \cdots, m)$, then M is generalized N-projective.*

Result 3. *Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.*

(1) M is lifting with the (finite) internal exchange property.

(2) M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.

(3) M_i is generalized M_j-projective for each $i \neq j$.

The reader is referred to [3, 5, 6, 7, 10, 13, 14] for research on lifting modules and exchange properties. We state the following for later use.

Lemma 1.1. (cf. [14, 41.7]) *Let X be a submodule of M. If M is an amply supplemented module, then M/X is amply supplemented.*

Lemma 1.2. (cf. [1, Proposition 5.17], [4, Lemma 2.5])

(1) *If* $N \ll M$ *and* $S \subseteq N$, *then* $S \ll M$.

(2) *Let* $f : M \to N$ *be a homomorphism. If* $S \ll M$, *then* $f(S) \ll f(M)$.

(3) *Let* X *be a co-closed submodule of* M. *If* $S \ll M$ *and* $S \subseteq X$, *then* $S \ll X$.

Lemma 1.3. (cf. [3, 3.2(1)]) *Let* $A \subseteq B \subseteq M$. *Then* $A \subseteq_c B$ *in* M *if and only if* $M = A + K$ *for any submodule* K *of* M *with* $M = B + K$.

Lemma 1.4. (cf. [8, Lemma 1.4]) *Let* $M = A + B$ *and* $A \subseteq C$. *If* $C \cap B \ll M$, *then* $A \subseteq_c C$ *in* M.

Lemma 1.5. *If* $S \ll M$ *and* $T \subseteq M$, *then* $T \subseteq_c T + S$ *in* M.

Proof. It is obvious. □

Lemma 1.6. (cf. [3, 3.2(7)], [9, Lemma 3.1])

(1) *Let* $f : M \to N$ *be an epimorphism. If* $X \subseteq_c Y$ *in* M, *then* $f(X) \subseteq_c f(Y)$ *in* N.

(2) *Let* $f : M \to N$ *be an epimorphism with* $\ker f \ll M$. *If* $A \subseteq_c B$ *in* N, *then* $f^{-1}(A) \subseteq_c f^{-1}(B)$ *in* M.

Proof. (1) Let $N = f(Y) + T$. Since f is a epimorphism, there exists a submodule K of M such that $f(K) = T$. So $M = Y + K + \ker f$. By Lemma 1.3, $M = X + K + \ker f$. Hence we see $N = f(M) = f(X) + f(K) = f(X) + T$. By Lemma 1.3, $f(X) \subseteq_c f(Y)$ in N.

(2) Let $M = f^{-1}(B) + K$. Since $A \subseteq_c B$ in N, $N = f(M) = B + f(K) = A + f(K)$. As $\ker f \ll M$, $M = f^{-1}(A) + K + \ker f = f^{-1}(A) + K$. Thus $f^{-1}(A) \subseteq_c f^{-1}(B)$ in M. □

Lemma 1.7. (cf. [3, 3.7(5)], [9, Lemma 3.2]) *Let* $f : M \to N$ *be an epimorphism with* $\ker f \ll M$. *Then*

(1) *If* X *is co-closed in* M, *then* $f(X)$ *is co-closed in* N.

(2) *If* $M = X \oplus Y$, *then* $f(X) \cap f(Y) \ll N$.

Proof. Let $f : M \to N$ be an epimorphism with $\ker f \ll M$.

(1) Let X be co-closed in M and let $T \subseteq_c f(X)$ in N. Then there exists a submodule K of X such that $f(K) = T$. By Lemma 1.6 (2), $f^{-1}(f(K)) \subseteq_c f^{-1}(f(X))$ in M. Since $f^{-1}(f(K)) = K + \ker f$, $f^{-1}(f(X)) = X + \ker f$ and $\ker f \ll M$, $K \subseteq_c K + \ker f \subseteq_c X + \ker f$ in M. Since $K \subseteq X \subseteq X + \ker f$, by [4, Lemma 2.5], $K \subseteq_c X$ in M and so $K = X$. Thus $T = f(K) = f(X)$.

(2) Let $M = X \oplus Y$. By Lemma 1.5, $X \subseteq_c X + \ker f$ in M. Hence $(X + \ker f) \cap Y \ll Y \subseteq M$ by [3, 2.3(i)]. So we see $f[(X + \ker f) \cap Y] \ll f(M) = N$ by Lemma 1.2 (2). Thus $f(X) \cap f(Y) = f[(X + \ker f) \cap Y] \ll N$. □

Lemma 1.8. *Let $f : M \to N$ be an epimorphism with $\ker f \ll M$. Then $S \ll N$ implies $f^{-1}(S) \ll M$.*

Proof. Let $S \ll N$ and let $M = X + f^{-1}(S)$. Then $N = f(M) = S + f(X) = f(X)$ since $S \ll N$. Thus $M = X + \ker f = X$ since $\ker f \ll M$. Therefore $f^{-1}(S) \ll M$. □

Lemma 1.9. (cf. [14, 19.3(1)]) *Let $f : M \to N$ and $g : N \to K$ be epimorphisms with $\ker f \ll M$ and $\ker g \ll N$. Then $\ker(g \circ f) \ll M$.*

Lemma 1.10. *If $M = M_1 \oplus M_2 = M_1 \oplus K$, then $K = \langle M_2 \xrightarrow{\alpha} M_1 \rangle$ for some $\alpha : M_2 \to M_1$.*

Proof. Let p_1 and p_2 be the projections $M \to M_1$, $M \to M_2$, respectively. Since $M_1 \oplus M_2 = M_1 \oplus K$, $p_2(K) = M_2$. Define $\alpha : M_2 \to M_1$ by $\alpha(p_2(k)) = -p_1(k)$, where $k \in K$. Then $K = \langle M_2 \xrightarrow{\alpha} M_1 \rangle$. □

2. Generalized Projectivity

Given modules A and B, A is said to be *generalized B-projective (B-cojective)* if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A' \oplus A''$, $B = B' \oplus B''$, a homomorphism $\varphi : A' \to B'$ and an epimorphism $\psi : B'' \to A''$ such that $g \circ \varphi = f|_{A'}$ and $f \circ \psi = g|_{B''}$ (cf. [11]). Note that every B-projective module is generalized B-projective. If A is generalized B-projective and B is generalized A-projctive, then A and B are said to be *relatively generalized projective*.

Proposition 2.1. (cf. [11]) *Let B^* be a direct summand of B. If A is generalized B-projective, then A is generalized B^*-projective.*

We consider the following condition $(*)$:

$(*)$ Any submodule of M has a co-closure in M.

Note that any module over a right perfect ring has the condition $(*)$ by Oshiro [12, Proposition 1.3].

Proposition 2.2. (cf. [8]) (1) *Let A be a module with the finite internal exchange property and let A^* be a direct summand of A. If A is generalized B-projective, then A^* is generalized B-projective.*

(2) *Let* $M = A \oplus B$ *be a weakly supplemented module with the condition* $(*)$ *and let* A^* *be a direct summand of* A*. If* A *is generalized* B*-projective, then* A^* *is generalized* B*-projective.*

Theorem 2.3. (cf. [8]) *Let* $M_1, \cdots, M_n$ *be lifting modules with the finite internal exchange property and put* $M = M_1 \oplus \cdots \oplus M_n$*. Then the following conditions are equivalent.*

(1) M *is lifting with the finite internal exchange property.*

(2) M *is lifting for* $M = M_1 \oplus \cdots \oplus M_n$*.*

(3) M_i *and* $\oplus_{j \neq i} M_j$ *are relatively generalized projective.*

A module A is said to be *im-small B-projective* if, for any epimorphism $g : B \to X$ and any homomorphism $f : A \to X$ with $\mathrm{Im} f \ll X$, there exists a homomorphism $h : A \to B$ such that $g \circ h = f$ (cf. [7]).

Proposition 2.4. (1) *Let* A *be a module and let* $\{B_i \mid i = 1, \cdots, n\}$ *be a family of modules. Then* A *is im-small* $\oplus_{i=1}^n B_i$*-projective if and only if* A *is im-small* B_i*-projective* $(i = 1, \cdots, n)$*.*

(2) *Let* I *be any set and let* $\{A_i \mid i \in I\}$ *be a family of modules. Then* $\oplus_I A_i$ *is im-small* B*-projective if and only if* A_i *is im-small* B*-projective for all* $i \in I$*.*

Proof. By similar arguments as the proofs of [2]. □

Proposition 2.5 (cf. [8, Proposition 2.5]) Let A be any module and let B be a lifting module. If A is generalized B-projective, then A is im-small B-projective.

3. Direct sums of quasi-discrete modules

A lifting module M is said to be *quasi-discrete* if M satisfies the following condition (D):

(D) If M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Any quasi-discrete module has the internal exchange property [12, Theorem 3.10].

Lemma 3.1. *Let* N *be a quasi-discrete module and let* $M = M_1 \oplus \cdots \oplus M_n$ *be lifting for* $M = M_1 \oplus \cdots \oplus M_n$*. Assume that* M_i *is generalized* N*-projective* $(i = 1, \cdots, n)$*. Then, for any epimorphism* $f : M \to X$ *with* $\ker f \ll M$ *and any epimorphism* $g : N \to X$ *with* $\ker g \ll N$*, there exist decompositions* $M = \overline{M} \oplus \overline{\overline{M}}$*,* $N = \overline{N} \oplus \overline{\overline{N}}$ *and epimorphisms* $\varphi : \overline{\overline{M}} \to \overline{N}$*,* $\psi : \overline{\overline{N}} \to \overline{M}$ *such that* $f|_{\overline{\overline{M}}} = g \circ \varphi$ *and* $g|_{\overline{\overline{N}}} = f \circ \psi$*.*

Proof. It is enough to prove the case of $M = M_1 \oplus M_2$. Let N be a quasi-discrete module and let M be lifting for $M = M_1 \oplus M_2$. Assume that $f : M \to X$ and $g : N \to X$ are epimorphisms such that $\ker f \ll M$ and $\ker g \ll N$, respectively. By Lemma 1.7, $f(M_i)$ is co-closed in X $(i = 1, 2)$ and $f(M_1) \cap f(M_2) \ll X$. Since N is lifting, there exists a decomposition $N = N_i \oplus N_i^*$ such that $N_i \subseteq_c g^{-1}(f(M_i))$ in N $(i = 1, 2)$. By Lemma 1.6, $g(N_i) \subseteq_c g(g^{-1}(f(M_i))) = f(M_i)$ in X and hence $g(N_i) = f(M_i)$. Then $g(N) = X = f(M) = f(M_1) + f(M_2) = g(N_1) + g(N_2)$. As $\ker g \ll N$, we see

$$N = N_1 + N_2 + \ker g = N_1 + N_2.$$

By Lemma 1.8, $f(M_1) \cap f(M_2) \ll X$ implies $g^{-1}(f(M_1) \cap f(M_2)) \ll N$. So we get

$$N_1 \cap N_2 \subseteq g^{-1}(f(M_1)) \cap g^{-1}(f(M_2)) = g^{-1}(f(M_1) \cap f(M_2)) \ll N.$$

Since N is quasi-discrete, we see

$$N = N_1 \oplus N_2.$$

By Proposition 2.1, M_i is generalized N_i-projective $(i = 1, 2)$. Hence there exist decompositions $M_i = M_i' \oplus M_i''$, $N_i = N_i' \oplus N_i''$, a homomorphism $\alpha_i : M_i'' \to N_i'$ and an epimorphism $\beta_i : N_i'' \to M_i'$ such that $f|_{M_i''} = g \circ \alpha_i$ and $g|_{N_i''} = f \circ \beta_i$.

Given $n_i' \in N_i'$, then there exists $m_i \in M_i$ with $g(n_i') = f(m_i)$. Express m_i in $M_i = M_i' \oplus M_i''$ as $m_i = m_i' + m_i''$, where $m_i' \in M_i'$ and $m_i'' \in M_i''$. As $f|_{M_i''} = g \circ \alpha_i$, $f(m_i'') = g \circ \alpha_i(m_i'')$. On the other hand, since β_i is an epimorphism, there exists $n_i'' \in N_i''$ such that $\beta_i(n_i'') = m_i'$. So $f(m_i') = f(\beta_i(n_i'')) = g(n_i'')$. Hence $g(n_i') = f(m_i'') + f(m_i') = g(\alpha_i(m_i'')) + g(n_i'')$. So $n_i' - \alpha_i(m_i'') - n_i'' \in \ker(g|_{N_i})$. Thus we see

$$N_i' \subseteq \alpha_i(M_i'') + N_i'' + \ker(g|_{N_i}).$$

As $\ker(g|_{N_i}) \ll N_i$, we see

$$N_i = N_i' \oplus N_i'' = \alpha_i(M_i'') + N_i'' + \ker(g|_{N_i}) = \alpha_i(M_i'') + N_i''.$$

Hence $\alpha_i(M_i'') = N_i'$, that is, α_i is an epimorphism.

Now define $\varphi : M_1'' \oplus M_2'' \to N_1' \oplus N_2'$ and $\psi : N_1'' \oplus N_2'' \to M_1' \oplus M_2'$ by $\varphi(m_1'' + m_2'') = \alpha_1(m_1'') + \alpha_2(m_2'')$, $\psi(n_1'' + n_2'') = \beta_1(n_1'') + \beta_2(n_2'')$. Then, for any $m_1'' + m_2'' \in M_1'' \oplus M_2''$, $f(m_1'' + m_2'') = g(\alpha_1(m_1'')) + g(\alpha_2(m_2'')) = g(\alpha_1(m_1'') + \alpha_2(m_2'')) = g \circ \varphi(m_1'' + m_2'')$. So $f|_{M_1'' \oplus M_2''} = g \circ \varphi$. Similarly, we see $g|_{N_1'' \oplus N_2''} = f \circ \psi$. The proof is completed. □

Proposition 3.2. *Let N be a quasi-discrete module and $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. If M_i' is generalized N-projective for any $M_i' <_\oplus M_i$ $(i = 1, \cdots, n)$, then M is generalized N-projective.*

Proof. It is enough to prove the case of $M = M_1 \oplus M_2$. Let $f : M \to X$ be a homomorphism and $g : N \to X$ be an epimorphism. Since N is lifting and M is lifting for $M = M_1 \oplus M_2$, we may assume $\ker f \ll M$ and $\ker g \ll N$. By Lemma 1.1, $X = g(N)$ is amply supplemented. So there exists a co-closure K of $f(M)$ in X. Let X' be a supplement of $f(M)$ in X. Then

$$X = X' + f(M) = X' + K \qquad \text{and} \qquad X' \cap f(M) \ll X'.$$

As $K \subseteq f(M)$, $f(M) = K + (X' \cap f(M))$. Since $f(M)$ is amply supplemented, there exists a co-closure S of $X' \cap f(M)$ in $f(M)$. Inasmuch as M is lifting for $M = M_1 \oplus M_2$, there exists a decomposition $M = L \oplus M_1' \oplus M_2'$ such that $L \subseteq_c f^{-1}(S)$ in M. By Lemma 1.6, $f(L) \subseteq_c f(f^{-1}(S)) = S$ in $f(M)$ and hence $f(L) = S$. Thus we see

$$f(M) = f(L) + f(M_1' \oplus M_2') = S + f(M_1' \oplus M_2').$$

By Lemma 1.2 (2), we see $S \cap f(M_1' \oplus M_2') = f(f^{-1}(S) \cap (M_1' \oplus M_2')) \ll f(M)$ since $f^{-1}(S) \cap (M_1' \oplus M_2') \ll M$.

Now let $A \subseteq_c f(M_1' \oplus M_2')$ in X. As $S \subseteq X'$ we see

$$X = S + f(M_1' \oplus M_2') + X' = f(M_1' \oplus M_2') + X' = A + X'.$$

Hence $f(M) = A + (X' \cap f(M)) = A + S$. By Lemma 1.4, $A \subseteq_c f(M_1' \oplus M_2')$ in $f(M)$. Since $f(M_1' \oplus M_2')$ is co-closed in $f(M)$, $A = f(M_1' \oplus M_2')$. Thus $f(M_1' \oplus M_2')$ is co-closed in X.

Since $f(L) = S \subseteq X' \cap f(M) \ll X$ and L is im-small N-projective, there exists a homomorphism $h : L \to N$ such that $f|_L = g \circ h$.

As N is lifting, there exists a decomposition $N = N^* \oplus N^{**}$ such that $N^* \subseteq_c g^{-1}(f(M_1' \oplus M_2'))$ in N. By Lemma 1.6, $g(N^*) \subseteq_c f(M_1' \oplus M_2')$ in X. This implies $g(N^*) = f(M_1' \oplus M_2')$. Thus, by Lemma 3.1, there exist decompositions $M_1' \oplus M_2' = \overline{T} \oplus \overline{\overline{T}}$, $N^* = \overline{N^*} \oplus \overline{\overline{N^*}}$ and epimorphisms $\varphi_1 : \overline{T} \to \overline{\overline{N^*}}$, $\varphi_2 : \overline{N^*} \to \overline{\overline{T}}$ such that $f|_{\overline{T}} = g \circ \varphi_1$ and $g|_{\overline{N^*}} = f \circ \varphi_2$. Now let α and β be the projections $N = \overline{N^*} \oplus \overline{\overline{N^*}} \oplus N^{**} \to \overline{N^*}$, $N = \overline{N^*} \oplus \overline{\overline{N^*}} \oplus N^{**} \to \overline{\overline{N^*}} \oplus N^{**}$, respectively. Put $\gamma = \varphi_2 \circ \alpha \circ h$. Define $\varphi^* : \langle L \xrightarrow{\gamma} \overline{\overline{T}} \rangle \to \overline{\overline{N^*}} \oplus N^{**}$ by $\varphi^*(l - \gamma(l)) = \beta \circ h \circ \delta(l - \gamma(l))$, where $\delta : \langle L \xrightarrow{\gamma} \overline{\overline{T}} \rangle \to L$ is the canonical homomorphism. Then $f|_{\langle L \xrightarrow{\gamma} \overline{\overline{T}} \rangle} = g \circ \varphi^*$.

Put $\varphi = \varphi^* + \varphi_1 : \langle L \xrightarrow{\gamma} \overline{\overline{T}} \rangle \oplus \overline{T} \to \overline{\overline{N^*}} \oplus N^{**}$ and $\psi = \varphi_2 : \overline{N^*} \to \overline{\overline{T}}$. Then

$$f|_{\langle L \xrightarrow{\gamma} \overline{\overline{T}} \rangle \oplus \overline{T}} = g \circ \varphi \qquad \text{and} \qquad g|_{\overline{N^*}} = f \circ \psi.$$

Therefore M is generalized N-projective. □

Proposition 3.3. *Let M be a quasi-discrete module and $N = N_1 \oplus \cdots \oplus N_m$ be lifting for $N = N_1 \oplus \cdots \oplus N_m$. If N_i and M are relatively generalized projective $(i = 1, \cdots, m)$, then M is generalized N-projective.*

Proof. It is enough to prove the case of $N = N_1 \oplus N_2$. Let $f : M \to X$ be a homomorphism and $g : N \to X$ be an epimorphism. Since M is lifting and N is lifting for $N = N_1 \oplus N_2$, we may assume $\ker f \ll M$ and $\ker g \ll N$. By Lemma 1.1, X is amply supplemented and so there exists a co-closure $f(M)^*$ of $f(M)$ in X. Let X' be a supplement of $f(M)$ in X. Then

$$X = f(M) + X' = f(M)^* + X' \qquad \text{and} \qquad X' \cap f(M) \ll X'.$$

So $f(M) = f(M)^* + (X' \cap f(M))$. By Lemma 1.1, $f(M)$ is amply supplemented and hence there exist a co-closure S of $X' \cap f(M)$ and co-closure T of $f(M)^*$ in $f(M)$. Thus

$$f(M) = f(M)^* + (X' \cap f(M)) = T + S.$$

Since M is lifting, there exists a decomposition $M = M_1 \oplus M_1^*$ such that $M_1 \subseteq_c f^{-1}(T)$ in M. By Lemma 1.6, $f(M_1) \subseteq_c f(f^{-1}(T)) = T$ in $f(M)$ and hence $f(M_1) = T$. Similarly, there exists a decomposition $M = M_2 \oplus M_2^*$ such that $f(M_2) = S$. As $f(M) = T + S = f(M_1) + f(M_2)$,

$$M = M_1 + M_2 + \ker f = M_1 + M_2.$$

Since $T \cap S \subseteq X' \cap f(M)^* \ll X' \subseteq X$ and $f(M)^*$ is co-closed in X, we see $T \cap S \ll f(M)^*$ by Lemma 1.2 (3). Hence $T \cap S \ll f(M)$. By Lemma 1.8, we see $f^{-1}(T \cap S) \ll M$. Thus

$$M_1 \cap M_2 \subseteq f^{-1}(T) \cap f^{-1}(S) = f^{-1}(T \cap S) \ll M.$$

Inasmuch as M is a quasi-discrete module, we see

$$M = M_1 \oplus M_2.$$

If $T' \subseteq_c T$ in X, then $X = f(M) + X' = (T + S) + X' = T + X' = T' + X'$. So $f(M) = T' + (X' \cap f(M)) = T' + S$. As $T \cap S \ll f(M)$, we obtain $T' \subseteq_c T$ in $f(M)$ by Lemma 1.4 and so $T' = T$. Thus T is co-closed in X. Now, since N is lifting for $N = N_1 \oplus N_2$, there exists a decomposition $N = K \oplus N_1' \oplus N_2'$ such that $K \subseteq_c g^{-1}(T)$ in N, where $N_1 = N_1' \oplus N_1''$ and $N_2 = N_2' \oplus N_2''$. By Lemma 1.6, $g(K) \subseteq_c T$ in X and so $g(K) = T$. Since M_2 is im-small N-projective and $S = f(M_2) \ll X$, there exists a homomorphism $h : M_2 \to N$ with $f|_{M_2} = g \circ h$.

By Propositions 2.1 and 2.2, K is generalized M_1-projective. By Lemma 3.1, for epimorphisms $g|_K : K \to T$ and $f|_{M_1} : M_1 \to T$, there exist decompositions

$K = K' \oplus K''$, $M_1 = M_1' \oplus M_1''$ and epimorphisms $\psi_1 : K' \to M_1''$, $\psi_2 : M_1' \to K''$ such that $g|_{K'} = f \circ \psi_1$ and $f|_{M_1'} = g \circ \psi_2$. Now let α and β be the projections $N = K' \oplus K'' \oplus N_1' \oplus N_2' \to K'$, $N = K' \oplus K'' \oplus N_1' \oplus N_2' \to K'' \oplus N_1' \oplus N_2'$, respectively. Put $\gamma = \psi_1 \circ \alpha \circ h$ and define $\varphi^* : \langle M_2 \xrightarrow{\gamma} M_1'' \rangle \to K'' \oplus N_1' \oplus N_2'$ by $\varphi^*(m_2 - \gamma(m_2)) = \beta \circ h \circ \delta(m_2 - \gamma(m_2))$, where $\delta : \langle M_2 \xrightarrow{\gamma} M_1'' \rangle \to M_2$ is the canonical homomorphism. For any $m_2 \in M_2$, we express $h(m_2)$ in $K' \oplus K'' \oplus N_1' \oplus N_2'$ as $h(m_2) = k' + y$, where $k' \in K'$, $y \in K'' \oplus N_1' \oplus N_2'$. Then

$$\begin{aligned} f(m_2 - \gamma(m_2)) &= f(m_2) - f\psi_1 \alpha h(m_2) = gh(m_2) - g\alpha h(m_2) \\ &= g(h(m_2) - k') = g(y) = g\beta h(m_2) \\ &= g\beta h\delta(m_2 - \gamma(m_2)) = g \circ \varphi^*(m_2 - \gamma(m_2)). \end{aligned}$$

Hence $f|_{\langle M_2 \xrightarrow{\gamma} M_1'' \rangle} = g \circ \varphi^*$. Put $\varphi = \varphi^* + \psi_2 : \langle M_2 \xrightarrow{\gamma} M_1'' \rangle \oplus M_1' \to K'' \oplus N_1' \oplus N_2'$ and $\psi = \psi_1 : K' \to M_1''$. Then

$$f|_{\langle M_2 \xrightarrow{\gamma} M_1'' \rangle \oplus M_1'} = g \circ \varphi \qquad \text{and} \qquad g|_{K'} = f \circ \psi.$$

Therefore M is generalized N-projective. □

Using the propositions above, we obtain the following.

Theorem 3.4. *Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.*

(1) *M is lifting with the (finite) internal exchange property.*

(2) *M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.*

(3) *M_i is generalized M_j-projective $(i \neq j)$.*

Proof. This is immediate from Propositions 2.2, 3.2, 3.3, and Theorem 2.3. □

A module H is said to be *hollow* if it is an indecomposable lifting module. Note that any hollow module is quasi-discrete. Hence we obtain the following:

Corollary 3.5. *Let $H_1, \cdots, H_n$ be hollow modules and put $M = H_1 \oplus \cdots \oplus H_n$. Then the following conditions are equivalent.*

(1) *M is lifting with the (finite) internal exchange property.*

(2) *M is lifting for $M = H_1 \oplus \cdots \oplus H_n$.*

(3) *H_i is generalized H_j-projective $(i \neq j)$.*

Acknowledgment. The author is thankful to Prof. Yoshitomo Baba and Prof. Derya Keskin Tütüncü for valuable comments. I also thank the referee for the many suggestions which improved the presentations of the paper.

References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, *Springer Verlag*, 1973.

[2] Y. Baba and M. Harada, *On almost M-projectives and almost M-injectives*, Tsukuba J. Math., 14 (1990), 53-69.

[3] J. Clark, C. Lomp, N. Vanaja and R. Wisbauer, Lifting modules. Supplements and Projectivity in Module Theory, *Frontiers in Math., Boston, Birkhäuser*, 2006.

[4] L. Ganesan and N. Vanaja, *Modules for which every submodule has a unique coclosure*, Comm. Algebra, 30 (2002), 2355–2377.

[5] K. Hanada, Y. Kuratomi and K. Oshiro, *On direct sums of extending modules and internal exchange property*, J. Algebra, 250 (2002), 115-133.

[6] M. Harada, Factor categories with applications to direct decomposition of modules, LN in Pure and Applied Math. 88, *Marcel Dekker, Inc., New York*, 1983.

[7] D. Keskin, *On lifting modules*, Comm. Algebra, 28 (2000), 3427-3440.

[8] Y. Kuratomi, *On direct sums of lifting modules and internal exchange property*, Comm. Algebra., 33 (2005), 1795-1804.

[9] Y. Kuratomi and C. Chaehoon, *Lifting modules over right perfect rings*, Comm. Algebra., 35 (2007), 3103-3109.

[10] S.H. Mohamed and B.J. Müller, Continuous and Discrete Modules, London Math. Soc., LN 147, *Cambridge Univ. Press*, Cambridge, 1990.

[11] S.H. Mohamed and B.J. Müller, *Cojective modules*, J. Egyptian Math. Soc. 12 (2004), 83-96.

[12] K. Oshiro, *Semiperfect modules and quasi-semiperfect modules*, Osaka J. Math. 20 (1983), 337-372.

[13] K. Oshiro, Theories of Harada in Artinian Rings and applications to classical Artinian Rings, International Symposium on Ring Theory, *Trends in Math., Birkhäuser*, (2001), 279-301.

[14] R. Wisbauer, Foundations of Module and Ring Theory, *Gordon and Breach Science Publications*, 1991.

Yosuke Kuratomi
Kitakyushu National College of Technology,
Sii, Kokuraminami-ku, Kitakyushu, Japan
E-mail: kuratomi@kct.ac.jp