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c-INJECTIVE ENVELOPE OF MODULES OVER A DEDEKIND DOMAIN

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Abstract. In this paper we prove that every module over a Dedekind domain has a c-injective envelope.

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1. Introduction

Throughout the paper module will mean a unital left R-module where R is an associative ring with identity, group will mean an abelian group, i.e. a $\mathbb{Z}$-module, where $\mathbb{Z}$ is the ring of integers. Given a submodule K of G, a submodule H of G is said to be K-high (or a complement of K) in G if H is maximal in G with respect to the property $H \cap K = 0$. Zorn's Lemma guarantees the existence of a K-high submodule of G for every $K \leq G$. For $R = \mathbb{Z}$ it is known (see Corollary of Proposition 8 in [9], see also [3] and [6]) that a subgroup H of a group G is K-high for some $K \leq G$ if and only if it is a neat subgroup of G, that is $H \cap pG = pH$ for every prime integer p. We give a direct proof of this important fact using the following lemma (Lemma 9.8 in [2]).

Lemma 1.1. *If B is a subgroup of A, and C is a B-high subgroup of A, then $a \in A$, $pa \in C$, (p a prime) implies $a \in B \oplus C \leq A$.*

Proposition 1.2. *H is a neat subgroup of G if and only if H is a K-high subgroup of G for some $K \leq G$.*

Proof. ($\Rightarrow$) Let H be a neat subgroup of G. We will prove that H is a K-high subgroup of G for some subgroup K of G. Applying Zorn's Lemma to the set $\Gamma = \{T \leq G : T \cap H = 0\}$, we find an H-high subgroup K of G. Now taking the set $\Gamma' = \{S \leq G : S \cap K = 0, H \leq S\}$ again by Zorn's Lemma we obtain a K-high subgroup M of G with $H \leq M$. We will show that $H = M$. Suppose on the contrary $M \neq H$. Then there exists $m \in M/H$. If $\langle m\rangle \cap H = 0$ then $(K + \langle m\rangle) \cap H = 0$. To see this let $h = k + tm$ for some $h \in H, k \in K, t \in \mathbb{Z}$. Then $k = h - tm \in K \cap M = 0$, i.e. $k = 0$, therefore $h = tm \in \langle m\rangle \cap H = 0$. So $h = 0$. Therefore $(K + \langle m\rangle) \cap H = 0$, which contradicts with maximality of K. Now if $\langle m\rangle \cap H \neq 0$, then there exists $h = sm \neq 0$ where $h \in H, s \in \mathbb{Z}$. $s = p_1p_2p_3...p_n$ for some primes $p_1p_2p_3, ..., p_n(s \neq 1$ since $m \notin H)$. Since $m \notin H$, but $(p_1p_2p_3...p_n)(m) \in H$, there exists $x \in M$ such that $x \notin H$ but $px \in H$ for some prime p. Then $px \in H \cap pG = pH$ i.e. $px = ph_1$ for some $h_1 \in H$ or $p(x - h_1) = 0$. Put $a = x - h_1 \in M\diagdown H$, so the order of a is p. Now $\langle a\rangle \cap H = 0$ (if $0 \neq ta \in H$ then $(t, p) = 1$ i.e. $tu + pv = 1$ for some $u, v \in \mathbb{Z}$, and $a = uta + vpa \in H$). Therefore $(K + \langle a\rangle) \cap H = 0$. Thus $M = H$.

($\Leftarrow$) Conversely, we assume that H is a K-high subgroup of G for some $K \leq G$ and prove that H is neat in G i.e. $pH = H \cap pG$ for every prime p. Now $pH \subseteq H \cap pG$ is always true. To prove the reverse inequality let $h = pa \in H \cap pG$ where $h \in H$ and $a \in G$. By Lemma 1.1, $a \in H \oplus K$, therefore $a = h' + k$ for some $h' \in H$ and $k \in K$. Hence $h = pa = ph' + pk$. Now $pk = h - ph' \in K \cap H = 0$, therefore $pk = 0$ and $h = ph' \in pH$. □

We give a proof of the following proposition from [9] .

Proposition 1.3. *Let L be a submodule of M. L is K-high for some K in M if and only if for every essential submodule H of M such that L is a submodule of H, H/L is essential in M/L.*

Proof. ($\Rightarrow$) Let H be an essential submodule of M with L a submodule in H. To show that H/L is essential in M/L, let $H/L \cap F/L = 0$, where F is a submodule of M containing L. This means that $H \cap F = L$, and we should show that $F = L$. If L is K-high in M, then $L \cap K = (H \cap F) \cap K = H \cap (F \cap K) = 0$ and hence $F \cap K = 0$. Since L is maximal, it follows that $F = L$. This means $F/L = 0$ and H/L is essential in M/L.

($\Leftarrow$) Conversely, to prove that L is maximal with respect to property $L \cap K = 0$, let $L \leq H$ and $H \cap K = 0$ for some $H \leq M$. Now $L + K$ is an essential submodule of M such that L is a submodule of $L + K$, so $L + K/L$ is essential in M/L by hypothesis. By Modular Law $(L + K) \cap H = L + (K \cap H) = L$, therefore $(L + K/L) \cap (H/L) = 0$. Since $L + K/L$ is essential in M/L therefore $H/L = 0$, i.e. $H = L$, so L is K-high. □

There are two generalizations of neat subgroups for modules. One of them, a neat submodule, is given by Stenström in [8] : A is a *neat submodule* of B if every simple object S is projective with respect to the canonical epimorphism $\sigma : B \longrightarrow B/A$. Another generalization is a *complement* (or a *closed*, or a *high*) submodule, that is a submodule H of a module M that is a complement of K (or K-high) for some submodule K of M. A module I is *c-injective* if for every closed submodule H of a module M every homomorphism from H into I can be extended to M (see [7]). We will study the second generalization of a neat subgroup and prove that over a Dedekind domain every module has a c-injective envelope.

2. c-Injective Envelopes.

It is well-known that every abelian group has a neat injective envelope. In [1] and [5] we have given the description of the neat-injective envelope of a group A in terms of its basic subgroups. We can easily generalize the notion of neat-injective envelope for a module over any ring R.

Definition 2.1. A monomorphism $\alpha : L \longrightarrow M$ is said to be *c-monomorphism* if $Im\alpha$ is a closed submodule of M. A module Q is called *c-injective* if for every c-monomorphism $\alpha : L \longrightarrow M$ and homorphism $\beta : L \longrightarrow Q$ there is a homomorphism $\gamma : M \longrightarrow Q$ such that $\gamma \circ \alpha = \beta$. A c-monomorphism $\alpha : L \longrightarrow M$ is called *c-essential* if every $\beta : M \longrightarrow N$, such that $\beta \circ \alpha$ is a c-monomorphism, is a monomorphism. A c-essential monomorphism $\alpha : L \longrightarrow M$ is a maximal c-essential monomorphism if every monomorphim $\beta : M \longrightarrow N$, with $\beta \circ \alpha$ c-essential, is an isomorphism. A c-essential monomorphism $\alpha : L \longrightarrow M$ with M being c-injective is called a *c-injective envelope*.

Proposition 2.2. *If $\alpha : L \longrightarrow M$ is a c-essential monomorphism and $\beta : L \longrightarrow Q$ is a c-monomorphism with Q c-injective, then there exists a monomorphism $\phi : M \longrightarrow Q$ such that $\phi \circ \alpha = \beta$.*

Proof. Since Q is c-injective, there is a homomorphism $\phi : M \longrightarrow Q$ such that $\phi \circ \alpha = \beta$. Since $\phi \circ \alpha = \beta$ is a c-monomorphism, ϕ is a monomorphism. □

Proposition 2.3. *If M is c-injective, then it is a maximal c-essential extension of itself.*

Proof. Clearly $1_M : M \longrightarrow M$ is a c-essential monomorphism. To prove that 1_M is maximal, let $\beta : M \longrightarrow N$ be a monomorphism with $\beta \circ 1_M = \beta$ being c-essential. Since M is c-injective, β is splitting, i.e. $\alpha \circ \beta = 1_M$ for some $\alpha : N \longrightarrow M$. Then α is an epimorphism. Since β is c-essential and $\alpha \circ \beta = 1_M$ is a c-monomorphism, α is a monomorphism. So α is an isomorphism and $\beta = \alpha^{-1}$ is also an isomorphism. Thus 1_M is maximal. □

For the rest of the section, we will assume that R is a *Dedekind domain.* In this case $\alpha : L \longrightarrow M$ is a c-monomorphism if and only if $\alpha \otimes 1_s : L \otimes S \longrightarrow M \otimes S$ is a monomorphism for every simple module S (see Theorem 5.2.2 in [6]). Since tensor product commutes with $\varinjlim$, if $\alpha_i : L_i \longrightarrow M_i$ is a direct system of c-monomorphisms (i.e. corresponding diagrams are commutative), then $\alpha = \varinjlim \alpha_i : \varinjlim L_i \longrightarrow \varinjlim M_i$ is a c-monomorphism that is a direct limit of c-monomorphisms is a c-monomorphism.

Theorem 2.4. *For every module M there is a maximal c-essential extension $\alpha : M \longrightarrow E$.*

Proof. Let Γ be the set of all c-essential extensions of M, mean to say

$$\Gamma = \{\alpha_i : M \longrightarrow E_i \mid \alpha_i \textit{ is a } c-\textit{essential monomorphism}\}.$$

Define order $\leq$ in Γ by $\alpha_i \leq \alpha_j$ if there is $\pi_i^j : E_i \longrightarrow E_j$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha_i} & E_i \\ \alpha_j & \searrow & \downarrow \pi_i^j \\ & & E_j \end{array}$$

is commutative. In this case π_i^j is a monomorphism since α_i is c-essential and α_j is a c-monomorphism. Clearly $\leq$ is a partially order "up to isomorphism", i.e. if $\alpha_i \leq \alpha_j$ and $\alpha_j \leq \alpha_i$ then $E_i \cong E_j$. Now if Λ is any chain in Γ, then $\left\{E_i, \pi_i^j, \alpha_i \in \Lambda\right\}$ is a direct system and since all α_i 's are monomorphisms, we have a monomorphism $\alpha' : M \longrightarrow \varinjlim_{\Lambda} E_i$. Without loss of generality we can assume that M and all modules E_i are contained in $E' = \varinjlim_{\Lambda} E_i$ and all monomorphisms α_i, π_i^j are inclusion maps. Now if M is contained in some essential submodule L of E', then $L \cap E_i$ is essential in E_i for every $\alpha_i \in \Lambda$. Since α_i is c-essential, $(L \cap E_i)/M$ is an essential submodule of E_i/M. Then it can be easily verified that $L/M = \bigcup_i (L \cap E_i)/M$ is essential in E/M. So α' is a c-essential monomorphism. Clearly α' is an upper bound for Λ. By Zorns lemma there is a maximal element $\alpha : M \longrightarrow E$ in Γ which clearly is a maximal c-essential extension of M. $\square$

Lemma 2.5. *If $\alpha : M \longrightarrow N$ is a c-monomorphism, then there is an epimorphism $\beta : N \longrightarrow K$ such that $\beta \circ \alpha : M \longrightarrow K$ is a c-essential monomorphism.*

Proof. Let $F = \{\beta_i : N \longrightarrow K_i \mid \beta_i \text{ is an epimorphism; } \beta_i \circ \alpha \text{ is a } c\text{-monomorphism}\}$ and let $\Gamma = \{i \mid \beta_i \in F\}$. Define $\leq$ on Γ as follows: $i \leq j$ if there is an epimorphism $\pi_i^j : K_i \longrightarrow K_j$ such that $\pi_i^j \circ \beta_i = \beta_j$.

Then $\leq$ is a partial order on Γ "up to isomorphism", i.e. if $i \leq j$ and $j \leq i$ then $K_i \cong K_j$. Let Λ be any chain in Γ. Then $\left\{K_i, \pi_i^j, i \in \Lambda\right\}$ is a direct system. Put $K' = \varinjlim_{\Lambda} K_i$ and define $\beta' : N \longrightarrow K'$ by $\beta'(n) = \overline{\beta_i(n)\quad} = \pi_i \circ \beta_i(n)$. Clearly β' is a well-defined homomorphism. Since all homomorphisms β_i, π_i^j are epimorphisms, β' is also an epimorphism and for each $i \in \Lambda$ the diagram

$$\begin{array}{ccc} N & \xrightarrow{\beta_i} & K_i \\ & \beta' \searrow & \downarrow \pi_i \\ & & K \end{array}$$

is commutative. Since the direct limit of c-monomorphisms is a c-monomorphism, $\beta' = \varinjlim_{\Lambda} \pi_i \beta_i$ is a c-monomorphism. Let $\beta' = \beta_{i_0}$ for $i_o \in \Gamma$. Clearly i_o is an upper bound for Λ. By Zorn's Lemma there is a maximal element in Γ, i.e. there is an epimophism $\beta : N \longrightarrow K$ such that $\beta \circ \alpha : M \longrightarrow K$ is a c-monomorphism and every epimorphism $\gamma : K \longrightarrow T$, for which $\gamma \circ \beta \circ \alpha : M \longrightarrow T$ is a c-monomorphism, is an isomorphism. Then for every homomorphism $\delta : K \longrightarrow S$ such that $\delta \circ \beta \circ \alpha : M \longrightarrow S$ is a c-monomorphism, the homomorphism $\gamma : K \longrightarrow \delta(k)$ defined by $\gamma(k) = \delta(k)$, is an epimorphism, and since $\delta \circ \beta \circ \alpha = \theta \circ \gamma \circ \beta \circ \alpha$, where $\theta : \delta(k) \longrightarrow S$ is an inclusion map, is a c-monomorphism, $\gamma \circ \beta \circ \alpha$ is also a c-monomorphism. Therefore γ is an isomorphism and so σ is a monomorphism. It means, that $\beta \circ \alpha$ is a c-essential monomorphism. □

Theorem 2.6. *If $\alpha : M \longrightarrow E$ is a maximal c-essential extension, then E is a c-injective module.*

Proof. Let $\beta : E \longrightarrow A$ be a c-monomorphism. Then $\beta \circ \alpha : M \longrightarrow A$ is a c-monomorphism and by Lemma 2.5 there is an epimorphism $\gamma : A \longrightarrow B$ such that $\gamma \circ \beta \circ \alpha : M \longrightarrow B$ is a c-essential monomorphism. Let $\delta = \gamma \circ \beta : E \longrightarrow B$. Then $\delta \circ \alpha = \gamma \circ \beta \circ \alpha$ is a c-essential monomorphism, therefore γ must be a monomorphism. Since α is maximal and $\delta \circ \alpha$ is c-essential, $\delta = \gamma \circ \beta$ is an isomorphism. Then β is a splitting monomorphism. So E is c-injective. □

Corollary 2.7. *Every module has a c-injective envelope which is unique up to isomorphism.*

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