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WEAKLY REGULAR SEMINEARRINGS

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Abstract. Weakly regular seminearrings are defined and characterized. The space of irreducible ideals is topologized and a sheaf representation is given for a class of distributive left weakly regular seminearrings.

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1. Introduction and Preliminaries

A right seminearring is a set R together with two binary operations "+"and "." such that $(R,+)$ and $(R,.)$ are semigroups and for all $a,b,c \in R : (a+b)c = ac+bc$ ([5]). A right seminearring R is said to have an absorbing zero 0 if $a+0 = 0+a = a$ and $a.0 = 0.a = 0$ hold for all $a \in R$. A non-empty subset I of a seminearring R is called a right (left) ideal if

(i) for all $x, y \in I$, $x + y \in I$ and

(ii) for all $x \in I$ and $r \in R$, $xr(rx) \in I$.

The word ideal will always mean a subset of R which is both a left and a right ideal of R. An element a of a seminearring R is called distributive if for all $x, y \in R, a(x + y) = ax + ay$; R will be called distributive if each of its element is distributive. A seminearring R is called distributively generated, or d.g. for short, if R contains a multiplicative subsemigroup D of distributive elements which generates $(R,+)$. If A, B are the non-empty subsets of a seminearring R, then AB will denote the set of all finite sums of the form $\sum a_k b_k$ with $a_k \in A$ and $b_k \in B$. In particular, for each $a \in R$, aR (Ra) will denote the set of all finite sums of the form $\sum ar_k$ $(\sum r_k a)$ with $r_k \in R$. Since R is right distributive, $Ra = \{ra : r \in R\}$. Clearly $aR(Ra)$ is a right (left) ideal of R.

For any subset A of R, $< A >$ will denote the ideal of R generated by A. If A and B are ideals of R then the product $AB = \{\sum_{k=1}^{n} a_k b_k : a_k \in A \text{ and } b_k \in B\}$ is not an ideal. However, if R is distributively generated seminearring then AB is an ideal of R. For ideals A, B of R, the sum $A + B$ is defined as the set of all finite sums $\sum(a_k + b_k)$ with $a_k \in A$ and $b_k \in B$.

If R is d.g. seminearring then $A + B$ is the smallest ideal of R containing both A and B. More generally, if $\{A_i : i \in I\}$ is an arbitrary family of ideals of a seminearring R with an absorbing zero, then the sum $\sum A_i$ is the set of all finite sums $\sum x_j$ where $x_j = \sum_{i \in I} a_{ij}$ such that $a_{ij} \in A_i$ and $a_{ij} = 0$ for all except finitely many $i \in I$. If R is d.g. seminearring, then $\sum_{i \in I} A_i$ is the smallest ideal of R containing all ideals $\{A_i : i \in I\}$. Moreover $\cap_{i \in I} A_i$ is the greatest ideal of R contained in all ideals $\{A_i : i \in I\}$.

Let R be a seminearring with multiplicative identity 1 (i.e. $1.x = x.1 = x$ for all $x \in R\}$. An additive semigroup $(M, +)$ with neutral element zero is called a left R-seminearmodule if there exist a function $\alpha : R \times M \longrightarrow M$ such that if $\alpha(r, m)$ is denoted by rm, then

(i) $(r_1 + r_2)m = r_1 m + r_2 m$

(ii) $(r_1 r_2)m = r_1(r_2 m)$

(iii) $1.m = m$

(iv) $r0 = 0m = 0$, for all $r_1, r_2, r \in R$ and $m \in M$.

A mapping $\alpha : A \longrightarrow B$ between left R-seminearmodules A and B is a left R-homomorphism if

(i) $\alpha(a + a_1) = \alpha(a) + \alpha(a_1)$

(ii) $\alpha(ra) = r\alpha(a)$ for all $a, a_1 \in A$ and $r \in R$.

Generally, the sum of two R-homomorphisms is not an R-homomorphism. Let R and L be two seminearrings. We shall say that L is an R-seminearring if L has the structure of R-seminearmodule and $r(xy) = (rx)y$, for all $x, y \in L$ and $r \in R$.

For two R-seminearrings L_1 and L_2, a seminearring homomorphism $f : L_1 \longrightarrow L_2$ is called a homomorphism of R-seminearrings if f is an R-homomorphism.

In [3], Brown and McCoy considered the notion of weakly regular rings. These rings were later studied by Ramamurthi [6], [4] and others. In this paper we initiate the study of weakly regular seminearrings. In Section 2, we define and characterize these seminearrings. In Section 3, we construct irreducible spectrum of a d.g. weakly regular seminearring. In Section 4, we prove a representation theorem for distributive left weakly regular seminearrings by sections in a presheaf.

2. Weakly Regular Seminearrings

A ring R is called left weakly regular if $x \in (Rx)^2$, for each $x \in R$. Adopting this definition, a seminearring R will be called left weakly regular if for each $x \in R$, $x \in (Rx)^2$. Every seminearring in this paper contains multiplicative identity.

2.1. Theorem. *The following assertions for a seminearring R are equivalent.*

(i) *R is left weakly regular.*

(ii) *Every left ideal of R is idempotent (i.e. $A^2 = A$ for every left ideal A of R).*

Proof. $(i) \Rightarrow (ii)$ Let A be a left ideal of R. Clearly $A^2 \subseteq A$. For the reverse inclusion, let $x \in A$, then $x \in (Rx)^2$. But $Rx \subseteq A$, so $(Rx)^2 \subseteq A^2 \Longrightarrow x \in A^2$.

$(ii) \Rightarrow (i)$ Suppose that every left ideal of R is idempotent. Let $x \in R$ then $x \in Rx$ implies that $x \in (Rx) = (Rx)^2$. Thus R is left weakly regular. □

2.2. Proposition. *Suppose R is distributively generated seminearring. Let $x \in R$ then RxR is a two sided ideal of R generated by x.*

Proof. Let $a, b \in RxR$ then $a = \sum_{finite} s_i x t_i$ and $b = \sum_{finite} s_i' x t_i'$ where s_i, s_i', t_i and $t_i' \in R$.

$$a + b = \left(\sum_{finite} s_i x t_i + \sum_{finite} s_i' x t_i' \right) \in RxR.$$

If $r \in R$ then

$$ar = (\sum_{finite} s_i x t_i) r = \sum_{finite} s_i x (t_i r) \in RxR.$$

Since R is d.g. so there exist distributive elements $d_1, d_2, ..., d_n$ such that $r = d_1 + d_2 + ... + d_n$. Thus

$$\begin{aligned} ra &= (d_1 + d_2 + ... + d_n)(\sum_{finite} s_i x t_i) \\ &= d_1(\sum_{finite} s_i x t_i) + d_2(\sum_{finite} s_i x t_i) + ... + d_n(\sum_{finite} s_i x t_i) \\ &= \sum_{finite} (d_1 s_i) x t_i + \sum_{finite} (d_2 s_i) x t_i + ... + \sum_{finite} (d_n s_i) x t_i \in RxR. \end{aligned}$$

Thus RxR is a two sided ideal of R. As R contains multiplicative identity, so $x \in RxR$. If A is any ideal of R containing x, then $sxt \in A$ for all $s, t \in R$. Also $\sum_{finite} s_i x t_i \in A$ implies that $RxR \subseteq A$. Hence RxR is the two sided ideal of R generated by x. □

2.3. Theorem. *The following assertions for a distributively generated seminearring R are equivalent.*

(i) *R is left weakly regular.*

(ii) *Every left ideal of R is idempotent.*

(iii) *For each (two sided) ideal I of R, $J \cap I = IJ$, for any left ideal J of R.*

Proof. $(i) \Leftrightarrow (ii)$ From Theorem 2.1.

$(ii) \Rightarrow (iii)$ Let I be an ideal and J be a left ideal of R. Since $IJ \subseteq J$ and $IJ \subseteq I \Longrightarrow IJ \subseteq J \cap I$. Let $x \in J \cap I \Longrightarrow x \in J$ and $x \in I$. By (ii) $x \in Rx = (Rx)^2$ which implies $x = \sum_{finite}(r_i x)(t_i x) = (\sum_{finite}(r_i x t_i))x \in IJ$. Since $x \in I$, so $\sum_{finite} r_i x t_i \in I$. Thus $J \cap I \subseteq IJ$. Hence $J \cap I = IJ$.

$(iii) \Rightarrow (i)$ Let $x \in R$ then $x \in Rx$ and $x \in RxR$. As Rx is a left ideal and RxR is an ideal of R, so by (iii), $Rx \cap RxR = (RxR)(Rx)$. As $x \in Rx \cap RxR = (RxR)(Rx) = (Rx)(Rx)$, therefore $x \in (Rx)^2$. Hence R is a left weakly regular. □

2.4. Proposition. *Each ideal of a left weakly regular seminearring is left weakly regular (as a seminearring).*

Proof. Let J be an ideal of a left weakly regular seminearring R. Let $x \in J$ then Jx is a left ideal of R (since $Jx = \{jx : j \in J\}$ if $j_1 x$ and $j_2 x \in Jx$ then $j_1 x + j_2 x = (j_1 + j_2)x \in Jx$, if $r \in R$ then $r(jx) = (rj)x \in Jx$. By Theorem 2.1, $(Jx)^2 = Jx$. As $x \in R$, so $x \in (Rx)^2$ that is

$$\begin{aligned} x &= r_1 x t_1 x + r_2 x t_2 x + \ldots + r_n x t_n x \\ &= (r_1 x t_1 + r_2 x t_2 + \ldots + r_n x t_n)x \in Jx, \text{ since } x \in J. \end{aligned}$$

As $Jx = (Jx)^2$, so $x \in (Jx)^2$. Thus J is a left weakly regular(as a seminearring). □

2.5. Definition. A two sided ideal I of a seminearring R is called left pure if for each $x \in I$ there exists $y \in I$ such that $x = yx$. In other words, I is left pure if and only if for each $a \in I$ the equation $a = xa$ has a solution in I.

2.6. Proposition. *A distributively generated seminearring R is left weakly regular if and only if every two sided ideal I of R is left pure.*

Proof. Suppose R is a left weakly regular seminearring and I a two sided ideal of R. Let $a \in I$. Then $a \in (Ra)^2$, that is

$$a = \sum_{finite} (r_i a)(t_i a) = (\sum_{finite} r_i a t_i)a = ya, \text{ where } y = \sum_{finite} r_i a t_i \in I.$$

Thus I is left pure.

Conversely, assume that every ideal of R is left pure. Let $a \in R$, then RaR is a two sided ideal of R generated by a. By hypothesis $a \in (RaR)a = (Ra)(Ra) = (Ra)^2$. Thus R is left weakly regular. □

2.7. Proposition. *For a distributively generated left weakly regular seminearring R, the set of all ideals of R (ordered by inclusion) form a complete lattice $\pounds_R$ under the sum and intersection of ideals with $I \cap J = IJ$ for ideals I, J of R.*

A lattice $\pounds$ is called Brouwerian if for any $a, b \in \pounds$, the set of all $x \in \pounds$ satisfying $a \wedge x \leq b$ contains a greatest element c, the pseudo-complement of a relative to b.

A (complete) Brouwerian lattice is distributive.

2.8. Proposition. *If R is a distributively generated left weakly regular seminearring, then the lattice $\pounds_R$ of all ideals of R (ordered by inclusion) is distributive.*

Proof. Follows from [2, Proposition 3.3]. □

2.9. Proposition. *Let R be a distributively generated left weakly regular seminearring. For an ideal P of R the following assertions are equivalent.*

(i) *For ideals I, J of R, $I \cap J = P$ implies $I = P$ or $J = P$,*

(ii) *$I \cap J \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$,*

(iii) *$< a > \cap < b > \subseteq P \Longrightarrow a \in P$ or, $b \in P$, for any $a, b \in R$.*

Proof. $(i) \Rightarrow (ii)$ Suppose $I \cap J \subseteq P$ for ideals I, J of R. Then $P = (I \cap J) + P = (I + P) \cap (J + P)$ by Proposition 2.8. Hence by hypothesis $I + P = P$ or $J + P = P$ that is $I \subseteq P$ or $J \subseteq P$.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Suppose I, J, are ideals of R containing P properly. Then there exist $a \in I \backslash P$ and $b \in J \backslash P$. By the contrapositivity of (iii), we have $< a > \cap < b > \nsubseteq P$. Hence $I \cap J \neq P$. □

2.10. Definition. An ideal P of R is called irreducible if it is proper (i.e. $P \neq R$) and satisfies one of the equivalent conditions of the above Proposition.

2.11. Proposition. *Let R be a distributively generated left weakly regular seminearring. If I is a proper ideal of R and $a \notin I$, then there exist an irreducible ideal J of R such that $I \subseteq J$ and $a \notin J$.*

Proof. By Zorn's Lemma, there exists an ideal J of R which is maximal with respect to the property that J is proper, $I \subseteq J$ and $a \notin J$. Then J is irreducible. For if $J = P \cap L$ but both P and L are properly contain J, then P and L are both contain a. Hence $a \in P \cap L$. Since $a \notin J$, this contradicts the assumption that $J = P \cap L$. □

The following is an immediate consequence of the above Proposition.

2.12. Proposition. *Let R be a distributively generated left weakly regular seminearring. Then each proper ideal of R is the intersection of all irreducible ideals which contain it.*

2.13. Definition. An ideal J of R is called a direct summand of R if there exists an ideal $J^{'}$, called Cosummand of J, such that $J + J^{'} = R$ and $J \cap J^{'} = \{0\}$.

2.14. Proposition. *Let R be a distributively generated left weakly regular seminearring. Then the set of direct summands of R is a Boolean sublattice of $\pounds_R$.*

3. Irreducible spectrum of a distributively generated left weakly regular seminearring

3.1. Definition. We denote by $\pounds_R$ the lattice of ideals of R and by $H(R)$ the set of irreducible ideals of R. For any ideal I of R, we define

$$\begin{aligned} \Theta_I &= \{J \in H(R) : I \nsubseteq J\}, \\ \Im(H(R)) &= \{\Theta_I : I \in \pounds_R\}. \end{aligned}$$

In the rest of this section, R will denote a d.g. left weakly regular seminearring.

3.2. Theorem. *The set $\Im(H(R))$ forms a topology on the set $H(R)$. Moreover, the assignment $I \longrightarrow \Theta_I$ is an isomorphism between the lattice $\pounds_R$ of ideals of R and the lattice of open subsets of $H(R)$.*

Proof. First we show that $\Im(H(R))$ forms a topology on the set $H(R)$. Note that $\Theta_0 = \{J \in H(R) : (0) \nsubseteq J\} = \emptyset$, since (0) is contained in every (irreducible) ideal. Thus Θ_0 is the empty subset of $\Im(H(R))$. On the other hand $\Theta_R = \{J \in H(R) : R \nsubseteq J\} = H(R)$. This is true since irreducible ideals are proper. So $\Theta_R = H(R)$ is an element of $\Im(H(R))$. Now let $\Theta_{I_1}, \Theta_{I_2} \in \Im(H(R))$ with $I_1, I_2 \in \pounds_R$. Then

$$\begin{aligned} \Theta_{I_1} \cap \Theta_{I_2} &= \{J \in H(R) : I_1 \nsubseteq J \text{ and } I_2 \nsubseteq J\} \\ &= \{J \in H(R) : I_1 \cap I_2 \nsubseteq J\} \\ &= \Theta_{I_1 \cap I_2}. \end{aligned}$$

This follows from Proposition 2.9.

Now consider an arbitrary family $\{I_\lambda\}_{\lambda\in\Lambda}$ of ideals of R. Then

$$\begin{aligned}\cup_{\lambda\in\Lambda}\Theta_{I_\lambda} &= \cup_{\lambda\in\Lambda}\{J\in H(R): I_\lambda \nsubseteq J\}\\ &= \{J\in H(R): \exists\lambda\in\Lambda \text{ such that } I_\lambda\nsubseteq J\}\\ &= \{J\in H(R): \sum_\lambda I_\lambda \nsubseteq J\}\\ &= \Theta_{\sum_\lambda I_\lambda}.\end{aligned}$$

Since $\sum_\lambda I_\lambda$ is an ideal of R it follows that $\cup_{\lambda\in\Lambda}\Theta_{I_\lambda}\in\Im(H(R))$. This shows that $\Im(H(R))$ is a topology on $H(R)$. Define

$$\Phi : \pounds_R \longrightarrow \Im(H(R))$$

by setting $\Phi(I)=\Theta_I$.

It is easily verified that Φ preserves finite intersection and arbitrary union. Hence Φ is a lattice homomorphism. Finally we show that Φ is an isomorphism. For this purpose we show that $I_1=I_2 \Longleftrightarrow \Theta_{I_1}=\Theta_{I_2}$ for I_1, I_2 in $\pounds_R$. Suppose $\Theta_{I_1}=\Theta_{I_2}$. If $I_1\neq I_2$, then $\exists\ x\in I_1$ such that $x\notin I_2$. Then by Proposition 2.11, there exists an irreducible ideal J such that $I_2\subseteq J$ and $x\notin J$. Hence $I_1\nsubseteq J$ and so $J\in\Theta_{I_1}$. By the assumption $\Theta_{I_1}=\Theta_{I_2}$ so $J\in\Theta_{I_2}$. Hence $I_2\nsubseteq J$. But this is a contradiction. Hence $I_1=I_2$. □

3.3. Definition. The set $H(R)$ of irreducible ideals of R will be called irreducible spectrum of R. The topology $\Im(H(R))$ in the above Theorem will be called the irreducible spectral topology on $H(R)$. We shall denote by $H(R)$ the corresponding topological space. H(R) will be called irreducible spectral space.

3.4. Proposition. *(i) H(R) is a compact space (but not, in general, Hausdorff)*

(ii) For $I\in\pounds_R$ Θ_I is open and closed in H(R) iff I is a direct summand of R.

Proof. (i) Suppose $\cup_{\lambda\in\Lambda}\Theta_I = H(R)$ be an open covering of $H(R)$. Then $\sum_\lambda I_\lambda = R$. Since $1\in R \Longrightarrow 1=\sum_{finite} x_i$ where $x_i=\sum_{\lambda\in\Lambda} a_{\lambda_i}$ such that $a_{\lambda_i}\in I_\lambda$ and $a_{\lambda_i}=0$ for all except finitely many $\lambda\in\Lambda$.

Suppose $1=x_1+x_2+...+x_n$ and each x_i is a sum of m_i non-zero a_{λ_i} then $1=\sum_{finite} I_\lambda$ where number of I_λ is not more than $m_1+m_2+...+m_n$. Thus the open cover $\{\Theta_I:\lambda\in\Lambda\}$ is reducible to a finite subcover. Thus H(R) is compact.

(ii) Suppose Θ_I $(I\in\pounds_R)\in\Im(H(R))$ is both open and closed. Then there exist Θ_J with $J\in\pounds_R$ so that $\Theta_I\cup\Theta_J=H(R)$ and $\Theta_I\cap\Theta_J=\emptyset$. This implies that $I+J=R$ and and $I\cap J=\{0\}$. Therefore, I is a direct summand of R. □

The following example shows that $H(R)$ need not be a Hausdorff space.

3.5. Example. Let $R = \{0, x, 1\}$ with the following operations

$+$	0	x	1
0	0	x	1
x	x	x	x
1	1	x	1

$\cdot$	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

R is of course a weakly regular seminearring, all of whose ideals are linearly ordered. $\pounds_R = \{\{0\}, \{0, x\}, \{0, x, 1\}\}$ and $H(R) = \{\{0\}, \{0, x\}\}$. The spectral space $H(R)$ is clearly not Hausdorff. Note that $H(R) = \{\emptyset, \{0\}, H(R)\}$.

4. Representation of distributive left weakly regular Seminearrings

In this section R will denote a distributive left weakly regular seminearring with multiplicative identity 1.

4.1. Proposition. *Let I and J be ideals of R with $J \subseteq I$. Then any R-homomorphism from J to I factors through J.*

Proof. Let $f : J \longrightarrow I$ be an R-homomorphism. If $a \in J$, then by Proposition 2.6, there exist $x \in J$ such that $a = xa$. Hence $f(a) = f(xa) = xf(a) \in J$, since $x \in J$. □

4.2. Proposition. *For each ideal I of R, $I^* = \{\sum_{finite} f_i : f_i \in End_R(I)\}$ is an R-seminearring.*

Proof. Clearly I^* is a seminearring with neutral element 0 with respect to pointwise addition and composition of mappings. Define the action of R on I^* by $(r \sum_{finite} f_i)(x) = (\sum_{finite} f_i(x))r$ for all $r \in R$. Now we show that I^* becomes an R-seminearmodule. If f is an R-endomorphism of I then we show that rf is also an R -homomorphism of I.

$$\begin{aligned} (rf)(x+y) &= (f(x+y))r = (f(x)+f(y))r \\ &= f(x)r + f(y)r = (rf)(x) + (rf)(y), \text{ and} \\ (rf)(ax) &= (f(ax))r = (af(x))r = a(f(x)r) = a((rf)(x)). \end{aligned}$$

Thus rf is an R- endomorphism of I. Now

$$(r \sum_{finite} fi)(x) = (\sum_{finite} fi(x))r = \sum_{finite} fi(x)r = \sum_{finite} (rfi)(x)$$

As $rf_i \in$End$R(I) \Longrightarrow r(\sum_{finite} f_i) \in I^*$. Let $r_1, r_2 \in R$. Then

$$\begin{aligned}((r_1+r_2)(\sum_{finite} f_i))(x) &= (\sum_{finite} f_i(x)(r_1+r_2) \\ &= (\sum_{finite} f_i(x))r_1 + (\sum_{finite} f_i(x))r_2, \; R \text{ is distributive} \\ &= \sum_{finite}(f_i(x)r_1) + \sum_{finite}(f_i(x)r_2) \\ &= \sum_{finite}(r_1f_i)(x) + \sum_{finite}(r_2f_i)(x) \\ &= (\sum_{finite}(r_1f_i) + \sum_{finite}(r_2f_i))(x)\end{aligned}$$

Thus

$$\begin{aligned}(r_1+r_2)\sum_{finite} f_i &= \sum_{finite} r_1f_i + \sum_{finite} r_2f_i \\ &= r_1(\sum_{finite} f_i) + r_2(\sum_{finite} f_i) \\ ((r_1r_2)(\sum_{finite} f_i))(x) &= (\sum_{finite} f_i(x))(r_1r_2) = \sum_{finite}(f_i(x)(r_1r_2) \\ &= \sum_{finite}(r_1r_2)f_i(x) = \sum_{finite} r_1(r_2f_i(x)) \\ &= r_1(\sum_{finite} r_2f_i(x)) = r_1(r_2\sum_{finite} f_i(x)) \\ &= (r_1(r_2(\sum_{finite} f_i)))(x).\end{aligned}$$

Thus

$$\begin{aligned}(r_1r_2)\sum_{finite} f_i &= r_1(r_2\sum_{finite} f_i) \\ 1.(\sum_{finite} f_i) &= (\sum_{finite} 1.f_i) = \sum_{finite} f_i \\ r.0 &= 0(\sum_{finite} f_i) = 0\end{aligned}$$

So I^* is an R-seminearmodule.

Further, let $r \in R$ and

$$\begin{aligned} \sum_{i=1}^{n} f_i, \sum_{j=1}^{m} g_j &\in I^*, \text{then} \\ r((\sum_{i=1}^{n} f_i)(\sum_{j=1}^{m} g_j)) &= r(\sum_{i=1}^{n}\sum_{j=1}^{m} f_i g_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} r(f_i g_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} (rf_i)g_j = (\sum_{i=1}^{n} rf_i)(\sum_{j=1}^{m} g_j). \end{aligned}$$

□

4.3. Definition. Let X be a topological space and $\Im(X)$ the category of open sets of X and inclusion maps. A presheaf P of R -seminearmodules on X is a contravariant functor from the category $\Im(X)$ to the category M_R of R-seminearmodules, that is, it consists of the data:

(a) for every open set $U \subseteq X$, an R-seminearmodule $P(U)$, and

(b) for every inclusion $V \subseteq U$ of open sets, an R-homomorphism $P_{\rho UV} : P(U) \longrightarrow P(V)$ subject to the following conditions:

(i) $P(\emptyset) = (0)$, where $\emptyset$ is the empty set,

(ii) $P_{\rho UU} : P(U) \longrightarrow P(U)$ is the identity map, and

(iii) If $W \subseteq V \subseteq U$ are three open sets then $P_{\rho UW} = P_{\rho VW} \circ P_{\rho UV}$: If P is a presheaf on X, $P(U)$ is called a section of the presheaf P on the open set U and the maps $P_{\rho UV}$ are called restriction maps, and often the notation $\alpha|_V$ is used instead of $P_{\rho UV}(\alpha)$ if $\alpha \in P(U)$.

4.4. Definition. A presheaf P on a topological space X is called a sheaf if the following additional conditions are satisfied.

(iv) If U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U, and if $\alpha|_{V_\lambda} = \beta|_{V_\lambda}$ for α, $\beta \in P(U)$ and for all V_λ , then $\alpha = \beta$.

(v) If U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U and if there are elements $\alpha_\lambda \in P(V_\lambda)$ for each $\lambda \in \Lambda$ with the properties that for each $\lambda, \mu \in \Lambda$, $\alpha_\lambda|_{V_\lambda \cap V_\mu} = \alpha_\mu|_{V_\lambda \cap V_\mu}$ then $\exists\ \alpha \in P(U)$ such that $\alpha|_{V_\lambda} = \alpha$ for each $\lambda \in \Lambda$. If a presheaf satisfies condition (iv) only, it is called separated.

4.5. Theorem. *Let R be a distributive left weakly regular seminearring. For every ideal I of R the assignment $\Theta_I \longrightarrow I^* = P_R(I)$ defines a separated presheaf P_R of R-seminearrings on $H(R)$. The seminearring of the global sections of this presheaf is isomorphic to R.*

Proof. First, we prepare the data for the existance of a presheaf. By Proposition 4.2, $P_R(I) = I^*$ is an R-seminearring for every ideal I of R. We need to define a restriction map $P_{\rho IJ} : I^* \longrightarrow J^*$, $\Theta_J \subseteq \Theta_I$, that is when $J \subseteq I$. By Proposition 4.1, if $f : I \longrightarrow I$ is an R-endomorphisms then $f|_J : J \longrightarrow J$. If $\sum_{finite} f_i \in I^*$ then $P_{\rho IJ}\ (\sum_{finite} f_i) = \sum_{finite} f_i|_J$. As

$$\begin{aligned}
P_{\rho IJ}(\sum_{finite} f_i + \sum_{finite} g_j) &= \sum_{finite} f_i|_J + \sum_{finite} g_j|_J \\
&= P_{\rho IJ}(\sum_{finite} f_i) + P_{\rho IJ}(\sum_{finite} g_j) \\
P_{\rho IJ}(\sum_{finite} f_i)(\sum_{finite} g_j) &= P_{\rho IJ}(\sum_{finite} f_i g_j) = \sum_{finite} (f_i g_j)|_J \\
&= \sum_{finite} (f_i|_J)(g_j|_J) = (\sum_{finite} f_i|_J)(\sum_{finite} g_i|_J) \\
&= P_{\rho IJ}(\sum_{finite} f_i) P_{\rho IJ}(\sum_{finite} g_j).
\end{aligned}$$

If $r \in R$ then

$$\begin{aligned}
F_{\rho IJ}(r \sum_{finite} f_i) &= F_{\rho IJ}(\sum_{finite} r f_i) = \sum_{finite} (r f_i)|_J \\
&= \sum_{finite} r(f_i|_J) = r \sum_{finite} (f_i|_J) = r F_{\rho IJ}(\sum_{finite} f_i).
\end{aligned}$$

Thus $P_{\rho IJ}$ is a homomorphism of R-seminearrings. Thus P_R satisfies the conditions of a presheaf. Thus, we have described the presheaf P_R. In order to show that P_R is separated, we verify condition (iv) in Definition 4.4. Let $I = \sum_{\lambda\in\Lambda} I_\lambda \in \pounds_R$, and suppose $\sum f_i, \sum g_i \in F_R(I) = I^*$ such that $(\sum f_i)|I_\lambda = (\sum g_i)|I_\lambda$ for all $\lambda \in \Lambda$. For each $x \in I$ we have $x = \sum_{finite} x_i$ where $x_i = \sum_{\lambda\in\Lambda} a_{\lambda i}$ such that $a_{\lambda i} \in I_\lambda$ and $a_{\lambda i} = 0$ for all except finitely many $\lambda \in \Lambda$.

$$\begin{aligned}
(\sum f_i)(x) &= \sum_{finite} f_i(x) = \sum_{finite} f_i(\sum_{finite} x_i) = \sum_{finite}\sum_{finite} f_i(x_i) \\
&= \sum_{finite}\sum_{finite} f_i(\sum_{\lambda\in\Lambda} a_{\lambda i}) = \sum_{finite}\sum_{finite}\sum_{\lambda\in\Lambda} f_i(a_{\lambda i}) \\
&= \sum_{finite}\sum_{finite}\sum_{\lambda\in\Lambda} g_j(a_{\lambda i}) = (\sum g_j)(x).
\end{aligned}$$

Hence $\sum f_i = \sum g_i$, and so P_R is separated. Now we show that $F_R(R) = R^* \cong R$. Define $h : R^* \longrightarrow R$ by $h(\sum_{finite} f_i) = \sum_{finite} f_i(1)$. Then h is homomorphism of R-seminearrings. Suppose $h(\sum_{finite} f_i) = h(\sum_{finite} g_i)$. Then $\sum_{finite} f_i(1) = \sum_{finite} g_i(1)$.

Let $r \in R$,

$$\begin{aligned}\sum_{finite} f_i(r) &= \sum_{finite} f_i(r.1) = \sum_{finite} rf_i(1) = r.\sum_{finite} f_i(1) \\ &= r.\sum_{finite} g_i(1) = \sum_{finite} r.g_i(1) = \sum_{finite} g_i(r.1) \\ &= \sum_{finite} g_i(r) \Longrightarrow \sum f_i = \sum g_i.\end{aligned}$$

So h is $1-1$. To show that h is surjective, let $t \in R$ and define $\alpha_t : R \longrightarrow R$ by $\alpha_t(r) = rt$. Clearly α_t is an R-homomorphism. Hence $\alpha_t \in R^*$ and $h(\alpha_t) = \alpha_t(1) = 1.t = t$. Thus h is also surjective and hence bijective. □

4.6. Theorem. *Let R be a distributive left weakly regular seminearring all of whose ideals are linearly ordered. For every ideal I of R, the assignment $\Theta_I \longrightarrow I^* = P_R(I)$ defines a sheaf P_R of R-seminearrings on $H(R)$.*

The seminearring of the global sections of this sheaf is isomorphic to R.

Proof. We need only to check condition (v) in Definition 4.4. Let $I = \sum_{\lambda\in\Lambda} I_\lambda \in \pounds_R$. Consider $\sum_{finite} f_\lambda \in I^*$ and $\sum_{finite} f_\mu \in I^*$ which coincides on $I_\lambda \cap I_\mu$. Since ideals of R are linearly ordered, $I_\lambda \subseteq I_\mu$ or $I_\mu \subseteq I_\lambda$. Hence $I_\lambda + I_\mu = I_\mu$ or $I_\lambda + I_\mu = I_\lambda$. Now define $f : I_\lambda + I_\mu$ by

$$f(x) = \begin{cases} \sum_{finite} f_\mu(x) \text{ if } I_\lambda + I_\mu = I_\mu \\ \sum_{finite} f_\lambda(x) \text{ if } I_\lambda + I_\mu = I_\lambda \end{cases}$$

Hence, f is an obvious extension of $\sum f_\lambda$ and $\sum f_\mu$. From this it follows that the family $\{I_\lambda\}_{\lambda\in\Lambda}$ is stable under finite sums. Hence $f = \sum f_i$ where $f_i : \sum_{\lambda\in\Lambda} I_\lambda \longrightarrow \sum_{\lambda\in\Lambda} I_\lambda$ can be defined with no ambiguity. Clearly, f extends each $\sum_{finite} f_k$. Hence, P_R is a sheaf. □

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