International Electronic Journal of Algebra
Volume 2 (2007) 28-53

MONOID ALGEBRAS OVER NON-COMMUTATIVE RINGS

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Received: 6 April 2007; Revised: 4 May 2007
Communicated by Stephen J. Pride

Abstract. We define on an arbitrary ring A a family of mappings $(\sigma_{x,y})$ subscripted with elements of a multiplicative monoid G. The assigned properties allow to call these mappings as derivations of the ring A. Beside the general situation it is given their description for the case of a partially ordered monoid. A monoid algebra of G over A is constructed explicitly, and the universality property of it is shown. The notion of a monoid algebra in our context extends those of a group ring, a skew polynomial ring, Weyl algebra and other related ones. The connection with crossed products is also shown.

Mathematics Subject Classification (2000): Primary 16S36; Secondary 13N15, 16S10.
Keywords: Derivations, monoid algebras, free algebras, skew polynomial rings.

1. Introduction

In his paper [15] Smits proposes an extension of the theory of skew polynomial rings elaborated and studied for the first by Ore [10]. In [15] skew polynomial rings are considered with a commutation rule defined by

$$x \cdot a = a_1 x + ... + a_r x^r,$$

where K is a field (in general, non-commutative), a_i $(i = 1, ..., r)$ are elements of K depending on a. The corresponding mappings $\delta_i : a \longrightarrow a_i$ $(i = 1, ..., r)$ must be endomorphisms of the additive group of the field K, and, due to their properties, they can be called as higher α-derivations. The special case of a commutation rule of the form

$$x \cdot a = \sigma(a)x + \delta(a) \ (a \in K)$$

leads to an Ore polynomial ring, σ being a field-endomorphism of K and δ a $(1, \sigma)$-derivation of K.

In the present paper we consider the following construction. First, we introduce axiomatically a family of mappings $\sigma = (\sigma_{x,y})$ defined on a ring A and subscripted with elements of a multiplicative monoid G. These mappings are defined in such a way that they can be called derivations of A. Thus the operations of differentiation defined traditionally on a ring [8], [4], [5] are particular examples in our case. We connect the structure of differentiation defined by means of the family σ with a monoid algebra $A\langle G\rangle$. The elements of $A\langle G\rangle$ are mappings α from G into A such that $\alpha(x) = 0$ for almost all $x \in G$. We make $A\langle G\rangle$ into an A - module by defining the (left) module operations in the natural way. But the law of multiplication is defined specifically, by involving the mappings of the family σ. Namely, we write the elements α of $A\langle G\rangle$ as sums $\alpha = \sum_{x\in G} a_x \cdot x$, where $a_x \cdot x$ denotes the function from G into A whose value is a_x at x and 0 at y different of x. For two elements $\alpha = \sum_{x\in G} a_x \cdot x$ and $\beta = \sum_{x\in G} b_x \cdot x$ we define the law of multiplication by the following formulas

$$\alpha\beta = \sum_{x,y\in G} (a_x \cdot x)(b_y \cdot y),$$

and

$$(a \cdot x)(b \cdot y) = \sum_{z\in G} a\sigma_{x,z}(b) \cdot zy \ (a, b \in A; x, y \in G).$$

In respect with this law of composition $A\langle G\rangle$ becomes to be a ring. This ring $A\langle G\rangle$ is also called a G-algebra over A (or simply a monoid algebra over A). It turns out that $A\langle G\rangle$ represents a free G-algebra over A. In order to prove this fact we construct a suitable category $\mathcal{C}$ in which the ring $A\langle G\rangle$ together with the canonical maps $\varphi_0 : G \longrightarrow A\langle G\rangle$, $\varphi_0(x) = 1 \cdot x$ $(x \in G)$ and $f_0 : A \longrightarrow A\langle G\rangle$, $f_0(a) = a \cdot e$ $(a \in A)$ is a universal object (see Section 5).

We note that free algebras over commutative rings (see, for instance, [8, Ch. V, p. 106]), group algebras (when G is a group) [2] (see also [11], [6] and [17]), Weyl algebras are concrete realizations of monoid algebras $A\langle G\rangle$ mentioned above. Certain special cases of crossed products (as, for example, twisted semigroup rings or skew group rings) [12] (see also [7] and [1]) can be considered as concrete situations of our approach as well. For instance, let σ be a monoid-homomorphism of G into $End(A)$, that is, $\sigma(xy) = \sigma(x) \circ \sigma(y)$ $(x, y \in G)$, and $\sigma(e) = 1$ (e designates the unit element of G). Thus for each $x \in G$ we have an endomorphism $\sigma(x)$ of A, and we can define $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $x \neq y$. Then, the law of multiplication in $A\langle G\rangle$ is given as follows

$$\left(\sum_{x\in G} a_x \cdot x\right)\left(\sum_{x\in G} b_x \cdot x\right) = \sum_{x\in G}\sum_{y\in G} a_x\sigma_{x,x}(b_y) \cdot xy.$$

Namely in this particular case the monoid algebra $A\langle G\rangle$ represents a crossed product [3], [12] of the multiplicative monoid G over the ring A with respect to the factors $\rho_{x,y} = 1$ $(x, y \in G)$. If G is a group, and $\sigma : G \longrightarrow End(A)$ is such that

$\sigma(x) = 1_A$ for all $x \in G$, then we evidently obtain an ordinary group ring [2] (the commutative case see also [6]).

The paper is organized as follows: the derivation mappings are defined in Section 2. In Section 3 is considered the case of a partially ordered monoid. In Sections 4 and 5 we define and study the monoid algebra $A\langle G\rangle$. In Section 4 we give some examples by which we show that the Weyl algebras (see Example 2.1) and group algebras (see Example 2.3) are special cases of our monoid algebras. The connection with crossed products is shown as well (see Example 2.2). Applications to skew polynomial rings are considered in Section 6. Concomitantly, we extend substantially results of T. H. M. Smits [14], [15], [16].

Algebraic structure of monoid algebras and some results related to [13] will be given in a subsequent publication.

2. Derivation mappings

Let A be a ring (in general non-commutative) and G a multiplicative monoid. Throughout the paper we consider $1 \neq 0$ (where 0 is the null element of A, and 1 is the unit element for multiplication), the unit element of G is denoted by e. In the following, it will be considered a family of mappings of A into itself which, due to their assigned properties, could be regarded as derivations of the ring A. This family is introduced by the following assumption.

(A) For each $x \in G$ there exists a unique family $\sigma_x = (\sigma_{x,y})_{y\in G}$ of mappings $\sigma_{x,y} : A \longrightarrow A$ which $\sigma_{x,y} = 0$ for almost all $y \in G$ (here and thereafter, almost all will mean all but a finite number, that is, $\sigma_{x,y} \neq 0$ only for a finite set of $y \in G$) and for which the following properties are fulfilled:

(i) $\sigma_{x,y}(a + b) = \sigma_{x,y}(a) + \sigma_{x,y}(b)$ $(a, b \in A; x, y \in G)$;

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$ $(a, b \in A; x, y \in G)$;

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$ $(x, y, z \in G)$;

(iv_1) $\sigma_{x,y}(1) = 0$ $(x \neq y; x, y \in G)$; (iv_2) $\sigma_{x,x}(1) = 1$ $(x \in G)$;

(iv_3) $\sigma_{e,x}(a) = 0$ $(x \neq e; x \in G)$; (iv_4) $\sigma_{e,e}(a) = a$ $(a \in A)$.

In (ii) the elements are multiplied as in the ring A, but in (iii) the symbol $\circ$ means the composition of maps. We have seen that the condition (iii) implies (for $y = x$) that

$$\sigma_{x^2,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{x,v} \ (x, z \in G). \tag{2.1}$$

Also, on the basis of the property (i), for three maps $\sigma', \sigma'', \sigma'''$ from the family $(\sigma_{x,y})_{x,y\in G}$ there holds the distributive law with respect to operation of the sum and the composition of maps, namely $\sigma^{'} \circ (\sigma^{''} + \sigma^{'''}) = \sigma^{'} \circ \sigma^{''} + \sigma^{'} \circ \sigma^{'''}$. Taking into account this fact from the condition (iii) and (2.1), we obtain

$$\sigma_{x^3,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{x^2,v} = \sum_{uv=z} \sigma_{x,u} \circ \Big(\sum_{st=v} \sigma_{x,s} \circ \sigma_{x,t} \Big)$$

$$= \sum_{ust=z} \sigma_{x,u} \circ \sigma_{x,s} \circ \sigma_{x,t},$$

and, in general, by induction on n

$$\sigma_{x^n,z} = \sum_{u_1 \cdot \ldots \cdot u_n = z} \sigma_{x,u_1} \circ \ldots \circ \sigma_{x,u_n} \ (x, z \in G;\ n = 1, 2, \ldots). \tag{2.2}$$

Next, if we form the matrix

$$\sigma(a) = [\sigma_{x,y}(a)]_{x,y \in G} \ (a \in A), \tag{2.3}$$

we see that the formula from (ii) of (A) can be written formally as follows

$$\sigma(ab) = \sigma(a)\sigma(b) \ (a, b \in A).$$

In addition, we note also (via of the property (i) of (A)) that

$$\sigma(a + b) = \sigma(a) + \sigma(b) \ (a, b \in A).$$

Thus, by means of the mapping σ we have a representation of the ring A. We formulate this important fact in the following

Proposition 2.1. *The mapping $\sigma : a \longrightarrow \sigma(a)$ determines a matrix representation of A (in general of infinite degree).*

Examples: 1. Let A be a ring and let G be a multiplicative monoid. Let us consider a monoid-homomorphism σ of G into $End(A)$, that is $\sigma(xy) = \sigma(x) \circ \sigma(y)$ $(x, y \in G)$ and $\sigma(e) = 1_A$. Thus for each $x \in G$ we have an endomorphism $\sigma(x)$ of A, and we can define $\sigma_{x,y} : A \longrightarrow A$ such that $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $y \neq x$. The properties (i)-(iv_4) of (A) are verified at once. We only note that the property (iii) becomes to be as follows

$$\sigma_{xy,xy} = \sigma_{x,x} \circ \sigma_{y,y} \ (x, y \in G)$$

that is true via the fact that σ is a monoid-homomorphism.

2. Let A be a ring, and let α be an endomorphism of A and δ be an α-differentiation of A, i.e.

$$\delta(a + b) = \delta(a) + \delta(b),\ \delta(ab) = \delta(a)b + \alpha(a)\delta(b)$$

for every $a, b \in A$. Denote by G the monoid of elements x_n $(n = 0, 1, ...)$ endowed with the law of composition defined by

$$x_n x_m = x_{n+m} \ (n, m = 0, 1, ...;\ x_0 := e).$$

We write σ_{nm} instead of σ_{x_n, x_m} by defining $\sigma_{nm} : A \longrightarrow A$ as the following mappings $\sigma_{00} = 1_A,\ \sigma_{10} = \delta,\ \sigma_{11} = \alpha,\ \sigma_{nm} = 0$ for $m > n$ and

$$\sigma_{nm} = \sum_{j_1+...+j_n=m} \sigma_{1j_1} \circ ... \circ \sigma_{1j_n} \quad (m = 0, 1, ..., n;\ n = 1, 2, ...),$$

where $j_k = 0, 1$ $(k = 1, ..., n)$. The family $\sigma = (\sigma_{nm})$ satisfies the axioms $(i) - (iv_4)$ of (A). In particular, if $\alpha = 1_A$, then the derivation mappings σ_{nm} for $n > m$ are given by

$$\sigma_{nm} = \binom{n}{m} \delta^{n-m} \quad (m = 0, 1, ..., n-1; n = 1, 2, ...),$$

and also if δ is a trivial derivation of A, i.e. $\delta(a) = 0$ for all $a \in A$, then $\sigma_{nm} = 0$ for $m < n$ too, and

$$\sigma_{nn} = \alpha^n \ (n = 0, 1, ...).$$

In Section 4 it will be presented other examples of derivations of the ring A. Note that the derivation mappings defined as in Example 1 determine a crossed product of A by G (cf. Example 2.2 of Section 4).

3. The case of a partially ordered monoid

In this section we continue our discussion by supposing that the monoid G is partially ordered, i.e. we assume

(O) G is a partially ordered monoid.

This means that G is partially ordered in such a manner that the partial ordering "$\leq$" is compatible with the algebraic structure of G, i.e.

(O_1) $x \leq x$ holds for all $x \in G$;

(O_2) If $x \leq y$ and $y \leq z$, then $x \leq z$ $(x, y, z \in G)$;

(O_3) $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ for every $z \in G$ $(x, y \in G)$;

In addition, we postulate

(O_4) $e \leq x$ for every $x \in G$.

If for any elements x and y of G we have $x \leq y$ and $x \neq y$, then we write $x < y$, and we agree to use $x \geq y$ and $x > y$ as alternative for $y \leq x$ and $y < x$. For two comparable elements $x, y \in G$ we denote by $[x, y]$ (in the case of $x \leq y$) the set of all elements $z \in G$ such that $x \leq z \leq y$, i.e.

$$[x, y] := \{z \in G \mid x \leq z \leq y\}.$$

The set $[x, y]$ is called the segment that connects the element x with y.

Let A be a ring for which the following assumption is fulfilled.

$(A^{'})$ The mappings $\sigma_{x,y} : A \longrightarrow A$ $(x, y \in G)$ defining by the assumption (A) are such that $\sigma_{x,y} = 0$ for either incomparable $x, y \in G$ or $x < y$.

The condition (ii) of (A) can be written then

$$(ii^{'}) \qquad \sigma_{x,y}(ab) = \sum_{z\in[y,x]} \sigma_{x,z}(a)\sigma_{z,y}(b) \ (a, b \in A;\ y \le x,\ x, y \in G).$$

It turns out that in the imposed conditions the mapping $\sigma_{x,x}$ for each $x \in G$ is a ring-homomorphism, i.e.

$$\sigma_{x,x}(a + b) = \sigma_{x,x}(a) + \sigma_{x,x}(b) \ (a, b \in A), \tag{3.1}$$

$$\sigma_{x,x}(ab) = \sigma_{x,x}(a)\sigma_{x,x}(b) \ (a, b \in A), \tag{3.2}$$

$$\sigma_{x,x}(1) = 1. \tag{3.3}$$

That $\sigma_{x,x}$ is a monoid-homomorphism for the additive structure of A it is postulated by the condition (i) of (A). Also, the property (3.3) is postulated by (iv_2) of (A). The property (3.2) follows from $(ii^{'})$ for $x = y$

Namely,

$$\sigma_{x,x}(ab) = \sum_{z\in[x,x]} \sigma_{x,z}(a)\sigma_{z,x}(b) = \sigma_{x,x}(a)\sigma_{x,x}(b)$$

for all $a, b \in A$.

In accordance with the law given by the condition $(ii^{'})$ the mappings $\sigma_{x,y}$ $(x, y \in G)$ can be called as generalized derivations of the ring A. In the particular case for which the segment $[e, x]$ consists only from two elements, that is, $[e, x] = \{e, x\}$, we have

$$\sigma_{x,e}(ab) = \sigma_{x,e}(a)\sigma_{e,e}(b) + \sigma_{x,x}(a)\sigma_{x,e}(b),$$

and since, due to the condition (iv_4) of (A), $\sigma_{e,e}(b) = b$, it follows

$$\sigma_{x,e}(ab) = \sigma_{x,x}(a)\sigma_{x,e}(b) + \sigma_{x,e}(a)b \ (a, b \in A). \tag{3.4}$$

Therefore, in this case the mapping $\sigma_{x,e}$ is a $(\sigma_{x,x}, 1)$ - derivation of A. For the notion of (α, β) - derivation, where α, β are homomorphisms, see for instance [4, pag. 24] (see also [9, Chapter 1, §2]).

Thus, we can formulate

Proposition 3.1. *Under the assumptions $(A^{'})$ and (O) for each $x \in G$ the mapping $\sigma_{x,x}$ is a ring-homomorphism, and the mappings $\sigma_{x,y}$ $(x, y \in G)$ satisfy the law of multiplication $(ii^{'})$. In particular, if the segment $[e, x]$ is reduced to two elements, i.e. $[e, x] = \{e, x\}$ the mapping $\sigma_{x,e}$ is a $(\sigma_{x,x}, 1)$ - derivation of A.*

An another situation is given by the following assumption.

$(A^{''})$ The mappings $\sigma_{x,y}$ $(x, y \in G)$ defined as in the assumption (A) are such that $\sigma_{x,y} = 0$ for either incomparable $x, y \in G$ or $x > y$.

In this case the condition (ii) of (A) is written as follows

$$(ii^{''}) \qquad \sigma_{x,y}(ab) = \sum_{z\in[x,y]} \sigma_{x,z}(a)\sigma_{z,y}(b) \ (a, b \in A;\ x \le y,\ x, y \in G).$$

Similar as in the previous case, we have that the mapping $\sigma_{x,x}$ for each $x \in G$ is a ring-homomorphism, as well.

In the particular case in which $G =< x >$ is cyclic, we have $e \le x$ and hence $x \le x^2$ or, in general $x^m \le x^n$ for $m \le n$ $(m, n = 0, 1, ...;\ x^0 = e)$. It follows that the monoid G is linearly ordered, i.e. every two elements of it are comparable. If we denote

$$\sigma_{nm} := \sigma_{x^n,x^m} \quad (n, m = 0, 1, ...), \tag{3.5}$$

the multiplication formula from the condition (ii) of (A) can be written as follows

$$\sigma_{nm}(ab) = \sum_{j=0}^{\infty} \sigma_{nj}(a)\sigma_{jm}(b) \quad (a, b \in A;\ n, m = 0, 1, ...),$$

where as before the sum is taken for almost all $j = 0, 1, ...$ Note that the matrix representation of A is given by the mapping $\sigma : a \longrightarrow \sigma(a)$, where

$$\sigma(a) = [\sigma_{nm}(a)]_{n,m=0}^{\infty} \quad (a \in A). \tag{3.6}$$

Subsequent properties of this special case together with applications to general skew polynomial rings will be presented further in Section 6.

4. The monoid algebra $A\langle G\rangle$

1. Let A be a ring and G a multiplicative monoid. We denote $A\langle G\rangle$ for the set of all mappings $\alpha : G \longrightarrow A$ such that $\alpha(x) = 0$ for almost all $x \in G$. $A\langle G\rangle$ can be treated as a left module over A. In this respect, we define the addition in $A\langle G\rangle$ to be the ordinary addition of mappings into the additive group of A and define the operation of A on $A\langle G\rangle$ by the map $(a, \alpha) \longrightarrow a\alpha$ $(a \in A\langle G\rangle)$, where $(a\alpha)(x) = a\alpha(x)$ $(x \in G)$. Following notations made in [8] we write an element $\alpha \in A\langle G\rangle$ as a sum

$$\alpha = \sum_{x\in G} a_x \cdot x, \tag{4.1}$$

where by $a \cdot x$ $(a \in A,\ x \in G)$ it is denoted the mapping whose value at x is a and 0 at elements different of x. Certainly, in (4.1) the sum is taken over almost all $x \in G$. With this notation, if $a \in A$ and $\alpha, \beta \in A\langle G\rangle$ are written as sums $\alpha = \sum_{x\in G} a_x \cdot x,\ \beta = \sum_{x\in G} b_x \cdot x$, the sum $\alpha+\beta$ and $a\alpha$ can be written

$$\sum_{x\in G} a_x \cdot x + \sum_{x\in G} b_x \cdot x = \sum_{x\in G} (a_x + b_x)\cdot x, \tag{4.2}$$

$$a\sum_{x\in G} a_x \cdot x = \sum_{x\in G} (aa_x)\cdot x \ (a \in A). \tag{4.3}$$

Note that the family of elements $(1\cdot x)_{x\in G}$ forms a basis of $A\langle G\rangle$ over A.

Let now the ring A satisfies the conditions of the assumption (A). Then $A\langle G\rangle$ becomes a ring if for elements of the form $a\cdot x$ $(a \in A; x \in G)$ we define their product by the rule

$$(a\cdot x)(b\cdot y) = \sum_{z\in G} a\sigma_{x,z}(b)\cdot zy \ (a, b \in A; x, y \in G) \tag{4.4}$$

and then extending for $\alpha, \beta \in A\langle G\rangle$ by

$$\Big(\sum_{x\in G} a_x \cdot x\Big)\Big(\sum_{x\in G} b_x \cdot x\Big) = \sum_{x\in G}\sum_{y\in G} (a_x\cdot x)(b_y \cdot y). \tag{4.5}$$

The verification that $A\langle G\rangle$ is a ring under the laws (4.4) and (4.5) is direct. Let us check the property of associativity, for instance. Clearly that it is enough to verify this property for the elements of the form $a\cdot x$ $(a \in A, x \in G)$. So, in virtue of the condition (iii) of (A), we can write

$$((a\cdot x)(b\cdot y))(c\cdot z) = \Big(\sum_{u\in G} a\sigma_{x,u}(b)\cdot uy\Big)(c\cdot z)$$

$$= \sum_{u\in G} (a\sigma_{x,u}(b)\cdot uy)(c\cdot z) = \sum_{u\in G}\Big(\sum_{v\in G} a\sigma_{x,u}(b)\sigma_{uy,v}(c)\cdot vz\Big)$$

$$= \sum_{u\in G}\sum_{v\in G}\sum_{ts=v} a\sigma_{x,u}(b)\sigma_{u,t}(\sigma_{y,s}(c))\cdot vz.$$

On the other hand, by the condition (iii) of (A), we have

$$(a\cdot x)((b\cdot y)(c\cdot z)) = (a\cdot x)\sum_{s\in G} b\sigma_{y,s}(c)\cdot sz = \sum_{s\in G}(a\cdot x)(b\sigma_{y,s}(c)\cdot sz)$$

$$= \sum_{s\in G}\sum_{t\in G} a\sigma_{x,t}(b\sigma_{y,s}(c))\cdot tsz = \sum_{s\in G}\sum_{t\in G} a\Big(\sum_{u\in G}\sigma_{x,u}(b)\sigma_{u,t}(\sigma_{y,s}(c)\Big)\cdot tsz$$

$$= \sum_{u\in G} a\sigma_{x,u}(b)\sum_{s\in G}\sum_{t\in G}\sigma_{u,t}(\sigma_{y,s}(c))\cdot tsz = \sum_{u\in G} a\sigma_{x,u}(b)\sum_{v\in G}\sum_{ts=v}\sigma_{u,t}(\sigma_{y,s}(c))\cdot vz$$

$$= \sum_{u\in G}\sum_{v\in G}\sum_{ts=v} a\sigma_{x,u}(b)\sigma_{u,t}(\sigma_{y,s}(c))\cdot vz,$$

and the property of associativity is proved.

It is easy to verify the following useful properties:

$$(a\cdot e)(b\cdot e) = ab\cdot e\ (a,b\in A), \tag{4.6}$$

$$(1\cdot x)(1\cdot y) = 1\cdot xy\ (x,y\in G), \tag{4.7}$$

$$(1\cdot e)(a\cdot x) = (a\cdot x)(1\cdot e) = a\cdot x\ (a\in A;\ x\in G). \tag{4.8}$$

We see that the unit element of $A\langle G\rangle$ is $1\cdot e$. Since by definition $a(1\cdot e) = a\cdot e$ and since the element $1\cdot e$ is the unit element $A\langle G\rangle$ it is naturally to identify the element a with $a\cdot e$, and by this the ring A can be embedded in the ring $A\langle G\rangle$. Taking into account the fact that

$$\Big(\sum_{x\in G} a_x\cdot x\Big)(a\cdot e) = \sum_{x\in G}(a_x\cdot x)(a\cdot e) = \sum_{x\in G}\Big(\sum_{z\in G} a_x\sigma_{x,z}(a)\cdot ze\Big)$$

$$= \sum_{z\in G}\Big(\sum_{x\in G} a_x\sigma_{x,z}(a)\Big)\cdot z = \sum_{x\in G}\Big(\sum_{y\in G} a_y\sigma_{y,x}(a)\Big)\cdot x,$$

we can define

$$\alpha a = \sum_{x\in G}\Big(\sum_{y\in G} a_y\sigma_{y,x}(a)\Big)\cdot x, \tag{4.9}$$

for $a\in A$ and $\alpha\in A\langle G\rangle$.

Consequently, we obtain an operation of A on $A\langle G\rangle$ and in such a way we make $A\langle G\rangle$ into a right A-module (the axioms of a right module are immediately verified). Thus, we may view $A\langle G\rangle$ as an algebra over A.

In the particular case in which G is a partially ordered monoid and the ring A satisfies the assumption $(A^{'})$ the formula of multiplication (4.4) can be written as

$$(a\cdot x)(b\cdot y) = \sum_{z\in[e,x]} a\sigma_{x,z}(b)\cdot zy\ (a,b\in A; x,y\in G) \tag{4.10}$$

and, respectively, the formula (4.9) becomes

$$\alpha a = \sum_{x\in G}\Big(\sum_{y\geq x} a_y\sigma_{y,x}(a)\Big)\cdot x. \tag{4.11}$$

Similarly, in the case of $(A^{''})$ the law of multiplication are determined by

$$(a\cdot x)(b\cdot y) = \sum_{x\leq z} a\sigma_{x,z}(b)\cdot zy\ (a,b\in A; x,y\in G), \tag{4.12}$$

and

$$\alpha a = \sum_{x\in G}\Big(\sum_{y\leq x} a_y\sigma_{y,x}(a)\Big)\cdot x. \qquad (4.13)$$

for $a \in A$ and $\alpha = \sum_{x\in G} a_x \cdot x \in A\langle G\rangle$.

2. Examples of monoid algebras.

2.1. Weyl Algebras. Let $K[t_1, ..., t_n]$ be the polynomial ring in the commuting indeterminates $t_1, ..., t_n$ over a field K (for the sake of tradition K is assumed to be a field, but, in general, K could be any nonzero associative noncommutative ring with identity 1). We denote by

$$\partial_j := \frac{\partial}{\partial t_j} \quad (j = 1, ..., n)$$

the usual partial derivative with respect to the indeterminate t_j $(j = 1, ..., n)$. Let G be the free commutative monoid (multiplicative) generated by $\{x_1, ..., x_n\}$, and with its identity element called e. We use vector notation and write $K[t]$ instead of $K[t_1, ..., t_n]$. For $\alpha = (\alpha_1, ..., \alpha_n)$, whose components are integers, we set $\alpha! = \alpha_1!...\alpha_n!$ and for $\beta = (\beta_1, ..., \beta_n)$ such that $\alpha_j \geq \beta_j$ (in this case we write $\alpha \geq \beta$), denote

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1!...\alpha_n!}{\beta_1!...\beta_n!(\alpha_1-\beta_1)!...(\alpha_n-\beta_n)!}.$$

For any $\alpha = (\alpha_1, ..., \alpha_n)$ we write $t^\alpha = t_1^{\alpha_1}...t_n^{\alpha_n}$ and also $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$ (in particular, we consider $t^0 = 1$ and $x^0 = e$ whenever $\alpha_j = 0$ $(j = 1, ..., n)$), and define

$$\partial^\alpha = \partial_1^{\alpha_1}...\partial_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}}.$$

Next, we let

$$\sigma_{x^\alpha,x^\beta} := \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\partial^{\alpha-\beta}$$

if $\alpha \geq \beta$ and $\sigma_{x^\alpha,x^\beta} = 0$ otherwise. So that in particular for any $\alpha = (\alpha_1, ..., \alpha_n)$ we take otherwise $\sigma_{x^\alpha,x^\alpha} = 1_{K[t]}$ including $\sigma_{e,e} = 1_{K[t]}$ and $\sigma_{x^\alpha,e} = \partial^\alpha$. The family $(\sigma_{x^\alpha,x^\beta})$ satisfies the conditions of the assumption (A). Indeed, due to the fact that the partial derivatives ∂_j are linear mappings of $K[t]$ into itself, the condition (i) is clear. The condition (ii) can be proved by using Leibniz's formula for derivations. For any $a, b \in K[t]$ we have

$$\sigma_{x^\alpha,x^\beta}(ab) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\partial^{\alpha-\beta}(ab) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\sum_{\kappa\leq\alpha-\beta}\begin{pmatrix} \alpha-\beta \\ \kappa \end{pmatrix}\partial^\gamma(a)\partial^{\alpha-\beta-\gamma}(b). \qquad (4.14)$$

On the other hand

$$\sum_{\beta\le\gamma\le\alpha}\sigma_{x^\alpha,x^\gamma}(a)\sigma_{x^\gamma,x^\beta}(b)=\sum_{\beta\le\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\begin{pmatrix}\gamma\\ \beta\end{pmatrix}\partial^{\alpha-\gamma}(a)\partial^{\gamma-\beta}. \tag{4.15}$$

If we put $\kappa=\alpha-\gamma$ we get $0\le\kappa\le\alpha-\beta$, and the right hand side of (4.15) then becomes

$$\sum_{\kappa\le\alpha-\beta}\begin{pmatrix}\alpha\\ \alpha-\kappa\end{pmatrix}\begin{pmatrix}\alpha-\kappa\\ \beta\end{pmatrix}\partial^{\kappa}(a)\partial^{\alpha-\beta-\kappa}(b) \tag{4.16}$$

Since

$$\begin{pmatrix}\alpha\\ \alpha-\kappa\end{pmatrix}\begin{pmatrix}\alpha-\kappa\\ \beta\end{pmatrix}=\frac{\alpha!}{(\alpha-\kappa)!\kappa!}\cdot\frac{(\alpha-\kappa)!}{\beta!(\alpha-\kappa-\beta)!}$$

$$=\frac{\alpha!}{\beta!(\alpha-\beta)!}\cdot\frac{(\alpha-\beta)!}{\kappa!(\alpha-\beta-\kappa)!}=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\begin{pmatrix}\alpha-\beta\\ \kappa\end{pmatrix},$$

we see that (4.16) is equal to the right hand side of (4.14), and thus (ii) is established.

In order to prove (iii) we take $\alpha=(\alpha_1,...,\alpha_n)$, $\beta=(\beta_1,...,\beta_n)$ arbitrary, and consider $\gamma=(\gamma_1,...,\gamma_n)$ such that $\gamma\le\alpha$ and $\gamma\le\beta$. We have

$$\sum_{\kappa\le\gamma}\sigma_{x^\alpha,x^\kappa}\circ\sigma_{x^\beta,x^{\gamma-\kappa}}=\sum_{\kappa\le\gamma}\begin{pmatrix}\alpha\\ \kappa\end{pmatrix}\begin{pmatrix}\beta\\ \gamma-\kappa\end{pmatrix}\partial^{\alpha-\kappa}\circ\partial^{\beta-\gamma+\kappa}$$

$$=\sum_{\kappa\le\gamma}\begin{pmatrix}\alpha\\ \kappa\end{pmatrix}\begin{pmatrix}\beta\\ \gamma-\kappa\end{pmatrix}\partial^{\alpha+\beta-\gamma}.$$

But

$$\sum_{\kappa\le\gamma}\begin{pmatrix}\alpha\\ \kappa\end{pmatrix}\begin{pmatrix}\beta\\ \gamma-\kappa\end{pmatrix}=\begin{pmatrix}\alpha+\beta\\ \gamma\end{pmatrix},$$

and thus we get

$$\sum_{\kappa\le\gamma}\sigma_{x^\alpha,x^\kappa}\circ\sigma_{x^\beta,x^{\gamma-\kappa}}=\begin{pmatrix}\alpha+\beta\\ \gamma\end{pmatrix}\partial^{\alpha+\beta-\gamma}=\sigma_{x^{\alpha+\beta},x^\gamma}$$

that is (iii).

The conditions $(iv_1)-(iv_4)$ are trivially fulfilled.

Now we consider the monoid algebra $K[t]\langle G\rangle$. We identify $x\in G$ with $1\cdot x\in K[t]\langle G\rangle$, and in this way we do, in fact, have $G\subset K[t]\langle G\rangle$ (the identity element of $K[t]$ is denoted by 1 as in K). Note that then the elements of G form a basis in $K[t]\langle G\rangle$. Furthemore, with this identification the formal sums and products become ordinary sums and products. In particular, for any element $a \in K[t]$ and any multi-index α we have

$$x^{\alpha}a = \sum_{\kappa\leq\alpha} \sigma_{x^{\alpha},x^{\kappa}}(a)x^{\kappa} = \sum_{\kappa\leq\alpha} \begin{pmatrix} \alpha \\ \kappa \end{pmatrix} \sigma^{\alpha-\kappa}(a)x^{\kappa}. \tag{4.17}$$

It follows easily that

$$x_i t_j = \frac{\partial t_j}{\partial t_i} + t_j x_i,$$

that is

$$x_i t_j - t_j x_i = \delta_{ij} \quad (i, j = 1, ..., n).$$

δ_{ij} denotes the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$ and $\delta_{i_j} = 0$ if $i \neq j$). Due to the commutativity of the indeterminates t_j $(j = 1, ..., n)$ and of G, we also have the following relations

$$t_i t_j - t_j t_i = x_i x_j - x_j x_i = 0 \quad (i, j = 1, ..., n).$$

Therefore, the monoid algebra $K[t]\langle G\rangle$ can be regarded as the nth Weyl algebra over K (for the concept see [9], for instance).

2.2. A crossed product. Given a ring R with identity, a multiplicative monoid G and a monoid-homomorphism of G into $End(A)$ (cf. Example 1 of Section 2), it is possible to construct a monoid algebra $A\langle G\rangle$ as follows. As in Example 1 of Section 2 we define the mappings $\sigma_{x,y} : A \longrightarrow A$ by assuming $\sigma_{x,x} = \sigma(x)$ for $x \in G$ and $\sigma_{x,y} = 0$ for $x \neq y,\ x, y \in G$. It is easily verified that the family $(\sigma_{x,y})_{x,y\in G}$ satisfies all conditions of (A). In the considered case the law of multiplication in $A\langle G\rangle$ is given by

$$\Big(\sum_{x\in G} a_x \cdot x\Big)\Big(\sum_{x\in G} b_x \cdot x\Big) = \sum_{x\in G}\sum_{y\in G} a_x \sigma_{x,x}(b_y) \cdot xy.$$

Therefore, $A\langle G\rangle$ represents a crossed product of A by G with respect to the factors $\rho_{x,y} = 1$ $(x, y \in G)$. We cite [3] and [11] for the notion of crossed product.

2.3. Group rings. If in Example 2.2 G is a group and $\sigma : G \longrightarrow Aut(A)$ is such that $\sigma(x) = 1_A$ for all $x \in G$, then the multiplication in $A\langle G\rangle$ is defined by

$$\Big(\sum_{x\in G} a_x \cdot x\Big)\Big(\sum_{x\in G} b_x \cdot x\Big) = \sum_{x\in G}\sum_{y\in G} a_x b_y \cdot xy = \sum_{z\in G} c_z \cdot z,$$

where

$$c_z = \sum_{xy=z} a_x b_y = \sum_{x\in G} a_x b_{x^{-1}z} = \sum_{y\in G} a_{zy^{-1}} b_y.$$

As above, in $A\langle G\rangle$ the addition and scalar multiplication are defined by (4.2) and (4.3), respectively.

Thus, in the considered case, $A\langle G\rangle$ represents an ordinary group ring (see, for instance, [2], [17] and also for the commutative case see [6]).

It can be continued with examples in order to describe other concrete monoid algebras as for instance skew group rings, skew polynomial rings and others or their generalizations. Particular cases of monoid algebras and applications to the theory of skew polynomial rings are given in Section 6.

5. $A\langle G\rangle$ as a free G - algebra over A

Let A be a ring (in general, non-commutative) and G a multiplicative monoid, and assume that on the ring A is defined a family of mappings $(\sigma_{x,y})_{x,y\in G}$ satisfying the conditions of assumption (A). We will say that on A is defined a differential structure σ. Given a ring-homomorphism $f : A \longrightarrow B$ it can be defined on the ring B a structure of A- module, defining the operation of A on B by the map $(a, b) \longrightarrow f(a)b$ for all $a \in A$ and $b \in B$. We denote this operation by $a * b$. The axioms for a module are trivially verified. Let now $\varphi : G \longrightarrow B$ be a multiplicative monoid-homomorphism. Denote by $\langle B; f, \varphi\rangle$ the module formed by all linear combinations of elements $\varphi(x)$ $(x \in G)$ over A (i.e. with coefficients in A) in respect with the operation $*$, that is,

$$\langle B; f, \varphi\rangle := \Big\{\sum_{a,x} a * \varphi(x) \mid a \in A, x \in G\Big\},$$

where the sum is taken over at the most a finite set of pairs (a, x) with $a \in A$ and $x \in G$. The axioms for a left A-module are trivially verified.

In what follows we assume that the homomorphisms f and φ satisfy the following assumption.

(E) $\varphi(G)f(A) \subset \langle B; f, \varphi\rangle$.

Here we denote XY for any sets X, Y of the ring B to be the set of products of the form xy with $x \in X$ and $y \in Y$, i.e. $XY = \{xy \mid x \in X, y \in Y\}$.

Thus, it is postulated that an element $\varphi(x)f(a)$ $(a \in A, x \in G)$ can be written as a linear combination of the form $\sum_{b\in B,y\in G} b\varphi(y)$. The coefficients b depend on $\varphi(x), \varphi(y)$ and $f(a)$. To designate this fact we denote the corresponding coefficients by $\sigma_{\varphi(x),\varphi(y)}(f(a))$. Therefore, it can be considered that there are defined a family of mappings $\sigma_{\varphi(x),\varphi(y)} : B \longrightarrow B$ such that

$$\varphi(x)f(a) = \sum_{y\in G} \sigma_{\varphi(x),\varphi(y)}(f(a))\varphi(y) \ (a \in A, x \in G). \tag{5.1}$$

Obviously, by these considerations, we may view $\langle B; f, \varphi\rangle$ as a right A-module. In order to make the module $\langle B; f, \varphi\rangle$ to be a ring in respect with the law of composition given by (5.1), we require the following additional assumption.

(F) The homomorphisms f and φ are such that the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \sigma_{x,y}} \uparrow & & \uparrow {\scriptstyle \sigma_{\varphi(x),\varphi(y)}} \\ A & \xrightarrow{f} & B \end{array}$$

is commutative for every $x, y \in G$, that is,

$$\sigma_{\varphi(x),\varphi(y)} \circ f = f \circ \sigma_{x,y} \ (x, y \in G). \tag{5.2}$$

We define multiplication in $\langle B; f, \varphi\rangle$ by the rules

$$\Big(\sum_{x\in G} a_x * \varphi(x)\Big)\Big(\sum_{x\in G} b_x * \varphi(x)\Big) = \sum_{x\in G}\sum_{y\in G}(a_x * \varphi(x))(b_y * \varphi(y)), \tag{5.3}$$

$$(a_x * \varphi(x))(b_y * \varphi(y)) = f(a_x)\sum_{z\in G}\sigma_{\varphi(x),\varphi(z)}(f(b_y))\varphi(zy). \tag{5.4}$$

The verification that $\langle B; f, \varphi\rangle$ is a ring under the above laws of composition is direct. We only give the proof for the property of associativity. It is clear that it is enough to be verified for expressions of the form $a * \varphi(x)$.

We have

$$[(a * \varphi(x))(b * \varphi(y))](c * \varphi(z)) = [f(a)\sum_{u\in G}\sigma_{\varphi(x),\varphi(u)}(f(b))\varphi(uy)](c * \varphi(z))$$

$$= [f(a)\sum_{u\in G} f(\sigma_{x,u}(b))\varphi(uy)](c * \varphi(z)) = \Big[\sum_{u\in G} a\sigma_{x,u}(b) * \varphi(uy)\Big](c * \varphi(z))$$

$$= \sum_{u\in G}\Big(\sum_{u\in G} f(a\sigma_{x,u}(b))\sigma_{\varphi(uy),\varphi(v)}(f(c))\varphi(vz)\Big) = \sum_{u\in G}\Big(\sum_{v\in G} f(a\sigma_{x,u}(b))f(\sigma_{uy,v}(c))\varphi(vz)\Big)$$

$$= \sum_{u\in G}\sum_{v\in G} f(a\sigma_{x,u}(b)\sigma_{uy,v}(c))\varphi(vz) = \sum_{u\in G}\sum_{v\in G} a\sigma_{x,u}(b)(\sigma_{uy,v}(c)) * \varphi(vz)$$

$$= \sum_{u\in G}\sum_{v\in G}\sum_{ts=v}(a\sigma_{x,u}(b))\sigma_{u,t}(\sigma_{y,s}(c)) * \varphi(vz).$$

and, similarly,

$$(a * \varphi(x))[(b * \varphi(y))(c * \varphi(z))] = (a * \varphi(x))\sum_{s\in G} f(b)\sigma_{\varphi(y),\varphi(s)}(f(c))\varphi(sz)$$

$$= \sum_{s\in G}(a*\varphi(x))[f(b\sigma_{y,s}(c))\varphi(sz)] = \sum_{s\in G}(a*\varphi(x))(b\sigma_{y,s}(c)*\varphi(sz))$$

$$= \sum_{s\in G} f(a)\sum_{t\in G}\sigma_{\varphi(x),\varphi(t)}(f(b\sigma_{y,s}(c))\varphi(tsz)) = \sum_{s\in G} f(a)\sum_{t\in G} f(\sigma_{x,t}(b\sigma_{y,s}(c))\varphi(tsz))$$

$$= \sum_{s\in G}\sum_{t\in G} a\sigma_{x,t}(b\sigma_{y,s}(c))*\varphi(tsz) = \sum_{s\in G}\sum_{t\in G} a\Big(\sum_{u\in G}\sigma_{x,u}(b)\sigma_{u,t}(\sigma_{y,s}(c))\Big)*\varphi(tsz)$$

$$= \sum_{u\in G}\sum_{v\in G}\sum_{ts=v} a\sigma_{x,u}(b)\sigma_{u,t}(\sigma_{y,s}(c))*\varphi(vz).$$

Thus, we have made $\langle B; f, \varphi\rangle$ into an algebra over A (in general, non-commutative).

Next, we define a category $\mathcal{C}$ whose objects are algebras $\langle B; f, \varphi\rangle$ constructed as above, and whose morphisms between two objects $\langle B; f, \varphi\rangle$ and $\langle B^{'}; f^{'}, \varphi^{'}\rangle$ are ring-homomorphisms $h : B \longrightarrow B^{'}$ making the diagrams commutative:

$$\begin{array}{ccc} G & === & G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ B & \xrightarrow{h} & B' \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ A & === & A \end{array}$$

The axioms for a category are trivially satisfied. We call a universal object in the category $\mathcal{C}$ a free G-algebra over A, or a free (A, G)-algebra. It turns out that the monoid algebra $A\langle G\rangle$ represents a free (A, G)-algebra. To this end, we observe that the mapping $\varphi_0 : G \longrightarrow A\langle G\rangle$ given by $\varphi_0(x) = 1 \cdot x$ $(x \in G)$ is a monoid-homomorphism. The mapping φ_0 is injective, and so it can be considered as an embedding of G into $A\langle G\rangle$. In addition, we have a ring-homomorphism $f_0 : A \longrightarrow A\langle G\rangle$ given by $f_0(a) = a \cdot e$ $(a \in A)$. Obviously, f_0 is also an embedding. We identify $A\langle G\rangle$ with the triple $\langle A\langle G\rangle; f_0, \varphi_0\rangle$ and in this sense we treat $A\langle G\rangle$ as an object of the category $\mathcal{C}$. The property of the universality of $A\langle G\rangle$ is formulated by the following assertion.

Theorem 5.1. *Let A be a ring, and G a multiplicative monoid for which the assumptions $(A), (E)$ and (F) are satisfied. Then for every object $\langle B; f, \varphi\rangle$ of the category $\mathcal{C}$ there exists a unique ring-homomorphism $h : A\langle G\rangle \longrightarrow B$ making the* *following diagram commutative*

$$\begin{array}{ccc} G & === & G \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi} \\ A\langle G\rangle & \xrightarrow{h} & B \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle f} \\ A & === & A \end{array}$$

Proof. From the fact that the monoid

$$1 \cdot G = \{1 \cdot x \mid x \in G\}$$

is a basis of the A-module $A\langle G\rangle$ it follows that there exists a unique module-homomorphism $h : A\langle G\rangle \longrightarrow B$ such that $h \circ \varphi_0 = \varphi$ (cf., for instance, [8], Theorem 1, III §4, pag. 84). The homomorphism h is defined by

$$h(\alpha) = \sum_{x \in G} f(a_x)\varphi(x),$$

where $\alpha = \sum_{x \in G} a_x \cdot x \in A\langle G\rangle$. It only remains to prove that h is a ring-homomorphism. Let $\beta = \sum_{x \in G} b_x \cdot x$ another element of $A\langle G\rangle$. Since

$$\alpha\beta = \sum_{x \in G}\sum_{y \in G}(a_x \cdot x)(b_y \cdot y) = \sum_{x \in G}\sum_{y \in G}\sum_{z \in G} a_x \sigma_{x,z}(b_y) \cdot zy,$$

it follows

$$h(\alpha\beta) = \sum_{x \in G}\sum_{y \in G}\sum_{z \in G} f(a_x \sigma_{x,z}(b_y))\varphi(zy) = \sum_{x \in G}\sum_{y \in G}\sum_{z \in G} f(a_x) f(\sigma_{x,z}(b_y))\varphi(zy).$$

Taking into account the property of the commutativity (5.2), we obtain

$$h(\alpha\beta) = \sum_{x \in G}\sum_{y \in G}\sum_{z \in G} f(a_x)\sigma_{\varphi(x),\varphi(z)}(f(b_y))\varphi(zy).$$

We obtain the same result in the calculation of $h(\alpha)h(\beta)$:

$$h(\alpha)h(\beta) = \sum_{x \in G}\sum_{y \in G} f(a_x)\varphi(x) f(b_y)\varphi(y) = \sum_{x \in G}\sum_{y \in G}\sum_{z \in G} f(a_x)\sigma_{\varphi(x),\varphi(z)}(f(b_y))\varphi(z)\varphi(y).$$

Hence h is a ring-homomorphism, and Theorem 5.1 is proved. □

6. Applications. Skew polynomials rings

In this section we study a particular case especially useful in the study of the rings of skew polynomials in one variable. This particular case refers to the situation in which the monoid $G = \langle x\rangle$ is cyclic. In what follows we postulate that $e \neq x$. As in Section 3 we denote

$$\sigma_{nm} := \sigma_{x^n,x^m} \quad (n, m = 0, 1, ...). \tag{6.1}$$

Recall that in this notation the formula from the condition (ii) of (A) is written as

$$\sigma_{nm}(ab)=\sum_{j=0}^{\infty}\sigma_{nj}(a)\sigma_{jm}(b)\quad (a,b\in A;\ n,m=0,1,...), \tag{6.2}$$

where the sum is taken for almost all $j=0,1,....$ Also, the formula (2.2) is written as follows

$$\sigma_{nm}=\sum_{j_1+...+j_n=m}\sigma_{1j_1}\circ...\circ\sigma_{1j_n}\ (n,m=0,1,...) \tag{6.3}$$

in which $j_k=1,2,...$ for $k=1,...,n$.

1. (The case $(A^{'})$). In this subsection we discuss the situation in supposing that the mappings σ_{nm} defined by (6.1) are such that $\sigma_{nm}=0$ for $n<m$. In this case the matrix representation σ given by (2.3) has a lower triangular form, and the formula (6.2) is written as

$$\sigma_{nm}(ab)=\sum_{j=m}^{n}\sigma_{nj}(a)\sigma_{jm}(b)\ (a,b\in A;\ n\geq m,\ n,m=0,1,2,...). \tag{6.4}$$

In particular,

$$\sigma_{n0}(ab)=\sum_{j=0}^{n}\sigma_{nj}(a)\sigma_{j0}(b)\ (a,b\in A;\ n=0,1,...). \tag{6.5}$$

For $n=1$ the formula (6.5) becomes

$$\sigma_{10}(ab)=\sigma_{10}(a)\sigma_{00}(b)+\sigma_{11}(a)\sigma_{10}(b),$$

and since, by the condition (iv_4) of (A), $\sigma_{00}(b)=b$ we have

$$\sigma_{10}(ab)=\sigma_{11}(a)\sigma_{10}(b)+\sigma_{10}(a)b\ (a,b\in A). \tag{6.6}$$

Taking into account that σ_{11} is a ring-homomorphism (cf. Proposition 3.1), we can formulate the following assertion.

Proposition 6.1. *Under the above assumptions the mapping* σ_{10} *is a* $(\sigma_{11},1)$ *- derivation of* A.

It is clear that, due to the assumption $(A^{'})$, the formula (6.3) can be written as follows

$$\sigma_{nm}=\sum_{j_1+...+j_n=m}\sigma_{1j_1}\circ...\circ\sigma_{1j_n}\ (m=1,...,n;\ n=0,1,2,...), \tag{6.7}$$

where $j_k=0,1\ \ (k=1,...,n)$.

In particular,

$$\sigma_{n0}=\sigma_{10}^{n}\ (n=0,1,2,...), \tag{6.8}$$

and

$$\sigma_{nn} = \sigma_{11}^n \ (n = 0, 1, 2, ...). \tag{6.9}$$

Further properties of the mappings σ_{nm} will obtained by supposing the following assumption.

(B) The mapping σ_{11} is an automorphism of the ring A.

Then, by formula (6.9), it follows immediately the following assertion.

Proposition 6.2. *The mappings σ_{nn} $(n = 0, 1, 2, ...)$ are automorphisms of the ring A.*

Next, we define

$$\delta_0 := \sigma_{21} \circ \sigma_{11}^{-1}, \tag{6.10}$$

where by σ_{11}^{-1} is denoted the inverse of σ_{11}. We observe that (by using the law of multiplication (6.6) and the relations (6.10))

$$\delta_0(ab) = (\sigma_{21} \circ \sigma_{11}^{-1})(ab) = \sigma_{21}(\sigma_{11}^{-1}(ab)) = \sigma_{21}(\sigma_{11}^{-1}(a)\sigma_{11}^{-1}(a))$$

$$= \sigma_{21}(\sigma_{11}^{-1}(a))\sigma_{11}(\sigma_{11}^{-1}(b)) + \sigma_{22}(\sigma_{11}^{-1}(a))\sigma_{21}(\sigma_{11}^{-1}(b))$$

$$= (\sigma_{21} \circ \sigma_{11}^{-1})(a)b + (\sigma_{22} \circ \sigma_{11}^{-1})(a)(\sigma_{21} \circ \sigma_{11}^{-1})(b)$$

$$= \delta_0(a)b + \sigma_{11}(a)\delta_0(b) = \sigma_{11}(a)\delta_0(b) + \delta_0(a)b,$$

i.e.

$$\delta_0(ab) = \sigma_{11}(a)\delta_0(b) + \delta_0(a)b \ (a, b \in A). \tag{6.11}$$

Thus, we have

Proposition 6.3. *Under the assumptions $(A^{'}), (B)$ and (O) the mapping δ_0 defined by (6.10) represents a $(\sigma_{11}, 1)$ - derivation of the ring A.*

Let us now consider the following mappings

$$\delta_n := \sigma_{21} \circ \sigma_{n-1\ n-1} \ (n = 1, 2, ...) \tag{6.12}$$

which in accordance with (6.9) can be represented as follows

$$\delta_n = \sigma_{21} \circ \sigma_{11}^{n-1} \ (n = 1, 2, ...). \tag{6.13}$$

We observe that

$$\delta_n = \sigma_{21} \circ (\sigma_{11}^{-1} \circ \sigma_{nn}) = (\sigma_{21} \circ \sigma_{11}^{-1}) \circ \sigma_{nn} = \delta_0 \circ \sigma_{nn},$$

so that

$$\delta_n = \delta_0 \circ \sigma_{nn} \ (n = 0, 1, 2, ...). \tag{6.14}$$

Proposition 6.4. *Under the hypothesis of Proposition 6.3 the mapping δ_n for each $n = 0, 1, 2, ...$ is a $(\sigma_{n+1\ n+1}, \sigma_{nn})$ - derivation of the ring A.*

Proof. By Propositions 6.2 and 6.3 we emphasize the fact that σ_{nn} is an automorphism of the ring A and that the mapping δ_0 is a $(\sigma_{11}, 1)$ - derivation of A.

We have

$$\delta_n(ab) = (\delta_0 \circ \sigma_{nn})(ab) = \delta_0(\sigma_{nn}(ab)) = \delta_0(\sigma_{nn}(a)\sigma_{nn}(b))$$

$$= \sigma_{11}(\sigma_{nn}(a))\delta_0(\sigma_{nn}(b)) + \delta_0(\sigma_{nn}(a))\sigma_{nn}(b)$$

$$= \sigma_{n+1\ n+1}(a)\delta_n(b) + \delta_n(a)\sigma_{nn}(b),$$

that is

$$\delta_n(ab) = \sigma_{n+1\ n+1}(a)\delta_n(b) + \delta_n(a)\sigma_{nn}(b) \ (a, b \in A;\ n = 0, 1, 2, ...), \tag{6.15}$$

and the assertion follows. □

Finally, we remark the following consequence of the formula (6.7).

Proposition 6.5. *Every mapping σ_{nm} $(m = 0, 1, ..., n;\ n = 0, 1, 2, ...)$ is expressed by the derivation σ_{10} and the homomorphism σ_{11}.*

Also from the formula (6.7), as it is easy to see, it can be concluded that the mapping σ_{nm} $(m = 0, 1, ..., n)$ represents the coefficient at t^m of the formal extension of the binomial $(\sigma_{10} + \sigma_{11}t)^n$. Thus, taking into account the relations (6.8) for the derivation of higher order σ_{10}^n we can write

$$\sigma_{10}^n(ab) = \sum_{m=0}^{n} \sigma_{nm}(a)\sigma_{10}^m(b) \ (a, b \in A;\ n = 0, 1, ...). \tag{6.16}$$

For the particular case $\sigma_{11} = 1$ from (6.16) it follows the classical formula of Leibnitz

$$\sigma_{10}^n(ab) = \sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} \sigma_{10}^{n-m}(a)\sigma_{10}^m(b) \ (a, b \in A;\ n = 0, 1, ...). \tag{6.17}$$

2. (The case (A'')). In this subsection we make remarks on the situation in which the assumptions (A''), (B) and (O) are supposed (G being cyclic). Some formulae and statements are similar to those from the previous case of (A'). We will restrict ourselves only their enumeration and, concomitantly, we will give other special properties for considered case.

Thus, in the case of $(A^{''})$, we have the following properties.

1) $\sigma_{nm} = 0$ for $n > m$ $(n, m = 0, 1, 2, ...)$;

2) The matrix representation σ of A has a upper triangular form at each element $a \in A$;

The formula of multiplication (6.2) becomes to be the form

$$\sigma_{nm}(ab) = \sum_{j=n}^{m} \sigma_{nj}(a)\sigma_{jm}(b) \ (a, b \in A;\ n \le m,\ n, m = 0, 1, ...). \tag{6.18}$$

4) σ_{nn} for each $n = 0, 1, 2, ...$ is a ring-homomorphism;

5) The formula (6.7) implies that

$$\sigma_{nn} = \sum_{j_1+...+j_n=n} \sigma_{1j_1} \circ ... \circ \sigma_{1j_n} = \sigma_{11} \circ ... \circ \sigma_{11} = \sigma_{11}^n,$$

i.e.

$$\sigma_{nn} = \sigma_{11}^n \ (n = 1, 2, ..). \tag{6.19}$$

If, in addition, it supposed the assumption (B), i.e. the mapping σ_{11} is an automorphism of the ring A, then

6) The mapping σ_{nn} $(n = 1, 2, ...)$ are also automorphisms of A;

7) The mapping

$$\gamma_0 := \sigma_{12} \circ \sigma_{11}^{-1} \tag{6.20}$$

is a $(1, \sigma_{11})$ - derivation of the ring A;

In general

8) For each $n = 1, 2, ...$ the mapping defined by

$$\gamma_n := \sigma_{12} \circ \sigma_{11}^{n-1} \tag{6.21}$$

is a $(\sigma_{nn}, \sigma_{n+1n+1})$ - derivation of the ring A, that is, the following formula

$$\gamma_n(ab) = \sigma_{nn}(a)\gamma_n(b) + \gamma_n(a)\sigma_{n+1n+1}(b) \ (a, b \in A) \tag{6.22}$$

holds for each $n = 0, 1, 2, ...$

Note that, by virtue of the relations (6.19), the mapping γ_n can be also expressed as

$$\gamma_n = \sigma_{12} \circ \sigma_{n-1\ n-1} \ (n = 1, 2, ...),$$

and that the formula (6.22) can be deduced via (6.2) by using the fact that σ_{nn} for each $n = 0, 1, 2, ...$ is a ring-homomorphism. Namely, we have

$$\gamma_n(ab) = \sigma_{12}(\sigma_{n-1\ n-1}(ab)) = \sigma_{12}(\sigma_{n-1n-1}(a)\sigma_{n-1n-1}(b)) =$$

$$= \sigma_{11}(\sigma_{n-1n-1}(a))\sigma_{12}(\sigma_{n-1n-1}(b)) + \sigma_{12}(\sigma_{n-1\ n-1}(a))\sigma_{22}(\sigma_{n-1n-1}(b)),$$

and since $\sigma_{22} = \sigma_{11}^2$, so that $\sigma_{22} \circ \sigma_{n-1n-1} = \sigma_{n+1n+1}$, (6.22) follows at once.

In what follows we require additional conditions on the mappings σ_{nm} $(n, m = 0, 1, ...)$.

(L) The mappings σ_{1j} $(j = 1, 2, ...)$ are A - independent from the left, that means that if

$$\sum_{j=1}^{\infty} c_j \sigma_{1j}(a) = 0 \tag{6.23}$$

for all $a \in A$, then $c_j = 0$ $(j = 1, 2, ...)$.

Note that, due to $\sigma_{1j}(1) = 0$ $(j = 2, 3, ...)$ and $\sigma_{11}(1) = 1$ (cf. (iv_1) and (iv_2) of (A)), from (6.23) for $a = 1$ it follows that $c_1 = 0$. Therefore, without loss of generality, it can be considered that j is changed beginning from 2 and so on going almost everywhere over the set of integer numbers. Thus, the mappings σ_{1j} $(j = 1, 2, ...)$ are A-independent from the left if and only if the condition

$$\sum_{j=2}^{\infty} c_j \sigma_{1j}(a) = 0 \ (\forall a \in A)$$

implies $c_j = 0$ $(j = 2, 3, ...)$.

Similarly, it can be introduced the notion of A - independent from the right for the mappings σ_{1j} $(j = 1, 2, ...)$. Namely, we say that the mappings σ_{1j} $(j = 1, 2, ...)$ are A-independent from the right if whenever from the condition

$$\sum_{j=2}^{\infty} \sigma_{1j}(a) c_j = 0 \ (\forall a \in A)$$

it follows $c_j = 0$ $(j = 2, 3, ...)$.

An alternative of the assumption (L) is the following one.

(R) The mappings σ_{1j} $(j = 1, 2, ...)$ are A-independent from the right.

Remark 6.6. *The assumption of the A-independence (L) (resp. (R)) is satisfied for instance if for the mappings σ_{1j} $(j = 1, 2, ...)$ there exists a family of the elements $a_j \in A$ $(j = 1, 2, ...)$ such that the elements $\sigma_{1j}(a_j)$ $(j = 1, 2, ...)$ are not left (resp. right) zero-divisors in A and $\sigma_{1k}(a_j) = 0$ for $j \neq k$ $(j, k = 1, 2, ...)$. In the particular case of an integral domain A it is sufficient to require that $\sigma_{1j}(a_j) \neq 0$ $(j = 1, 2, ...)$ and $\sigma_{1k}(a_j) = 0$ for $j \neq k$ $(j, k = 1, 2, ...)$.*

The next assumption will be useful for our further discussion.

(C) There exists a positive integer r such that $\sigma_{1j} = 0$ for $j > r$.

It turns that the assumption (C) together with the assumption of the independence (R) implies the following property.

Proposition 6.7. $\sigma_{kj} = 0$ $(k = 1, ..., r;\ j = r+1, r+2, ...)$.

Proof. For $k = 1$ the assertion is true via the assumption (C). For the other values the assertion follows from the independence conditions given by the assumption (R). This is can be shown as follows. First we observe that (cf. (6.18))

$$\sigma_{1j}(ab) = \sum_{k=1}^{j} \sigma_{1k}(a)\sigma_{kj}(b)\ (a, b \in A).$$

Then, since $\sigma_{1j}(ab) = 0$ for $j = r+1, r+2, ...$, we can write

$$\sum_{k=1}^{j} \sigma_{1k}(a)\sigma_{kj}(b) = 0\ (j = r+1, r+2, ...).$$

For $j = r+1$, we obtain

$$\sum_{k=1}^{r} \sigma_{1k}(a)\sigma_{kr+1}(b) = 0$$

from which, by the assumption (R), we get

$$\sigma_{1r+1}(b) = ... = \sigma_{rr+1}(b) = 0\ (\forall b \in A),$$

i.e. $\sigma_{kr+1} = 0$ $(k = 1, ..., r)$.

By similar arguments it can be concluded the same for the other values. □

Further, we note the relations (cf. (iii) of (A))

$$\sigma_{n+ms} = \sum_{k+j=s} \sigma_{nk} \circ \sigma_{mj} = \sum_{k+j=s} \sigma_{mj} \circ \sigma_{nk}\ (s = n+m, n+m+1, ...), \quad (6.24)$$

where $k = n, n+1, ...$ and $j = m, m+1, ...$

From Proposition 6.7 we have $\sigma_{rk} = 0$ for $k = r+1, r+2, ...$ Hence, by applying the relations (6.24) for $n = r$ and $m = j$ $(j = 1, ..., r)$, one has

$$\sigma_{rr} \circ \sigma_{1j} = \sigma_{1j} \circ \sigma_{rr}\ (j = 1, ..., r), \quad (6.25)$$

or, taking into account the relations (6.19),

$$\sigma_{11}^{r} \circ \sigma_{1j} = \sigma_{1j} \circ \sigma_{11}^{r}\ (j = 1, ..., r). \quad (6.26)$$

Note that the relations (6.25) can be deduced also by observing that

$$\sigma_{r+1r+j} = \sum_{k+l=r+j} \sigma_{rk} \circ \sigma_{1l} = \sigma_{rr} \circ \sigma_{1j},$$

$$\sigma_{r+1r+j} = \sum_{k+l=r+j} \sigma_{1k} \circ \sigma_{rl} = \sigma_{1j} \circ \sigma_{rr}.$$

Now some special properties of the mappings σ_{nm} will be discussed further.

Again, in virtue of the condition *(iii)* of (A), we see

$$\sigma_{rr+j-1} = \sum_{k+l=r+j-1} \sigma_{r-1k} \circ \sigma_{1l} = \sigma_{r-1r-1} \circ \sigma_{1j} + \sigma_{r-1r} \circ \sigma_{1j-1} + \sigma_{r-1r+1} \circ \sigma_{1j-2} + ...$$

and since $\sigma_{rr+j-1} = 0$ for $j = 2, 3, ...$ and also $\sigma_{r-1r+1} = \sigma_{r-1r+2} = ... = 0$, it follows the relations

$$0 = \sigma_{r-1r-1} \circ \sigma_{1j} + \sigma_{r-1r} \circ \sigma_{1j-1} \ (j = 2, 3, ...). \tag{6.27}$$

Similarly

$$\sigma_{rr+k} = \sum_{s+t=r+k} \sigma_{1s} \circ \sigma_{r-1t} = \sigma_{1k+1} \circ \sigma_{r-1r-1} + \sigma_{1k} \circ \sigma_{r-1r},$$

and hence

$$\sigma_{1k+1} \circ \sigma_{r-1r-1} + \sigma_{1k} \circ \sigma_{r-1r} = 0 \ (k = 1, 2, ...) \tag{6.28}$$

In virtue of the relations (6.27) and (6.28), we obtain

$$(\sigma_{1k} \circ \sigma_{r-1r-1}) \circ \sigma_{1j} = \sigma_{1k} \circ (\sigma_{r-1r-1} \circ \sigma_{1j}) = -\sigma_{1k} \circ (\sigma_{r-1r} \circ \sigma_{1j-1})$$

$$= -(\sigma_{1k} \circ \sigma_{r-1r}) \circ \sigma_{1j-1} = (\sigma_{1k+1} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1},$$

i.e.

$$(\sigma_{1k} \circ \sigma_{r-1r-1}) \circ \sigma_{1j} = (\sigma_{1k+1} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \ (j = 2, 3, ...;\ k = 1, 2, ...). \tag{6.29}$$

In particular, for $k = 1$, the formula (6.29) becomes to be as follows

$$\sigma_{rr} \circ \sigma_{1j} = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \ (j = 2, 3, ...)$$

and thus, by the commutative relations (6.25), we have

$$\sigma_{1j} \circ \sigma_{rr} = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-1} \ (j = 2, 3, ...). \tag{6.30}$$

Then, we can continue

$$\sigma_{1j} \circ \sigma_{rr}^2 = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ (\sigma_{1j-1} \circ \sigma_{rr}) = (\sigma_{12} \circ \sigma_{r-1r-1}) \circ [(\sigma_{12} \circ \sigma_{r-1r-1}) \circ \sigma_{1j-2}]$$

$$= (\sigma_{12} \circ \sigma_{r-1\ r-1})^2 \circ \sigma_{1j-2},$$

and, by further iteration, we obtain

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\sigma_{12} \circ \sigma_{r-1\ r-1})^{j-1} \circ \sigma_{11} \ (j = 2, 3, ...). \tag{6.31}$$

Substituting in (6.31) $j = r + 1$ and taking into account that $\sigma_{1r+1} = 0$, we obtain

$$(\sigma_{12} \circ \sigma_{r-1r-1})^r \circ \sigma_{11} = 0. \tag{6.32}$$

Since, by the assumption (B), σ_{11} is an automorphism, from (6.32) it follows

$$(\sigma_{12} \circ \sigma_{r-1r-1})^r = 0.$$

Therefore $\gamma_r = \sigma_{12} \circ \sigma_{r-1r-1}$ is a nilpotent mapping of index r.

In virtue of (6.31), we see that the mappings σ_{1j} $(j = 2, 3, ...)$ can be expressed by σ_{11} and σ_{12}, namely

$$\sigma_{1j+1} = \gamma_r^j \circ \sigma_{11}^{-rj+1} \ (j = 1, 2, ...). \qquad (6.33)$$

In addition, we note that the mapping $\gamma_0 = \sigma_{12} \circ \sigma_{11}^{-1}$ commutates with σ_{rr}. In fact, from (6.25), in particular, it follows

$$\sigma_{rr} \circ \sigma_{12} = \sigma_{12} \circ \sigma_{rr},$$

from which, multiplying from the right by σ_{11}^{-1}, we have

$$\sigma_{rr} \circ (\sigma_{12} \circ \sigma_{11}^{-1}) = (\sigma_{12} \circ \sigma_{11}^{-1}) \circ (\sigma_{r+1r+1} \circ \sigma_{11}^{-1})$$

or

$$\sigma_{rr} \circ \gamma_0 = \gamma_0 \circ \sigma_{rr}. \qquad (6.34)$$

Further, we change in the formula (6.31) the expression $\sigma_{12} \circ \sigma_{r-1\ r-1}$ by $\gamma_0 \circ \sigma_{rr}$, and we get

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\gamma_0 \circ \sigma_{rr})^{j-1} \circ \sigma_{11}.$$

This relation together with the commutative property (6.34) implies

$$\sigma_{1j} \circ \sigma_{rr}^{j-1} = (\gamma_0^{j-1} \circ \sigma_{11}) \circ \sigma_{rr}^{j-1}.$$

Since σ_{rr} is an automorphism, we conclude

$$\sigma_{1j} = \gamma_0^{j-1} \circ \sigma_{11} \ (j = 2, 3, ...). \qquad (6.35)$$

In particular, from (6.35) it follows that

$$\gamma_0^r \circ \sigma_{11} = \sigma_{1r+1} = 0,$$

that is $\gamma_0^r = 0$. Moreover, if $\sigma_{1r} \neq 0$, then $\gamma_0^{r-1} \neq 0$.

The obtained results are assumed as follows.

Proposition 6.8. *Under the assumptions* (A''), (B), (C), (O) *and* (R) *the following assertions hold.*

1) The derivation γ_0 *is a nilpotent mapping of index* r*, that is,* $\gamma_0^r = 0$*, and* $\gamma_0^{r-1} \neq 0$ *whenever* $\sigma_{1r} \neq 0$*;*

2) $\gamma_r = \sigma_{12} \circ \sigma_{r-1r-1}$ *is also a nilpotent mapping of index* r*;*

3) The mappings σ_{1j} $(j = 2, 3, ...)$ *are expressed by* σ_{11} *and* σ_{12}*, and, moreover, for them there hold the relations (6.35) and the commutation relations (6.25).*

3. Given a ring A, an element x, it is possible to construct a monoid algebra $A\langle G\rangle$ as described above, where G is the monoid of all elements formally written in the form x^n $(n = 0, 1, ...;\ x^0 := e)$ subject to the law of composition $x^n \cdot x^m = x^{n+m}$ $\ (n, m = 0, 1, ...)$. We identify $x^n \in G$ with $1 \cdot x^n \in A\langle G\rangle$, and write ax^n instead of $a \cdot x^n$ $(n = 0, 1, ...)$. Let $\sigma_{nm} := \sigma_{x^n,x^m}$ $\ (n, m = 0, 1, ...)$ are mappings from A into itself satisfying the conditions of the assumption (A). Then the monoid algebra $A\langle G\rangle$ becomes to be a skew polynomial ring. The elements of it can be written uniquely in the form

$$\sum_{j=0}^{n} a_j x^j \ (a_j \in A,\ j = 0, 1, ..., n)$$

and the law of multiplication is defined by

$$(ax^n)(bx^m) = \sum_j a\sigma_{nj}(b)x^{m+j} \tag{6.36}$$

for two monomials, and then extending distributively to arbitrary polynomials (cf. (4.4) and (4.5)).

In the case when $\sigma_{nm} = 0$ for $n < m$ (this is the case of $(A^{'})$) the index j in the formula (6.36) ranges over integers from 0 up to n (cf. (4.10)). In particular, we have

$$xa = \sigma_{10}(a) + \sigma_{11}(a)x \quad (a \in A).$$

By virtue of Proposition 3.1 σ_{11} is a ring-homomorphism and the mappings σ_{10} is a $(\sigma_{11}, 1)$ - derivation of A. Therefore, this is the case of an Ore polynomial ring [10]. As usually it is denoted by $A[x; \sigma_{11}, \sigma_{10}]$. If $\sigma_{10} = 0$ this is written as $A[x; \sigma_{11}]$, and if $\sigma_{11} = 1_A$, as $A[x; \sigma_{10}]$.

In case of $(A^{''})$, that is, if $\sigma_{nm} = 0$ for $n > m$ the sum (6.36) is taken over $j \geq n$ (cf. (4.12)). Thus, in this case, the monoid algebra $A\langle G\rangle$ is considered with a multiplication defined by

$$xa = \sum_{j\geq 1} \sigma_{1j}(a)x^j \quad (a \in A)$$

in which only a finite number of the coefficients $\sigma_{1j}(a)$ are different of zero. If, in addition, it is supposed that there exists an integer r such that $\sigma_{1j} = 0$ for $j > r$ (cf. the assumption (C)) we obtain a skew polynomial ring with the law of multiplication defined by $xa = \sum_{j=1}^{r} \sigma_{1j}(a)x^j$ $\ (a \in A)$. Such rings were studied for the first by T. H. M. Smits in [14], [15] and [16] (for related results see also [4]).

Acknowledgements. The author is grateful to B. Gardner and Yu. M. Ryabukhin for valuable discussions.

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