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# A NOTE ON CHARACTERIZATION OF $\mathcal{N}_{\mathcal{U}}(D_n)$

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Abstract. In this paper construction of normalizer of $D_n$ in $V(\mathbb{Z}D_n)$ is reduced to construction of integral group ring of its cyclic subgroup. In a better expression, we have shown that $\mathcal{N}_{\mathcal{U}}(D_n) = D_n \times F$, where $F$ is a free abelian group with rank $\rho = \frac{1}{2}\varphi(n) - 1$.

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**Introduction.** Let $G$ be a finite group and $\mathbb{Z}G$ be its integral group ring. Let $\mathcal{U}(\mathbb{Z}G)$ denote the unit group of $\mathbb{Z}G$. Let $\mathcal{U} = V(\mathbb{Z}G)$ denote the group of normalized units of $\mathbb{Z}G$ and $\mathcal{Z}(\mathcal{U})$ denote the subgroup of the central units of $\mathcal{U}$. Let $\mathcal{N}_{\mathcal{U}}(G)$ denote the normalizer of $G$ in $V(\mathbb{Z}G)$. Problem 43 in [9] asks if the following normalizer property holds:

$$\mathcal{N}_{\mathcal{U}}(G) = G\mathcal{Z}(\mathcal{U})$$

First time, the normalizer property was proved for finite nilpotent groups by Coleman[2]. After that, Jackowski and Marciniak [3] extended this property to finite groups of odd and the groups having normal Sylow 2-subgroup. Later, Li, Parmenter and Sehgal[7] showed that if the intersection of non-normal subgroups of $G$ is non-trivial then, $G$ satisfies the normalizer property. Next, the normalizer property is verified by Li[6] for some metabelian groups. And also, Hertweck[5] studied for a family of Frobenius groups with abelian Sylow subgroups.

In this study, we will give some results and construction of $\mathcal{N}_{\mathcal{U}}(D_n)$; normalizer of $D_n = \langle a, b : a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, dihedral group of order $2n$, in its normalized units of $\mathbb{Z}D_n$. Let us recall the theorem related to dihedral groups [7].

**Theorem 1.** *Let $G = \langle H, g\rangle$ where $H$ is an Abelian subgroup of index 2. Then the normalizer property holds for $G$.*

It is clear that finite dihedral groups satisfy the normalizer property. In other words, Theorem 1 reduces the construction of the normalizer of a finite dihedral group in its normalized units of $\mathbb{Z}D_n$ to the construction of subgroup of central units of $\mathcal{U}$.

In this study, we have characterized the normalizer of a dihedral group $D_n$ in the normalized units of its integral group ring as follows:

**Theorem 2.** *$\mathcal{N}_{\mathcal{U}}(D_n) = D_n \times F$, where $F$ is an free abelian group with rank*

$$\rho = \frac{1}{2}\varphi(n) - 1.$$

At the end we have given some concrete examples for some dihedral groups.

**Reduction to Cyclic Groups.** Before to prove previous theorem let us recall some basic facts about the unit group of the integral group ring of a cyclic group and definition of Bass cyclic units [8].

**Lemma 3.** *Let $C_n$ be a cyclic group of order $n$. The unit group of $\mathbb{Z}C_n$ is trivial if and only if the order of $C_n$ is 1,2,3,4 or 6.*

**Lemma 4.** *Let $G$ be a group such that the units of $\mathbb{Z}G$ are trivial and let $C_2$ be a cyclic group of order 2. Then the units of $\mathbb{Z}(G \times C_2)$ are also trivial.*

On the other hand, Bass cyclic units are playing an important role in the characterization.

**Definition 5.** *Let $a$ be an element of order $n$ in a group $G$. A Bass cyclic unit is an element of the group ring $\mathbb{Z}G$ of the form*

$$u_i = (1 + a + \cdots + a^{i-1})^{\varphi(n)} + \frac{1 - i^{\varphi(n)}}{n}(1 + a + \cdots + a^{n-1}),$$

*where $i$ is an integer such that $1 < i < n - 1$ and $(i, n) = 1$.*

**Corollary 6.** *Let $C_n = \{a : a^n = 1\}$ be a cyclic group with order $n$, Then the Bass cyclic units $\langle u_i : 1 < i < n/2, (i, n) = 1\rangle$ of the $\mathbb{Z}C_n$ generate a subgroup of finite index in $\mathcal{U}(\mathbb{Z}C_n)$.*

At first, we'll reduce construction of normalizer of a finite dihedral group in its normalized units of $\mathbb{Z}D_n$ to the construction of $V(\mathbb{Z}\langle a\rangle)$. Now let us remember the conjugate classes of $D_n$. If $n = 2k - 1$ then

$$D_n = \{1\} \cup (\bigcup_{i=1}^{i=k-1} \{a^i, a^{-i}\}) \cup \{ba^i : i = 0, ..., 2k - 2\},$$

and if $n = 2k$ then,

$$D_n = \{1\} \cup \{a^k\} \cup (\bigcup_{i=1}^{i=k-1} \{a^i, a^{-i}\}) \cup \{ba^{2i} : i = 0, ..., k-1\} \cup \{ba^{2i+1} : i = 0, ..., k-1\}.$$

In this case, for an arbitrary $\gamma \in \mathcal{Z}(\mathbb{Z}D_n)$ can be written

$$\gamma = \gamma_0 + \sum_{i=1}^{i=k-1} \gamma_i(a^i + a^{-i}) + \lambda b\hat{a}, \tag{1}$$

where $n = 2k-1$, $\lambda, \gamma_i \in \mathbb{Z}$ and $\hat{a} = 1 + a + \cdots + a^{2k-2}$. Otherwise

$$\gamma = \gamma_0 + \gamma_k a^k + \sum_{i=1}^{i=k-1} \gamma_i(a^i + a^{-i}) + \lambda_1 b(\hat{a}^2) + \lambda_2 ba(\widehat{a^2}), \tag{2}$$

where $n = 2k$, $\lambda_1, \lambda_2, \gamma_i \in \mathbb{Z}$ and $\widehat{a^2} = 1 + a^2 + \cdots + a^{2k-2}$.

If we denote the commutator subgroup by $D'_n$ then, $D'_n = \langle a^2 \rangle$ and we have

$$D_n/D'_n \cong \begin{cases} C_2 & n = 2k-1 \\ C_2 \times C_2 & n = 2k. \end{cases} \tag{3}$$

Now, let us introduce a well-known subring of $\mathbb{Z}C_n$ and show both of their normalized units have the same rank.

**Remark 7.** $\mathbb{Z}[a + a^{-1}] = \{\gamma \in \mathbb{Z}C_n : \gamma_i = \gamma_{n-i}, 0 < i < n\}$*is a subring of* $\mathbb{Z}C_n$.

**Proposition 8.** *The rank of the torsion-free part of $V(\mathbb{Z}C_n)$ is equal to the rank of the torsion-free part of $V(\mathbb{Z}[a + a^{-1}])$.*

**Proof.** It is enough to demonstrate that each Bass cyclic unit can be embedded into $V(\mathbb{Z}[a + a^{-1}])$. Let us consider the following mapping :

$$\psi : \mathcal{B} \longrightarrow V(\mathbb{Z}[a + a^{-1}])$$
$$u_i \mapsto a^{-\frac{i-1}{2}} u_i.$$

where $u_i$ is a Bass cyclic unit. Here, let us denote $\hat{a} = 1+a+\cdots+a^{n-1}$. If $i = 2m+1$ then

$$\psi(u_i) = (a^{-m} + \cdots + 1 + \cdots + a^m)^{\varphi(n)} + \frac{1 - i^{\varphi(n)}}{n}\hat{a} \in V(\mathbb{Z}[a + a^{-1}]).$$

On the other hand, if $i = 2m$ then we get

$$\psi(u_i) = (a^{-\frac{2m-1}{2}} + \cdots + a^{-\frac{1}{2}} + a^{\frac{1}{2}} + \cdots + a^{\frac{2m-1}{2}})^{\varphi(n)} + \frac{1 - i^{\varphi(n)}}{n}\hat{a}.$$

Since $n$ is odd, say $n = 2t-1$ for a fixed $t \in \mathbb{N}$ than $a^{\frac{1}{2}} = a^t$. By rewriting, we get

$$\psi(u_i) = (a^{-(2m-1)t} + \cdots + a^{-t} + a^t + \cdots + a^{(2m-1)t})^{\varphi(n)} + \frac{1 - i^{\varphi(n)}}{n}\hat{a}.$$

So, for each Bass cyclic unit with even index $i, \psi(u_i) \in V(\mathbb{Z}[a + a^{-1}])$. □

**Definition 9.** *Let* $\widetilde{\mathbb{Z}}[a+a^{-1}] = \begin{cases} \{\gamma \in \mathbb{Z}[a+a^{-1}] : \gamma_k \in 2\mathbb{Z}\} & , \quad n = 2k \\ \mathbb{Z}[a+a^{-1}] & , \quad n = 2k-1. \end{cases}$

**Remark 10.** *It is clear that* $\widetilde{\mathbb{Z}}[a+a^{-1}]$ *is a subring of* $\mathbb{Z}[a+a^{-1}]$ *and* $V(\mathbb{Z}[a+a^{-1}]) = V(\widetilde{\mathbb{Z}}[a+a^{-1}]) \times \mathcal{Z}(D_n)$.

Now, we can prove the main theorem

**Proof. (Theorem 2)** Let us consider the natural ring homomorphism:

$$\varphi : \mathbb{Z}D_n \longrightarrow \mathbb{Z}(D_n/D'_n)$$
$$\sum \gamma_g g \mapsto \sum \gamma_g (gD'_n)$$

For $\gamma \in \mathcal{Z}(\mathcal{U})$, by using (1) and (2) we have

$$\varphi(\gamma) = \begin{cases} (\sum_{i=1}^{i=k-1} \gamma_i)D'_n + ([2k-1]\lambda)bD'_n & , \quad n = 2k-1 \\ \\ \alpha D'_n + \beta a D'_n + k\lambda_1 bD'_n + k\lambda_2 baD'_n & , \quad n = 2k \end{cases}$$

where $\alpha$ is the sum of $\gamma_i$'s with even indices and $\beta$ is the sum of $\gamma_i$'s with odd indices.

If $k = 1$ then, in both cases $D_n$ is an abelian group itself. So, the result is trivial. If we take $k > 1$ then neither $[2k-1]\lambda$ nor $k\lambda_1$ and $k\lambda_2$ can be equal to 1. On the other hand by (3), Lemma 3 and Lemma 4, we have $\varphi(\gamma) \in D_n/D'_n$. Therefore,

$$\gamma \in V(\mathbb{Z}[a+a^{-1}]).$$

Since

$$D_n \cap V(\mathbb{Z}[a+a^{-1}]) = \mathcal{Z}(D_n),$$

Here if we choose $F = V(\widetilde{Z}[a+a^{-1}])$ and regard as Remark 10, then we get $\mathcal{N}_{\mathcal{U}}(D_n) = D_n \times F$. Hence, the rank of $F$ is :

$$\begin{aligned} \rho(F) &= \rho(V(\widetilde{Z}[a+a^{-1}]) \\ &= \rho(V(\mathbb{Z}[a+a^{-1}])) \\ &= \rho(V(\mathbb{Z} < a >)) \\ &= \frac{1}{2}\varphi(n) - 1. \end{aligned}$$

□

**Corollary 11.** *If we denote* $\mathcal{B} = \langle a^{-\frac{i-1}{2}} u_i : u_i$ *is a Bass cyclic unit*$\rangle$ *then* $\mathcal{N}_{\mathcal{U}}(D_n) \supseteq D_n \times \mathcal{B}$, *where* $(F : \mathcal{B}) < \infty$.

In order to give a complete characterization of $F$, we need some basic results. For a finite abelian group A, we write [8] :

**Theorem 12.** *$V(\mathbb{Z}A) = A \times F$, where $F$ is a finitely generated abelian group.*

**Definition 13.** *In Theorem 12, if $F = \langle u_1, \ldots, u_r \rangle$ then the set $\{u_1, \ldots, u_r\}$ is called **a fundamental system of units** of $V(\mathbb{Z}A)$.*

Karpilovsky [4] have given some characterizations of fundamental system for some groups of small order as follows :

$$V(\mathbb{Z}C_5) = C_5 \times \langle -1 + a^2 + a^3 \rangle,$$

$$V(\mathbb{Z}C_8) = C_8 \times \langle -1 - a - a^2 + a^4 + 2a^5 + a^6 \rangle,$$

$$V(\mathbb{Z}C_7) = C_7 \times \langle 1 - a + a^2 \rangle \times \langle 2 + 2a - a^3 - a^4 - a^5 \rangle,$$

and Bilgin[9] characterized for $n = 12$ as follows :

$$V(\mathbb{Z}C_{12}) = C_{12} \times \langle 3 + 2a + a^2 - a^4 - 2a^5 - 2a^6 - 2a^7 - a^8 + a^{10} + 2a^{11} \rangle.$$

By using the fundamental system of units we can write the normalizer of $D_n$ respectively.

$$\mathcal{N}_{\mathcal{U}}(D_5) = D_5 \times \langle -1 + (a^2 + a^{-2}) \rangle,$$

$$\mathcal{N}_{\mathcal{U}}(D_8) = D_8 \times \langle -1 - (a + a^{-1}) + (a^3 + a^{-3}) + 2a^4 \rangle,$$

$$\mathcal{N}_{\mathcal{U}}(D_7) = D_7 \times \langle -1 + (a + a^{-1}) \rangle \times \langle -1 - (a + a^{-1}) + 2(a^3 + a^{-3}) \rangle,$$

$$\mathcal{N}_{\mathcal{U}}(D_{12}) = D_{12} \times \langle 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}) - 2(a^5 + a^{-5}) - 2a^6 \rangle$$

However, in order to give a complete characterization of $\mathcal{N}_{\mathcal{U}}(D_n)$ for each $n$, first of all, *Problem A* in [4, p.267] must be solved.

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