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$nil-$INJECTIVE RINGS

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Abstract. A ring R is called left $nil-$injective if every R–homomorphism from a principal left ideal which is generated by a nilpotent element to R is a right multiplication by an element of R. In this paper, we first introduce and characterize a left $nil-$injective ring, which is a proper generalization of left $p-$injective ring. Next, various properties of left $nil-$injective rings are developed, many of them extend known results.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and R–modules are unital. For $a \in R$, $r(a)$ and $l(a)$ denote the right annihilator and the left annihilator of a, respectively. We write $J(R)$, $Z_l(R)$ ($Z_r(R)$), $N(R)$, $N_1(R)$ and $S_l(R)$ ($S_r(R)$) for the Jacobson radical, the left (right) singular ideal, the set of nilpotent elements, the set of non-nilpotent elements and the left (right) socle of R, respectively.

2. Characterizations of left $nil-$injective rings

Call a left $R-$module M $nil-$injective if for any $a \in N(R)$, any left $R-$ homomorphism $f : Ra \longrightarrow M$ can be extended to $R \longrightarrow M$, or equivalently, $f = \cdot m$ where $m \in M$. Clearly, every left $p-$injective module ($c.f.$[8] or [16]) is left $nil-$injective. If R_R is $nil-$injective, then we call R a left $nil-$injective ring. Hence every left $p-$injective ring (c.f [16]) is left $nil-$injective. Our interest here is in left $nil-$injective rings. The following theorem is an application of [16, Lemma 1.1].

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Theorem 2.1. *The following conditions are equivalent for a ring R.*

(1) R is a left $nil-$injective ring.

(2) $rl(a) = aR$ for every $a \in N(R)$.

(3) $b \in aR$ for every $a \in N(R), b \in R$ with $l(a) \subseteq l(b)$.

(4) $r(l(a) \cap Rb) = r(b) + aR$ for all $a, b \in R$ with $ba \in N(R)$.

Proof. (1) $\Rightarrow$ (2) $aR \subseteq rl(a)$ is clear. Now let $x \in rl(a)$. Then $f : Ra \longrightarrow R$ defined by $f(ra) \longmapsto rx$, $r \in R$ is a left $R-$homomorphism. Since R is a left $nil-$injective ring and $a \in N(R)$, there exists a $c \in R$ such that $f = \cdot c$. Therefore $x = f(a) = ac \in aR$. Hence $rl(a) \subseteq aR$ and so $aR = rl(a)$.

(2) $\Rightarrow$ (3) Assume that $a \in N(R), b \in R$ and $l(a) \subseteq l(b)$. By (2), $b \in rl(b) \subseteq rl(a) = aR$.

(3) $\Rightarrow$ (4) Obviously, $r(b) + aR \subseteq r(l(a) \cap Rb)$ always holds. Let $x \in r(l(a) \cap Rb)$. Then $l(ba) \subseteq l(bx)$. By (3), $bx \in baR$ because $ba \in N(R)$. Write $bx = bac, c \in R$. Then $x - ac \in r(b)$ and so $x \in r(b) + aR$. Therefore $r(l(a) \cap Rb) = r(b) + aR$.

(4) $\Rightarrow$ (1) Let $a \in N(R)$ and $f : Ra \longrightarrow R$ be any left $R-$homomorphism. Since $l(a) \subseteq l(f(a))$ and $a \in N(R)$, $f(a) \in rl(f(a)) \subseteq rl(a) = r(l(a) \cap R1) = r(1) + aR = aR$ by (4). This shows that R is left $nil-$injective ring. □

Example 2.2. *The ring $\mathbb{Z}$ of integers is left $nil-$injective ring which is not $p-$injective.*

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 2.3. *Let $R = \prod_{i \in I} R_i$ be a direct product of rings. Then R is left $nil-$injective if and only if R_i is left $nil-$injective for all $i \in I$.*

Recall that a ring R is left universally mininjective [14] if $kR = rl(k)$ for every minimal left ideal Rk of R. R is called right minannihilator [14] if Rk being minimal left ideal of R always implies that kR is a minimal right ideal. R is called left universally mininjective [14] if Rk being a minimal left ideal of R implies that $Rk = Re, e^2 = e \in R$. Call a ring R left $MC2$ [19] if $aRe = 0$ implies $eRa = 0$, where $a, e^2 = e \in R$ and Re is a minimal left ideal of R. A ring R is called left Johns [15] if it is left noetherian and every left ideal is an annihilator. R is said to be a left $CEP-$ring [15] if every cyclic left $R-$module can be essentially embedded in a projective module.

Corollary 2.4. *Let R be a left $nil-$injective ring. Then*

(1) R is a left mininjective ring.

(2) R is left minsymmetric ring.

(3) R is left $MC2-$ring.

(4) R is a right minannihilator ring.

(5) If R is a left Johns ring, then R is quasi-Frobenius.

(6) If R is a left $CEP-$ring, then R is quasi-Frobenius.

(7) If R is a left noetherian ring with essential left socle, then R is left artinian.

(8) If R is a left continuous ring and $R/S_r(R)$ is left Goldie, then R is quasi-Frobenius.

Proof. (1) Assume that Rk is any minimal left ideal of R. If $(Rk)^2 = 0$, then $k \in N(R)$. By hypothesis and Theorem 2.1, $Rk = rl(k)$; we are done. If $(Rk)^2 \neq 0$, then $Rk = Re, e^2 = e \in R$. Write $e = ck, c \in R$. Then $k = ke = kck$. Set $g = kc$. Then $g^2 = g, k = gk$ and $kR = gR$. Hence $l(k) = l(g)$ and so $kR = gR = rl(g) = rl(k)$; we are also done. Therefore R is a left mininjective ring.

(2) It follows from [14, Theorem 1.14].

(3) Assume that $Re, e^2 = e \in R$ is a minimal left ideal of R and $a \in R$ with $aRe = 0$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Since $eba \in N(R)$, $ebaR = rl(eba)$ by hypothesis. Clearly, $l(e) = l(eba)$, so $ebaR = eR$. Therefore $eR = eReR = ebaReR = 0$, which is a contradiction. Hence $eRa = 0$ and so R is a left $MC2$ ring.

(4) Assume that kR is a minimal right ideal of R. If $(kR)^2 \neq 0$, then $kR = eR, e^2 = e \in R$. So $rl(k) = rl(kR) = rl(eR) = rl(e) = eR = kR$. If $(kR)^2 = 0$, then $k \in N(R)$ so, by hypothesis, $rl(k) = kR$

(5) It follows from [15, Theorem 4.6].

(6) Since any left $CEP-$ ring is left Johns, (6) follows from (5).

(7) According to [17, Theorem 2], any left noetherian left minsymmetric ring with essential left socle is left artinian, so we derive (7).

(8) This is an immediate consequence of [18, Corollary 1]. $\square$

Example 2.5. *Let $V = Fv \oplus Fw$ be a two-dimensional vector space over a field F. The trivial extension $R = T(F, V) = F \oplus V$ is a commutative, local, artinian ring with $J(R)^2 = 0$ and $J(R) = Z_l(R)$. Since $(0, v) \in N(R)$ and $rl((0, v)) \neq (0, v)R$, R is not a left $nil-$injective ring.*

Example 2.6. *If R is not a left $nil-$injective, then the polynomial ring $R[x]$ is not $nil-$injective (In fact, there exists $\neq a \in N(R)$ such that $r_R l_R(a) \neq aR$. Hence $a \in N(R[x]$ and $r_{R[x]} l_{R[x]}(a) = (r_R l_R(a))[x] \neq (aR)[x] = a(R[x])$). On the other hand, since $S_l(R[x]) = 0, R[x]$ is a left mininjective ring. Hence there exists a left mininjective ring which is not left $nil-$injective.*

Hence we have: {left $p-$injective rings} $\subsetneqq$ { left $nil-$injective rings} $\subsetneqq$ {left mininjective rings}.

A ring R is said to be **NI** if $N(R)$ forms an ideal of R. A ring R is said to be $2-prime$ if $N(R) = P(R)$, where $P(R)$ is the prime radical of R. Clearly, every $2-prime$ ring is NI.

A ring R is called zero commutative (briefly ZC)[4] if for $a, b \in R$, $ab = 0$ implies $ba = 0$. A ring R is called ZI [4] if for $a, b \in R$ $ab = 0$ implies $aRb = 0$. According to [4], every ZC ring is ZI. A ring R is called reduced if $N(R) = 0$. Clearly, reduced rings are ZC. A ring R is Abelian if every idempotent of R is central.

Corollary 2.7. *Let R be a left $nil-$injective ring. Then the following statements hold:*

(1) If $a \in N(R)$ and ${}_RRa$ is projective, then $Ra = Re$ with $e^2 = e \in R$.

(2) $P(R) \subseteq Z_l(R)$.

(3) If R is an NI ring, then $N(R) \subseteq Z_l(R)$.

(4) If R is a $2-prime$ ring, then $N(R) \subseteq Z_l(R)$.

(5) The following conditions are equivalent:

(a) R is a reduced ring.

(b) R is a ZC left nonsingular ring.

(c) R is a ZI left nonsingular ring.

(d) R is a left nonsingular $2-prime$ ring.

(e) R is a left nonsingular NI ring.

Proof. (1) Since ${}_RRa$ is projective, $l(a) = Rg, g^2 = g \in R$. By hypothesis and Theorem 2.1, $(1-g)R = r(Rg) = rl(a) = aR$. Write $1-g = ac$ and $e = ca$. Then $a = (1-g)a = aca = ae$, $e^2 = e$ and $Ra = Re$.

(2) If $b \in P(R)$ and $b \notin Z_l(R)$, then there exists a nonzero left ideal I of R such that $I \cap l(b) = 0$. Let $0 \neq c \in I$. Then $cb \neq 0$. Set $f : Rcb \longrightarrow R$ *via* $rcb \longmapsto rc, r \in R$. Then f is a well-defined left $R-$homomorphism. Since $cb \in P(R) \subseteq N(R)$, $f = \cdot u, u \in R$. Therefore $c = f(cb) = cbu$ and so $c(1-bu) = 0$ and so $c = 0$ because $1-bu$ is invertible. This is a contradiction. Hence $b \in Z_l(R)$ and so $P(R) \subseteq Z_l(R)$

(3) The proof is similar to that of (2).

(4) Follows by (3).

(5) Since every ZI ring is a $2-prime$, $(c) \Rightarrow (d)$ holds.

$(e) \Rightarrow (a)$ Since R is an NI ring, by Corollary 2.4, $N(R) \subseteq Z_l(R)$. Hence $N(R) = 0$ because $Z_l(R) = 0$ by (e). This shows that R is a reduced ring.

The rest of the implications are clear. □

Recall that a ring R is left PP if every principal left ideal of R is projective as a left $R-$module. A ring R is left PS [13] if every minimal left ideal is projective as a left $R-$module. A ring R is said to be left NPP if ${}_RRa$ is projective for all $a \in N(R)$. Hence left PP rings, Von Neumann regular rings and reduced rings are left NPP.

Example 2.8. *Since there exists a reduced left $p-$injective ring which is not Von Neumann regular, there exists a left $p-$injective left NPP ring which is not Von Neumann regular. Since a ring R is Von Neumann regular if and only if R is left $p-$injective left PP ring, there exists a left $p-$injective left NPP ring which is not left PP. Therefore there exists a left NPP ring which is not left PP.*

Theorem 2.9. *The following conditions are equivalent for a ring R.*

(1) R is a reduced ring.

(2) R is a left NPP ZC ring.

(3) R is a left NPP ZI ring.

(4) R is a left NPP Abelian ring.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are obvious.

$(4) \Rightarrow (1)$ Let $a \in R$ with $a^2 = 0$. Since R is a left NPP ring, ${}_RRa$ is projective, so $l(a) = R(1-e)$ where $e^2 = e \in R$. Hence $a = ea = ae$ because R is an Abelian ring. Since $a \in l(a) = R(1-e)$, $a = a(1-e)$. Thus $a = ae = a(1-e)e = 0$, which implies that R is reduced. □

Theorem 2.10. *(1) R is a left NPP ring if and only if every homomorphic image of any $nil-$injective left $R-$module is $nil-$injective.*

(2) If R is a left NPP ring, then R is a left nonsingular ring.

(3) If R is a left NPP ring, then R is a left PS ring.

(4) Let R be a ring such that the polynomial ring $R[x]$ is left NPP ring. Then R is a left NPP ring.

Proof. (1) Assume that R is a left NPP ring and $f : Q \longrightarrow W$ is an $R-$epic where ${}_RQ$ is left $nil-$injective. Let $a \in N(R)$ and $g : Ra \longrightarrow W$ be a left $R-$homomorphism. Since ${}_RRa$ is projective, there exists a left $R-$homomorphism $h : Ra \longrightarrow Q$ such that $fh = g$. Since ${}_RQ$ is $nil-$injective and $a \in N(R)$, there exists a left $R-$homomorphism $\gamma : R \longrightarrow Q$ such that $\gamma i = h$ where $i : Ra \hookrightarrow R$ is the inclusion map. Set $\sigma = f\gamma : R \longrightarrow W$. Then $\sigma i = f\gamma i = fh = g$. This shows that ${}_RW$ is $nil-$injective.

Conversely, suppose that every homomorphic image of any $nil-$injective left $R-$module is $nil-$injective and $a \in N(R)$. In order to show that Ra is projective, let $g : E \longrightarrow W$ be an epimorphism of left $R-$module and $h : Ra \longrightarrow W$ an $R-$homomorphism where ${}_RE$ is any injective module. By hypothesis, ${}_RW$ is $nil-$injective, so there exists a left $R-$homomorphism $\gamma : R \longrightarrow W$ such that $\gamma i = h$. Therefore there exists a left $R-$homomorphism $\sigma : R \longrightarrow E$ such that $g\sigma = \gamma$. Set $f = \sigma i : Ra \longrightarrow E$. Then $gf = g\sigma i = \gamma i = h$. This implies that ${}_RRa$ is projective.

(2) Let $0 \neq a \in Z_l(R)$ with $a^2 = 0$. Since R is a left NPP ring, ${}_RRa$ is projective. So $l(a)$ is a direct summand of R as a left $R-$module. But $a \in Z_l(R)$, $l(a)$ must be essential in ${}_RR$, which is a contradiction. Hence $Z_l(R) = 0$.

(3) By (2), $Z_l(R) = 0$ and so $S_l(R) \cap Z_l(R) = 0$. By [2], R is a left PS ring.

(4) Assume that $a \in N(R)$. Then $a \in N(R[x])$ and so $l_{R[x]}(a) = R[x]e$ where $e^2 = e \in R[x]$ by hypothesis. Let $e = e_0 + e_1x + e_2x^2 + \cdots + e_nx^n$ where $e_i \in R, i = 1, 2, \cdots, n$. Thus $e_0^2 = e_0$ and $l_R(a) = Re_0$, which implies that R is a left NPP ring. □

Example 2.11. *The trivial extension $R = T(\mathbb{Z}, \mathbb{Z}_{2^\infty})$ is a commutative ring for which $Z_r = J \neq 0$ and $S_l(R)$ is simple and essential in R. Hence $S_l(R) \cap Z_l(R) = S_l(R) \neq 0$, so R is not left PS by* [13]*. By Theorem 2.10(3), R is not left NPP. Hence $R[x]$ is not left NPP. But $R[x]$ is left PS because $S_l(R[x]) = 0$. Therefore there exists a left PS ring which is not left NPP.*

Example 2.12. *Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $N(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $0 \neq u \in F$. Then $\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} R = \begin{pmatrix} 0 & uF \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. On the other hand, $rl(\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} R$. Hence R is not left $nil-$injective. Since R is left PP, R is left NPP. Therefore there exists a left NPP ring which is not left $nil-$injective.*

Example 2.13. *If ${}_RV_R$ is a bimodule over a ring R, then the trivial extension $R = T(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a commutative ring with $J(R) = Z_l(R) \neq 0$ and $S_l(R) = 0$. Hence R is a left PS ring that is not left nonsingular.*

Remark 2.14. *Since there exists a commutative nonsingular semiprimary ring R which is not semisimple. Hence R is not left NPP ring by Theorem 2.9. This implies that there exists a left nonsingular ring which is not left NPP. Hence we have:*

$\{$left PP rings$\}$ $\subsetneqq$ $\{$ left NPP rings$\}$ $\subsetneqq$ $\{$left nonsingular rings$\}$ $\subsetneqq$ $\{$left PS rings$\}$.

Recall that a ring R is $I-$finite if it contains no infinite orthogonal family of idempotents. Evidently, every semiperfect ring is $I-$finite. Call a nonzero right ideal I of R right weakly essential if $I \cap aR \neq 0$ for all $0 \neq a \in N(R)$. Clearly, every essential right ideal of R is right weakly essential.

Theorem 2.15. *Let R be a left $nil-$injective ring and let $a \in N(R)$, $b \in R$.*

(1) If $\sigma : {}_R Ra \longrightarrow {}_R Rb$ is epic, then bR_R can be embedded in aR_R.

(2) Let R be a ZC ring or $N(R) \subseteq c(R)$. If ${}_R Ra \cong {}_R Rb$, then $aR_R \cong bR_R$.

(3) If R is a left $Kasch$ ring, then $r(J)$ is weakly essential as a right ideal of R.

(4) If K is a singular simple left ideal of R, then KR is the homogeneous component of $S_l(R)$ containing K.

(5) If R is $I-$finite, then $R = R_1 \times R_2$, where R_1 is semisimple and every simple left ideal of R_2 in nilpotent.

Proof. (1) Let $\sigma = \cdot u, u \in R$. Then $au = \sigma(a) = vb, v \in R$. Set $\varphi : bR \longrightarrow aR$ defined by $\varphi(br) = vbr = aur \in aR$. Then φ is a right $R-$homomorphism. If $\varphi(br) = 0$, then $aur = vbr = 0$. Since $b = \sigma(ca), c \in R, b = cau$. Hence $br = caur = 0$, which implies that φ is a monic.

(2) Let φ, u, v as (1). Under the hypothesis, we can show that $\sigma(a) \in N(R)$. Since $l(a) = l(\sigma(a)), \sigma(a)R = rl(\sigma(a)) = rl(a) = aR$ by Theorem 2.1. Thus $aR = auR$, which implies that φ is epic.

(3) Assume that $0 \neq a \in N(R)$ and M is a maximal submodule of ${}_R Ra$. Then there exists a left $R-$monic $\sigma : Ra/M \longrightarrow R$ because R is a left $Kasch$ ring. Let $\rho : Ra \longrightarrow R$ defined by $\rho(ra) = \sigma(ra+M)$. Then $\rho = \cdot u, u \in R$ because R is a left $nil-$injective ring. Clearly $au = \rho(a) = \sigma(a+M) \neq 0$ and $Jau = J\sigma(a+M) = \sigma(Ja+M) = 0$ because $JRa \subseteq M$. Hence $0 \neq au \in aR \cap r(J)$, which shows that $r(J)$ is right weakly essential.

(4) Let $K = Rk, k \in R$ and $\sigma : K \longrightarrow S$ be a left $R-$isomorphism, where S is a left ideal of R. Since K is a singular simple left ideal of R, $K^2 = 0$, and so $k \in N(R)$. By hypothesis, $kR = rl(k) = rl(\sigma(k)) = \sigma(k)R$ because $l(k) = l(\sigma(k))$

and $\sigma(k) \in N(R)$. Hence $S = R\sigma(k) \subseteq RkR = KR$, so the $K-$component is in KR. The other inclusion always holds.

(5) By Corollary 2.4, this is an immediate consequence of [14, Theorem 1.12]. $\square$

Remark 2.16. *If R is a commutative nil$-$injective ring, then the singular homogeneous component of $S_l(R)$ are simple. We generalize this fact as follows: If $A \cap B = 0$ where A and B are left ideals of left nil$-$injective ring R. If A is a nil ideal of R, then $Hom_R(A, B) = 0$ by Theorem 2.1.*

Theorem 2.17. *Let R be a left nil$-$injective ring. If $ReR = R$ where $e^2 = e \in R$, then eRe is left nil$-$injective.*

Proof. Assume that $a \in N(S)$ where $S = eRe$. Then $a \in N(R)$ and so $aR = r_R l_R(a)$ by Theorem 2.1. Let $x \in r_S l_S(a)$. Then $l_S(a) \subseteq l_S(x) \subseteq l_R(x)$. Now let $y \in l_R(a)$. Then $ya = 0$. Write $1 = \Sigma_{i=1}^n u_i e v_i, u_i, v_i \in R$. Clearly $ev_i yx = ev_i yex = 0$ for all i because $l_S(a) \subseteq l_S(x)$. Therefore $yx = \Sigma_{i=1}^n u_i e v_i yx = 0$, so $y \in l_R(x)$. This implies that $l_R(a) \subseteq l_R(x)$ and so $x \in r_R l_R(x) \subseteq r_R l_R(a) = aR$. Therefore $x = xe \in aRe = aeRe = aS$, which shows that $r_S l_S(a) \subseteq aS$. Hence $aS = r_S l_S(a)$ and so $eRe = S$ is a left *nil*$-$injective ring by Theorem 2.1. $\square$

Call a ring R $n-$regular if $a \in aRa$ for all $a \in N(R)$. Examples include Von Neumann regular rings and reduced rings. Clearly, every $n-$regular ring is semiprime.

Theorem 2.18. *The following conditions are equivalent for a ring R.*

(1) R is a $n-$regular ring.

(2) Every left $R-$module is nil$-$injective.

(3) Every cyclic left $R-$module is nil$-$injective.

(4) R is left nil$-$injective left NPP ring.

Proof. (1) $\Rightarrow$ (2) Assume that M is left $R-$module and $f : Ra \longrightarrow M$ is any left $R-$homomorphism for $a \in N(R)$. By (1), $a = aba$, $b \in R$. Write $e = ba$. Then $e^2 = e$ and $a = ae$. Set $m = f(e)$. Then $f = \cdot m$, which implies that ${}_RM$ is *nil*$-$injective.

(2) $\Leftrightarrow$ (3) It is clear.

(3) $\Rightarrow$ (4) Clearly R is a left *nil*$-$injective ring by (3). Assume that $a \in N(R)$. Then ${}_RRa$ is *nil*$-$injective by (3), so $I = \cdot c$, $c \in Ra$ where $I : {}_RRa \longrightarrow {}_RRa$ is the identity map. Therefore $a = I(a) = ac \in aRa$. Write $c = ba, b \in R$. Then $a = ac = aba, c^2 = baba = ba = c$ and $Ra = Rc$ is a projective left $R-$module.

(4) $\Rightarrow$ (1) Suppose that $a \in N(R)$. By (4) and Theorem 2.1, $aR = rl(a)$. Since R is left NPP ring, $l(a) = R(1 - e), e^2 = e \in R$. Therefore $aR = eR$. Write

$e = ac, c \in R$. Then $a = ea = aca \in aRa$, which implies that R is $n-$regular ring. $\square$

Remark 2.19. *Since there exists a reduced ring which is not Von Neumann regular ring, there exists an $n-$regular ring which is not Von Neumann regular ring. For example, $\mathbb{Z}$.*

Remark 2.20. *The ring introduced in Example 2.12 is left NPP which is not $n-$regular because R is not left $nil-$injective.*

According to [14], a ring R is left universally mininjective if and only if R is left mininjective left PS. Since $n-$regular rings are semiprime, every $n-$regular ring is left universally mininjective. On the other hand, the ring $R[x]$ in Example 2.11 is left universally mininjective which is not left NPP and so is not $n-$regular by Theorem 2.18. It is well known that there exists a semiprime ring R such that $Z_l(R) \neq 0$. Hence there exists a semiprime ring which is not left NPP by Theorem 2.10(2). Therefore there exists a semiprime ring which is not $n-$regular. Since there exists a polynomial ring $R[x]$ which is not semiprime and all polynomial rings are left universally mininjective, we have:

{Von Neumann regular rings} $\subsetneqq$ { $n-$regular rings} $\subsetneqq$ {semiprime rings} $\subsetneqq$ {left universally mininjective rings}.

Call a ring R left $NC2$ if ${}_RRa$ projective implies $Ra = Re, e^2 = e \in R$ for all $a \in N(R)$. Clearly, every left $C2$ ring (c.f. [15]) is left $NC2$. By Corollary 2.7(1), we know that every left $nil-$injective ring is left $NC2$.

Example 2.21. *The trivial extension $R = T(\mathbb{Z}, \mathbb{Z}_{2^\infty})$ is a commutative ring with $Z_l(R) = J(R) \neq 0$ which is not left $C2$ by* [15, Example 3.2]. *Since $N(R) \subseteq J(R)$, R is left $NC2$. Therefore there exists a left $NC2$ ring which is not left $C2$.*

The ring $\mathbb{Z}$ of integers is also left $NC2$ ring which is not left $C2$.

The ring R in Example 2.5 is left $NC2$ which is not left $nil-$injective.

Theorem 2.22. *(1) If R is a left $NC2$ ring, then R is left $MC2$ ring.*

(2) If $R[x]$ is a left $NC2$ ring, then so is R.

Proof. (1) Assume that Rk is a minimal projective left ideal R. If $(Rk)^2 \neq 0$, then $Rk = Re, e^2 = e \in R$, we are done; If $(Rk)^2 = 0$, then $k \in N(R)$. Since R is a left $NC2$ ring, $Rk = Rg, g^2 = g \in R$.

(2) Suppose that $a \in N(R)$ and ${}_RRa$ is projective. Then $l_R(a) = Re, e^2 = e \in R$. Since $l_{R[x]}(a) = R[x]e$ and $a \in N(R[x])$, ${}_{R[x]}R[x]a$ is projective. Therefore $R[x]a =$

$R[x]h, h^2 = h \in R[x]$ by hypothesis. Let $h = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n$ where $h_i \in R, i = 1, 2, \cdots, n$. Clearly, $Ra = Rh_0, h_0^2 = h_0$. □

Remark 2.23. *Let F be a division ring and let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R is not left $MC2$ ring, so R is not left $NC2$ by Theorem 2.22 (1). Therefore $R[x]$ is not left $NC2$ by Theorem 2.22 (2). But $R[x]$ is left mininjective, so $R[x]$ is left $MC2$. Hence there exists a left $MC2$ ring which is not left $NC2$. Hence we have:*

$\{$*left* $C2$ *rings*$\} \subsetneqq \{$ *left* $NC2$ *rings*$\} \subsetneqq \{$*left* $MC2$ *rings*$\}$

Theorem 2.24. *(1) R is $n-$regular ring if and only if R is left $NC2$ left NPP ring.*

(2) If R is $n-$regular ring, then $N(R) \cap J(R) = 0$.

Proof. (1) By Theorem 2.18, every $n-$regular ring is left $NC2$ left NPP ring. Conversely, let $a \in N(R)$. Since R is left NPP, ${}_RRa$ projective. Since R is left $NC2$ ring, $Ra = Re, e^2 = e \in R$. Thus $a = ae \in aRa$. Hence R is left $n-$regular ring.

(2) If $a \in N(R) \cap J(R)$, then $a = aba, b \in R$. Hence $a(1 - ba) = 0$. Since $a \in J(R)$, $ba \in J(R)$. Hence $1 - ba$ is invertible and so $a = 0$. □

According to [15], a ring R is said to be left weakly continuous if $Z_l(R) = J(R)$, $R/J(R)$ is Von Neumann regular ring and idempotents can be lifted modulo $J(R)$. Every Von Neumann regular ring is left weakly continuous. Since every Von Neumann regular ring is left PP, we have the following corollary.

Corollary 2.25. *The following conditions are equivalent for a ring R.*

(1) R is an Von Neumann regular ring.

(2) R is a left weakly continuous left PP ring.

(3) R is a left weakly continuous left NPP ring.

(4) R is a left weakly continuous left nonsingular ring.

3. $Wnil-$injective Modules

Call a left $R-$module M $Wnil-$injective if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and any left $R-$morphism $f : Ra^n \longrightarrow M$ can be extends to $R \longrightarrow M$, or equivalently, $f = \cdot m$ where $m \in M$. Clearly, every left $YJ-$injective module (c.f. [3], [20] or [4]) and left $nil-$injective modules are all left $Wnil-$injective. The following theorem is a proper generalization of [10, Proposition 1].

Theorem 3.1. *Let R be a ring whose every simple singular left $R-$module is $Wnil-$injective. If R satisfies one of the following conditions, then the following statements hold.*

(1) $Z_r(R) \cap Z_l(R) = 0$.

(2) $Z_l(R) \cap J(R) = 0$.

(3) If R is left $MC2$ ring, then $Z_r(R) = 0$.

(4) R is a left PS ring.

Proof. (1) If $Z_r(R) \cap Z_l(R) \neq 0$, then there exists a $0 \neq b \in Z_r(R) \cap Z_l(R)$ such that $b^2 = 0$. We claim that $RbR + l(b) = R$. Otherwise there exists a maximal essential left ideal M of R containing $RbR + l(b)$. So R/M is a simple singular left $R-$module, and then it is left $Wnil-$injective by hypothesis. Set $f : Rb \longrightarrow R/M$ defined by $f(rb) = r + M$. Then f is well-defined left $R-$homomorphism. Hence $f = \cdot\bar{c}, c \in R$ and so $1 - bc \in M$. Since $bc \in RbR \subseteq M, 1 \in M$, which is a contradiction. Therefore $1 = x + y, x \in RbR, y \in l(b)$, and so $b = xb$. Since $RbR \subseteq Z_r(R)$, $x \in Z_r(R)$. Thus $r(1 - x) = 0$ and so $b = 0$, which is a contradiction. This shows that $Z_r(R) \cap Z_l(R) = 0$.

(2) can be done with an argument similar to that of (1).

(3) Suppose that $Z_r(R) \neq 0$. Then there exists $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. If there exists a maximal left ideal M of R containing $RaR + l(a)$. then M must be an essential left ideal. Otherwise $M = l(e), e^2 = e \in R$. Hence $aRe = 0$. We claim that $eRa = 0$. Otherwise there exists a $c \in R$ such that $eca \neq 0$. Since ${}_RRe \cong {}_RReca$, ${}_RReca$ is projective. Thus $Reca = Rg, g^2 = g \in R$, which implies that $reca = Rg = RgRg = RecaReca = Rec(aRe)ca = 0$. This is a contradiction. Therefore $eRa = 0$ and so $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Hence M is essential and so R/M is $Wnil-$injective by hypothesis. As proved in (1), there exists a $c \in R$ such that $1 - ac \in M$. Since $ac \in RaR \subseteq M$, $1 \in M$, which is also a contradiction. Thus $RaR + l(a) = R$ and so $1 = x + y, x \in RaR, y \in l(a)$. Hence $a = xa$. and so $a = 0$ because $x \in RaR \subseteq Z_r(R)$. This is also a contradiction, which shows that $Z_r(R) = 0$.

(4) Let Rk be minimal left ideal of R. If $(Rk)^2 \neq 0$, then $Rk = Re, e^2 = e \in R$, so ${}_RRk$ is projective. If $(Rk)^2 = 0$, then $l(k)$ is a summand of ${}_RR$. Otherwise $l(k)$ is a maximal essential left ideal. So $R/l(k)$ is a $Wnil-$injective left $R-$module by hypothesis. Therefore the left $R-$homomorphism $f : Rk \longrightarrow R/l(k)$ defined by $f(rk) = r + l(k), r \in R$ can be extended to $R \longrightarrow R/l(k)$. This implies that there exists a $c \in R$ such that $1 - kc \in l(k)$. Since $RkRkR = 0$, $1 - kc$ is invertible.

Hence $l(k) = R$, which is a contradiction. Thus $l(k)$ is a summand of ${}_RR$ and so ${}_RRk$ is projective. Consequently, R is a left PS ring. □

Recall that R is a left $GQ-$injective ring [9] if, for any left ideal I isomorphic to a complement left ideal of R, every left $R-$homomorphism of I into R extends to an endomorphism of ${}_RR$. It is clear that left $GQ-$injective rings generalize left continuous rings. We know that if R is left $GQ-$injective, then $J(R) = Z_l(R)$ and $R/J(R)$ is Von Neumann regular ring. Since every left module over a Von Neumann regular ring is $p-$injective, the following corollary to Theorem 3.1 generalizes [3, Theorem 2] and [10, Corollary 1.2].

Corollary 3.2. *(1) R is a Von Neumann regular ring if and only if R is a left weakly continuous ring whose simple singular left $R-$modules are $Wnil-$injective.*

(2) Let R be a left $GQ-$injective ring whose simple singular left $R-$module is $Wnil-$injective. Then R is a Von Neumann regular ring.

(3) Let R be a ring whose simple singular left $R-$module is $Wnil-$injective. Then $Z_l(R) = 0$ if and only if $Z_l(R) \subseteq J(R)$.

(4) Let R be a ring whose simple singular left $R-$module is $nil-$injective. Then $Z_l(R) = 0$ if and only if $Z_l(R) \subseteq Z_r(R)$.

(5) If R is a left $MC2$ right $GQ-$injective ring such that every simple singular left $R-$module is $Wnil-$injective. Then R is a Von Neumann regular ring

(6) If R is a left $MC2$ right weakly continuous ring such that every simple singular left $R-$module is $Wnil-$injective. Then R is a Von Neumann regular ring

According to [6], a left $R-$module M is called Small injective if every homomorphism from a small left ideal to ${}_RM$ can be extended to an $R-$homomorphism from ${}_RR$ to ${}_RM$.

A left $R-$module M is said to be left weakly principally small injective (or, $WPSI$) if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any left $R-$homomorphism from $Ra^n \longrightarrow M$ can be extended to $R \longrightarrow M$. Evidently, left Small injective modules are left $WPSI$. We do not know whether the converse is true. A ring R is called left $WPSI$ if ${}_RR$ is a left $WPSI$. It is easy to show that R is a left $WPSI$ ring if and only if for every $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and $rl(a^n) = a^nR$. Clearly, every left $YJ-$injective ring is left $WPSI$. The following corollary generalizes [10, Corollary 1.3].

Corollary 3.3. *(1) If R is a left $WPSI$ ring, then the following statements hold:*

(a) $J(R) \subseteq Z_l(R)$.

(b) R is a left mininjective ring.

(c) If $e^2 = e \in R$ is such that that $ReR = R$, then eRe is a left $WPSI$ ring.

(d) If R is an NI, then R is a left $nil-$injective ring.

(2) If R is a left $WPSI$ ring whose every simple singular left $R-$module is $Wnil-$injective, then

(a) $J(R) = 0 = Z_r(R)$.

(b) If R is a right $GQ-$injective, then R is Von Neumann regular ring.

(c) If R is a right weakly continuous, then R is Von Neumann regular ring.

(3) If R is a right $WPSI$ left $MC2$ ring whose every simple singular left $R-$module is $Wnil-$injective, then $J(R) = 0$.

Proof. (1) (a) If there exists a $b \in J(R)$ with $b \notin Z_l(R)$. Then there exists a nonzero left ideal I of R such that $I \cap l(b) = 0$. Let $0 \neq a \in I$. Then $ab \neq 0$. Evidently, $ab \in J(R)$. Hence there exists a positive integer n such that $(ab)^n \neq 0$ and $(ab)^n R = rl((ab)^n)$. Since $l((ab)^{n-1}a) = l((ab)^n), (ab)^{n-1}a \in rl((ab)^{n-1}a) = rl((ab)^n) = (ab)^n R$. Write $(ab)^{n-1}a = (ab)^n c, c \in R$. Then $(ab)^{n-1}a(1 - bc) = 0$ and so $(ab)^{n-1}a = 0$ because $1 - bc$ is invertible. Hence $(ab)^n = (ab)^{n-1}ab = 0$, which is a contradiction. Hence $J(R) \subseteq Z_l(R)$.

(b) Assume that Rk is a minimal left ideal of R. If $(Rk)^2 \neq 0$, then $Rk = Re, e^2 = e \in R$. Set $e = ck, c \in R$ and $g = kc$. Then $k = ke = kck = gk$, $g^2 = kckc = kc = g$ and $kR = gR$. Therefore $kR = gR = rl(g) = rl(k)$, we are done. If $(Rk)^2 = 0$, then $k \in J(R)$. Since R is a left $WPSI$ ring, $rl(k) = kR$. Hence R is a left mininjective ring.

(c) Similar to the proof of Theorem 2.24.

(2) (a) By Theorem 3.1 and (1), $J(R) = Z_l(R) \cap J(R) = 0$ and R is a left mininjective ring, so R is a left $MC2$ by [1]. Hence $Z_r(R) = 0$ by Theorem 3.1.

(b) Since R is a right $GQ-$injective, $R/J(R)$ is Von Neumann regular. Hence R is Von Neumann regular ring because $J(R) = 0$ by (a).

(c) Similar to (b).

(3) Similar to (1), we have $J(R) \subseteq Z_r(R)$. So $J(R) = 0$ because $Z_r(R) = 0$ by Theorem 3.1. □

It is well known that if every simple left $R-$module is injective, then R is semiprime. Since every simple singular left $R-$module is injective, R must not be semiprime.

According to [3], a ring R is idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $e^2 = e, a \in R$. Clearly, every idempotent reflexive ring is left $MC2$.

Proposition 3.4. *(1) If every simple left $R-$module is $Wnil-$injective, then R is semiprime.*

(2) If every simple singular left $R-$module is $Wnil-$injective, then R is semiprime if it satisfies any one of the following conditions.

(a) R is a left $MC2$.

(b)R is an idempotent reflexive.

(c) R is a left $NC2$.

Proof. (1) Assume that $a \in R$ such that $aRa = 0$. Then $RaR \subseteq l(a)$. If $a \neq 0$, then there exists a maximal left ideal M containing $l(a)$. By hypothesis, R/M is $Wnil-$injective. So there exists a $c \in R$ such that $1 - ac \in M$. Hence $1 \in M$, which is a contradiction. So $a = 0$ and then R is a semiprime ring.

(2) (a) Since R is left $MC2$, as proved in Theorem 3.1(3), we know that M as in (1) are essential in ${}_RR$. The rest proof containing (b), (c) are similar to (1). □

Corollary 3.5. *Suppose that every simple singular left $R-$module is $Wnil-$injective. Then the following conditions are equivalent.*

(1) R is a reduced ring.

(2) R is a ZC ring.

(3) R is a ZI ring.

(4) R is an Abelian $2-$prime ring.

(5) R is an idempotent reflexive $2-$prime ring.

(6) R is a left $MC2$ $2-$prime ring.

Proof. (1) ⇒ (2) ⇒ (3) and (4) ⇒ (5) ⇒ (6) are obvious.

(3) ⇒ (1) If R is a ZI ring, then R is an Abelian ring, and so R is a left $MC2$ ring. By Proposition 3.4, R is semiprime. Now, let $a \in R$ with $a^2 = 0$. then $aRa = 0$ because R is a ZI ring. Hence $a = 0$. Therefore R is a reduced ring.

(6) ⇒ (1) By (6) and Proposition 3.4, R is a semiprime ring. So $N(R) = P(R) = 0$ because R is a $2-$prime ring. □

Call an element $k \in R$ left (right, resp) minimal if Rk (kR, resp) is a minimal left (right, resp) ideal of R. Call an element $e \in R$ is called left minimal idempotent if $e^2 = e$ is a left minimal element.

According to [14], if R is a left minsymmetric ring, then $S_l(R) \subseteq S_r(R)$.

Call a ring R reflexive [3] if $aRb = 0$ implies $bRa = 0$ for all $a, b \in R$. Clearly, every semiprime ring is reflexive and every reflexive ring is idempotent reflexive. Hence we have the following corollary.

Corollary 3.6. *Suppose that every simple singular left $R-$module is $Wnil-$injective. Then the following conditions are equivalent.*

(1) R is a semiprime ring.

(2) R is a reflexive ring.

(3) R is an idempotent reflexive ring.

(4) R is a left $MC2$ ring.

(5) $S_l(R) \subseteq S_r(R)$.

(6) R is a left minsymmetric ring.

(7) R is a left mininjective ring.

(8) R is a left universally mininjective ring.

(9) Every left minimal idempotent of R is right minimal.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (5) are obvious. By Proposition 3.4, we have (4) $\Rightarrow$ (1).

(5) $\Rightarrow$ (4) First, we assume that Rk, Re are minimal left ideals of R with ${}_RRk \cong {}_RRe$ where $e^2 = e, k \in R$. It is easy to show that there exists an idempotent $g \in R$ such that $k = gk$ and $l(k) = l(g)$. Hence, by hypothesis, $gR \supseteq mR$ where mR is a minimal right ideal of R, so $l(g) = l(m)$. It suffices to show that $(Rm)^2 \neq 0$. For, if $(Rm)^2 \neq 0$, then $(mR)^2 \neq 0$. So $mR = hR, h^2 = h \in R$. Consequently, $gR = rl(g) = rl(m) = rl(h) = hR = mR$ is a minimal right ideal of R and so $kR = gkR = gR$. Write $g = kc, c \in R$ and $u = ck$. Then $k = gk = kck = ku$, $u^2 = ckck = ck = u$ and $Rk = Ru$, we are done; Assume to the contrary $(Rm)^2 = 0$. Then there exists a right ideal I of R such that $RmR \oplus I$ is essential in R_R. So $S_l(R) \subseteq S_r(R) \subseteq RmR \oplus I$. Since $RmR \oplus I \subseteq l(m)$, $g \in S_l(R) \subseteq l(m) = l(g)$, which is a contradiction. Next, let $a, e^2 = e \in R$ with $aRe = 0$, where e is a left minimal element of R. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Since ${}_RRe \cong {}_RReba$, $Reba = Rh, h^2 = h \in R$ by the proof above. Therefore $Rh = RhRh = RebaReba = 0$, which is a contradiction. This shows that $eRa = 0$ and so R is a left $MC2$ ring.

(4) $\Rightarrow$ (9) Assume that $e \in R$ is a left minimal idempotent. Let $a \in R$ be such that $ea \neq 0$. Since Rea is a minimal left ideal and $l(e) \subseteq l(ea)$, $l(e) = l(ea)$. If $(Rea)^2 = 0$, then $eaR \subseteq l(ea) = l(e)$, so $eaRe = 0$. Since $(Rea)^2 \neq 0$ and so $Rea = Rg$, $g^2 = g \in R$. Therefore $eaR = hR$ for some $h^2 = h \in R$. Since

$l(h) = l(ea) = l(e)$, $eR = rl(e) = rl(ea) = rl(h) = hR = eaR$, which implies that eR is a minimal right ideal of R, e.g. e is a right minimal element.

(9) $\Rightarrow$ (4) Assume that Rk, Re, $e^2 = e, k \in R$ are minimal left ideals of R with ${}_RRk \cong {}_RRe$. Then there exists an idempotent $g \in R$ such that $k = gk$ and $l(k) = l(g)$. Hence, by hypothesis, gR is a minimal right ideal of R. Therefore $kR = gkR = gR$ and so $Rk = Rh$ for some $h^2 = h \in R$. □

Now we give some characteristic properties of reduced rings in terms of the $Wnil-$injectivity.

Theorem 3.7. *The following conditions are equivalent for a ring R.*

(1) R is a reduced ring.

(2) R is an Abelian ring whose every left $R-$module is $Wnil-$injective.

(3) R is an Abelian ring whose every cyclic left $R-$module is $Wnil-$injective.

(4) $N(R)$ forms a right ideal of R and every left $R-$module is $Wnil-$injective.

(5) $N(R)$ forms a right ideal of R and every cyclic left $R-$module is $Wnil-$injective.

(6) $N(R)$ forms a right ideal of R and every simple left $R-$module is $Wnil-$injective.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are clear.

(6) $\Rightarrow$ (1) Assume that $a \in R$ such that $a^2 = 0$. If $a \neq 0$, then let M be a maximal left $R-$submodule of Ra . Then Ra/M is a simple left $R-$module. By (6), Ra/M is a $Wnil-$injective. So the canonical homomorphism $\pi : Ra \longrightarrow Ra/M$ can be expressed as $\pi = \cdot ca + M, c \in R$. Hence $a - aca \in M$. By (6), $ac \in N(R)$ so $1 - ac$ is invertible. Thus $a = (1 - ac)^{-1}(1 - ac)a = (1 - ac)^{-1}(a - aca) \in M$, which is a contradiction. So $a = 0$ and then R is a reduced ring. □

A ring R is called $MELT$ [5] if every maximal essential left ideal of R is an ideal. The following theorem is a generalization of [10, Proposition 9].

Theorem 3.8. *Let R be ring whose every simple singular left $R-$module is $Wnil-$injective. If R satisfies one of the following conditions, then $Z_l(R) = 0$.*

(1) R is an $MELT$ ring.

(2) R is a ZI ring.

(3) $N(R) \subseteq J(R)$.

Proof. Suppose that $Z_l(R) \neq 0$. Then $Z_l(R)$ contains a nonzero element z such that $z^2 = 0$. Therefore $l(z) \neq R$. Let M be a maximal left ideal of R containing $l(z)$. Then M is an essential left ideal of R which implies that R/M is a left $Wnil-$injective. Define a left $R-$homomorphism $f : Rz \longrightarrow R/M$ by $f(rz) = r+M$ for all $r \in R$. Since R/M is $Wnil-$injective and $z^2 = 0$, there exists a $c \in R$ such that $1 - zc \in M$. If R is $MELT$, then M is an ideal of R. Since $z \in l(z) \subseteq M$, $zc \in M$. If R is ZI, then $zRz = 0$ because $z^2 = 0$, so $zc \in l(z) \subseteq M$. If $N(R) \subseteq J(R)$, then $zc \in J(R) \subseteq M$. Hence we always have $1 \in M$, contradicting that $M \neq R$. This proves that $Z_l(R) = 0$. □

Corollary 3.9. *Let R be an $MELT$ ring whose every simple singular left $R-$module is $Wnil-$injective. Then:*

(1) If R is a left $GQ-$injective ring, then R is Von Neumann regular.

(2) If R is a left weakly continuous ring, then R is Von Neumann regular.

A left $R-$module M is said to be $Wjcp-$injective if for each $a \notin Z_l(R)$, there exists a positive integer n such that $a^n \neq 0$ and every left $R-$homomorphism from Ra^n to M can be extended to one of R to M. If ${}_RR$ is $Wjcp-$injective, we call R is a left $Wjcp-$injective ring. Evidently, every left $YJ-$injective ring is $Wjcp-$injective.

It is easy to show that R is left $Wjcp-$injective if and only if for any $0 \neq a \notin Z_l(R)$, there exists a positive integer n such that $a^n \neq 0$ and $rl(a^n) = a^nR$.

The ring in Example 2.5 is a left $Wjcp-$injective which is not left $YJ-$injective.

Theorem 3.10. *(1) Let R be a left $Wjcp-$injective ring. Then:*

(a) $Z_l(R) \subseteq J(R)$.

(b) R is a left $C2$ ring.

(c) If R is also a left $WPSI$ ring, then $Z_l(R) = J(R)$.

(d) If every simple singular left $R-$module is $Wnil-$injective, then $Z_l(R) = 0$. Hence R is a semiprime left $YJ-$injective ring.

(2) R is a left $YJ-$injective ring if and only if R is a left $WPSI$ left $Wjcp-$injective ring.

Proof. (1) (a) Assume that $a \in Z_l(R)$. Then $1 - a \notin Z_l(R)$ because $l(1 - a) = 0$. Therefore $rl((1 - a)^n) = (1 - a)^nR$, so $R = (1 - a)^nR$. This shows that a is a right quasi-regular element of R. Since $Z_l(R)$ is an ideal of R, $a \in J(R)$. Hence $Z_l(R) \subseteq J(R)$.

(b) Let $e^2 = e, a \in R$ be such that ${}_RRa \cong_R Re$. Then there exists a $g^2 = g \in R$ such that $a = ga$ and $l(a) = l(g)$. Therefore $a \notin Z_l(R)$ and so $aR = rl(a) = rl(g) = gR$. Then there exists $h^2 = h \in R$ such that $Ra = Rh$. This shows that R is a left $C2$ ring.

(c) Since R is left $WPSI$ ring, $J(R) \subseteq Z_l(R)$ by Corollary 3.3. By (a), $Z_l(R) = J(R)$.

(d) By Theorem 3.1, $Z_l(R) \cap J(R) = 0$. By (a), $Z_l(R) = 0$. By (b) and Corollary 3.6, R is semiprime.

(2) Follows from (1). □

[10, Proposition 3] shows that if R is a reduced ring whose every simple left module is either $YJ-$injective or flat, then R is a biregular ring. We can generalize the result as follows.

Theorem 3.11. *Let R be a reduced ring whose every simple singular left module is either $Wjcp-$injective or flat. Then R is a biregular ring.*

Proof. For any $0 \neq a \in R$, $l(RaR) = r(RaR) = r(a) = l(a)$. If $RaR \oplus l(a) \neq R$, then there exists a maximal left ideal M of R containing $RaR \oplus l(a)$. If M is not essential in ${}_RR$, then $M = l(e), e^2 = e \in R$. Therefore $ae = 0$. Since R is Abelian, $ea = 0$. Hence $e \in l(a) \subseteq l(e)$, which is a contradiction. So M is essential in ${}_RR$. By hypothesis, R/M is either $Wjcp-$injective or flat. First we assume that R/M is $Wjcp-$injective. Since R is reduced, $Z_l(R) = 0$. Hence there exists a positive integer n such that $a^n \neq 0$ and any left $R-$homomorphism $Ra^n \longrightarrow R/M$ can be extended to $R \longrightarrow R/M$. Set $f : Ra^n \longrightarrow R/M$ defined by $f(ra^n) = r + M, r \in R$. Then f is a well defined left $R-$homomorphism. Hence there exists a $g :_R R \longrightarrow_R R/M$ such that $1 + M = f(a^n) = g(a^n) = a^n g(1) = a^n c + M$ where $g(1) = c + M$, so $1 - a^n c \in M$. Since $a^n c \in RaR \subseteq M$, $1 \in M$, which is a contradiction. So we assume that R/M is flat. Since $a \in M$, $a = ac$ for some $c \in M$. Now $1 - c \in r(a) = l(a) \subseteq M$ which implies that $1 \in M$, again a contradiction. Hence $RaR \oplus l(a) = R$ and so $RaR = Re, e^2 = e \in R$. Since R is an Abelian ring, R is a biregular ring. □

In [6, Proposition 2.3] , semiprimitive rings are characterized in terms of Small injective modules. In the next theorem, we obtain a similar result.

Theorem 3.12. *The following conditions are equivalent for a ring R.*

(1) $J(R) = 0$.

(2) Every left $R-$module is $WPSI$.

(3) Every cyclic left $R-$module is $WPSI$.

(4) Every simple left $R-$module is Small injective.

Proof. (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4) are clear.

(4) ⇒ (1) If $J(R) \neq 0$, then there exists $0 \neq a \in J(R)$. By (4), Ra/M is a left Small injective $R-$module where M is a maximal $R-$submodule of Ra. Hence any left $R-$homomorphism $Ra \longrightarrow Ra/M$ extends to $R \longrightarrow Ra/M$. Therefore the left

$R-$homomorphism $f : Ra \hookrightarrow Ra/M$ defined by $f(ra) = ra+M$ can be extended to $R \longrightarrow Ra/M$. So there exists a $c \in R$ such that $a-aca \in M$. Hence $(1-ac)a \in M$ and so $a \in M$ because $1-ac$ is invertible, which is a contradiction. Therefore $J(R) = 0$.

(3) $\Rightarrow$ (1) If $J(R) \neq 0$, then there exists $0 \neq a \in J(R)$. By (3), Ra is a left $WPSI$ $R-$module. Hence there exists a positive integer n such that $a^n \neq 0$ and any left $R-$homomorphism $Ra^n \longrightarrow Ra$ extends to $R \longrightarrow Ra$. Therefore the left $R-$homomorphism $f : Ra^n \longrightarrow Ra$ defined by $f(ra^n) = ra^n, r \in R$ can be extended to $R \longrightarrow Ra$. So there exists $c \in R$ such that $a^n = a^nca = 0$. Hence $a^n(1-ca) = 0$ and so $a^n = 0$ because $1-ca$ is invertible, which is a contradiction. Therefore $J(R) = 0$. □

Theorem 3.13. *The following conditions are equivalent for a ring R.*

(1) R is a left universally mininjective.

(2) Every minimal left ideal of R is left $WPSI$.

(3) Every small minimal left ideal of R is left $WPSI$.

Proof. (1) $\Rightarrow$ (2) Assume that Rk is a minimal left ideal of R and $0 \neq a \in J(R)$. For any positive integer n with $a^n \neq 0$, if $f : Ra^n \longrightarrow Rk$ is any left $R-$homomorphism, we claim that $f = 0$. Otherwise f is an epic. Since R is a left universally mininjective ring, $Rk = Re, e^2 = e \in R$ is a projective left $R-$module. Therefore $Ra^n = kerf \oplus I$, where I is a minimal left ideal of R which is isomorphic to Rk as a left $R-$module. Therefore $I = Rg, g^2 = g \in R$ because R is left universally mininjective. But $I \subseteq Ra^n \subseteq J(R)$ which is a contradiction. Hence $f = 0$. Certainly, f can be extended to $R \longrightarrow Rk$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) Let Rk be a minimal left ideal of R. If $(Rk)^2 \neq 0$, we are done; If $(Rk)^2 = 0$, then $Rk \subseteq J(R)$. Hence Rk is left $WPSI$ module. Thus the identity map $I : Rk \longrightarrow Rk$ can be extended to $R \longrightarrow Rk$, which implies that there exists a $c \in R$ such that $k = kck \in RkRk$. Therefore $k = 0$ which is a contradiction. Hence R is a left mininjective ring. □

The next theorem can be proved with an argument similar to [10, Theorem 4].

Theorem 3.14. *The following conditions are equivalent for a ring R.*

(1) R is a division ring.

(2) R is a prime left $Wjcp-$injective ring containing a non-zero reduced right ideal which is a right annihilator.

(3) R is a prime left $Wjcp-$injective ring containing a non-zero reduced right ideal which is a left annihilator.

Theorem 3.15. *R is a Von Neumann regular ring if and only if R is a left PP left $Wjcp-$injective ring.*

Proof. One direction is obvious. Suppose that R is a left PP left $Wjcp-$injective ring. Let $0 \neq a \in R$. Then $a \notin Z_l(R)$ because $Z_l(R) = 0$. Then there exists $n > 0$ such that $a^n \neq 0$ and $rl(a^n) = a^nR$ because R is left $Wjcp-$injective. Since R is a left PP ring, $l(a^n) = l(e), e^2 = e \in R$. Thus $eR = rl(e) = rl(a^n) = a^nR$. This implies that a^n is a regular element of R. If $a^2 = 0$, the argument above shows that a is a regular element. so by [2, Theorem 2.2], R is a Von Neumann regular ring. □

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