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ON PRINCIPALLY LIFTING MODULES

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Abstract. Discrete (quasi-discrete) modules form an important class in module theory, they are studied extensively by many authors. The class of lifting modules is obtained by considering only one of the defining conditions of quasi-discrete modules, namely the condition (D_1). Here we focus on and study principally lifting modules, or modules with the condition (PD_1). These modules are generalizations of lifting modules. We also study direct sums of P-hollow (semi-hollow) modules. It is known that relative projectivity is essential to study direct sums of quasi-discrete modules. Here we introduce the definition of relative Pprojectivity, which is essential to examine direct sums of hollow, and of P-hollow (semi hollow), modules for being principally lifting. Quasi-discrete module are always direct sums of hollow submodules, we show that finite dimensional modules with the condition (PD_1) are direct sums of P-hollow (semi-hollow) submodules. We also obtain some properties for modules with (PD_1), which are in analogy with the known properties for lifting modules.

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1. Introduction

Discrete modules were defined in S. Mohamed and S. Singh [7] under the name "*dual-continuous modules*", and in Oshiro [8], *quasi-discrete modules* were given under the name "*quasi-semiperfect modules*". *Lifting* modules, over Dedekind domains, were studied by S. Singh [9]; and *lifting* $\mathbb{Z}$-modules were characterized by Mohamed and Abdul-Karim [4], under the name "*Semi-dual-continuous modules*". It is known that every *quasi-discrete* module is a direct sum of hollow submodules, unique up to isomorphism, and is fully relatively projective. On the other hand, a direct sum $\underset{i\in I}{\oplus} H_i$ of hollow modules is a *quasi-discrete* module if it complements direct summands, every local summand is a summand, and the H_i is $\underset{j\in I-\{i\}}{\oplus} H_j$ *-projective*, for every $i \in I$.

Here we study *principally lifting* modules (or modules with (PD_1) for short). They are considered as generalizations of *lifting* modules. In [1], the concept of *finitely lifting* (*f-lifting* for short) modules were given. In fact, *f-lifting* modules were originally introduced in Wisbauer [10], where a characterization of *f-lifting* modules was given, and it was shown that a module M is *f-lifting* if and only if it is *principally lifting* (or has (PD_1)). This interesting result was also mentioned, without proof, in [1]. Here we show that the condition (PD_1) for modules is inherited by summands. In the presence of the condition (D_3), we prove that the condition (PD_1) is equivalent to *quasi-discreteness*, and this has been done for finite direct sums of local modules. *Semi-hollow* modules, which are obviously equivalent to *P-hollow* modules, were introduced and characterized in [1]. We start section 2 by giving the definition of *P-hollow* (*Semi-hollow*) modules, and their characterizations. In fact, it was shown in [1] that there are two types of *P-hollow* modules, namely the *local* modules and modules which are their own radicals. It is known that *lifting* modules (or modules satisfying the condition (D_1)) are *supplemented* and each *supplement* is a summand. Here we show that modules which satisfy the condition (PD_1) need not be even *principally supplemented* (*P-supplemented*), while over principal ideal rings they are *P-supplemented.* Section 3 is devoted to relative *Pprojective* modules, this concept is used to study direct sums of *P-hollow* modules, and to examine them for being *principally lifting* modules. We show that if $M = A \oplus B$ is a module over a local ring, where A and B are relatively *Pprojective*, then M has (PD_1) if and only if both of A and B have (PD_1). It is known that a finite direct sum of hollow modules which in pairs are relatively *projective* is a *quasi-discrete* module. Here we show that an arbitrary direct sum of hollow modules which in pairs are relatively *projective* must have the condition (PD_1). Over arbitrary rings, direct sums of *P-hollow* modules which are in pairs relatively *Pprojective* are also dealt with.

A submodule A of a module M is called *small* in M (denoted by $A \ll M$) if $A + B \neq M$ for any proper submodule B of M. A module H is called *hollow* if every proper submodule of H is small in H. A submodule P of a module M is called a supplement in M of a submodule A if P is a minimal with the property that $M = A + P$, or equivalently, if $M = A + P$ and $A \cap P \ll P$. A submodule C of M is called a supplement submodule if it is a supplement in M for some submodule of M. M is called a *supplemented* module if for any submodules A and B of M with $M = A + B$, then B contains a *supplement* of A in M. A module M is *weakly supplemented* if every submodule of M has a supplement in M. A module M is lifting (or has the condition (D_1)) if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A \ll M$. M is said to have the condition (D_2) in case of if A is a submodule of M such that M/A is isomorphic to a summand of M, then A is a summand of M. M is said to have the condition (D_3) in case of if M_1 and M_2 are summands of M with $M_1 + M_2 = M$, then $M_1 \cap M_2$ is a summand of M. A module M is called a *discrete* module if it has the conditions (D_1) and (D_2). A module M is called a *quasi-discrete* module if it has the conditions (D_1) and (D_3). A decomposition $M = \underset{i\in I}{\oplus} M_i$ of a module M is said to complement direct summands if for every summand submodule A of M there exists $J \subseteq I$ such that $M = A \oplus \underset{i\in J}{\oplus} M_i$.

All modules M here are unitary right modules over a ring R (not necessary commutative) with unity. By $A \leq M$ we mean A is a submodule of M. $A \leq^{\oplus} M$ stands for A is a direct summand of M.

2. P-hollows and the condition (PD$_1$)

Definition 2.1. ([1], 2.12) A nonzero module M is called *semi-hollow* if every proper finitely generated submodule is small in M.

A nonzero module M is called *P-hollow* if every proper cyclic submodule is small in M.

Hollow modules are obviously *semi-hollow* (*P-hollow*) modules. From the fact that finite sums of *small* submodules is *small*, one can easily show that a module M is a *semi-hollow* module if and only if it is a *P-hollow* module.

Lemma 2.2. ([1], 2.15) *Let M be a module, then:*

(1) If M is (semi-) hollow, then every factor module is (semi-) hollow.

(2) If $K \ll M$ and M/K is (semi-) hollow, then M is (semi-) hollow.

(3) M is semi-hollow if and only if M is local or $Rad(M) = M$.

Proposition 2.3. *The following are equivalent for a module M :*

(1) M is P-hollow

(2) $N \ll M$ whenever M/N is a nonzero cyclic module.

Proof. (1)⇒(2) Let $0 \neq M/B$ be cyclic, then by (1) of Lemma 2.2 M/B is *P-hollow*. By (3) of Lemma 2.2 M/B is local; and hence $B \ll M$.

(2)⇒(1) Let mR be a proper cyclic submodule of M. If mR is not small in M then $mR + B = M$ for some proper submodule B of M and since $(mR + B)/B$ is cyclic, then by (2) we have B is small in M which yields $mR = M$, which is a contradiction. Therefore $B = M$; i. e. $mR \ll M$. □

An immediate consequence of Proposition 2.3 is that for cyclic modules the concepts of *hollow* and of *P-hollow* coincide. In particular a ring R (as a right R-module) is *P-hollow* if and only if it is *local*.

Remark 2.4. *1- P-hollow modules need not be hollows, as it is explained in [1] by considering the set $\mathbb{Q}$ of all rational numbers as a $\mathbb{Z}$-module ($\mathbb{Q}/\mathbb{Z}$ is not hollow, while $\mathbb{Q}/N$ is not cyclic for all proper submodules N of $\mathbb{Q}$).*

2- Hollow modules are indecomposable modules, so direct sums of hollow modules are not hollows, while, according to Lemma 2.2, if $M = \bigoplus_{i\in I} P_i$, where the P_i are non cyclic P-hollows for all $i \in I$, then M is P-hollow.

Definition 2.5. ([1], 22.7) A module M is called *finitely* lifting (or *f-lifting* for short) if for any finitely generated submodule A, there is a decomposition $M = N \oplus S$ with $N \leq A$ and $A \cap S \ll M$. M is *principally lifting* (or has (PD_1) for short) if for all $m \in M$, M has a decomposition $M = N \oplus S$ with $N \leq mR$ and $mR \cap S \ll M$.

Observe that every *P-hollow* module satisfies the condition (PD_1).

Lemma 2.6. ([10], 41.13) *The following are equivalent for a module M:*

(1) M is f-lifting;

(2) M is principally lifting.

Proposition 2.7. *The condition (PD_1) is inherited by summands.*

Proof. Let M have the condition (PD_1) and $K \leq^{\oplus} M$, if $k \in K$, then M has a decomposition $M = N \oplus S$ with $N \leq kR$ and $kR \cap S \ll M$. It follows that $K = N \oplus (K \cap S)$, and $kR \cap (K \cap S) \leq kR \cap S \ll M$, so $kR \cap (K \cap S) \ll K$ (due to $K \leq^{\oplus} M$). Therefore K has (PD_1). □

It is known that an indecomposable module has (D_1) if and only if it is a *hollow* module, the following Lemma gives an analogy to this fact.

Lemma 2.8. *The following are equivalent for an indecomposable module M:*

(1) M has (PD_1).

(2) M is a P-hollow module.

Proof. Follows directly from the defining condition of (PD_1). □

Lemma 2.9. *If M has (PD_1), then every cyclic submodule C has a supplement S which is a summand, and C contains a complementary summand of S in M.*

Proof. Follows directly from the defining condition of (PD_1), and the fact that a small submodule of M is small in any summand of M. $\square$

Lemma 2.10. *The following are equivalent for a module M.*

(1) M has (PD_1);

(2) Every cyclic submodule C of M can be written as $C = N \oplus S$ with $N \leq^{\oplus} M$ and $S \ll M$.

(3) For each $m \in M$, there exist principal ideals I and J of R such that $mR = mI \oplus mJ$, where $mI \leq^{\oplus} M$ and $mJ \ll M$.

Proof. (1)⇒(2) It is clear.

(2)⇒(1) Let C be a cyclic submodule of M, then by (2) $C = N \oplus S$ with $N \leq^{\oplus} M$ and $S \ll M$. Write $M = N \oplus N'$, it follows that $C = N \oplus C \cap N'$. Now let $\pi : N \oplus N' \to N'$ be the natural projection, we have $C \cap N' = \pi(C) = \pi(N \oplus S) = \pi(S) \ll M$. Therefore M has (PD_1).

(2)⇔ (3) Clear. $\square$

Corollary 2.11. *Proper cyclic summands of a P-hollow module are the zero submodule.*

Lemma 2.12. ([5], Corollary 4.50) *Let $M = \overset{n}{\underset{i=1}{\oplus}} M_i$ where M_i is hollow and M_j-projective whenever $j \neq i$. Then M is a quasi-discrete module.*

In the following proposition we show that an arbitrary direct sum of hollow modules, which in pairs are relatively projective, satisfies the condition (PD_1).

Proposition 2.13. *Let $M = \underset{i\in I}{\oplus} H_i$ where every H_i is a hollow module, and is H_j-projective ($j \neq i$). Then M has (PD_1).*

Proof. Let C be a cyclic submodule of M, then there exists a finite subset F of I such that $C \subseteq \underset{i\in F}{\oplus} H_i$. By Lemma 2.12, $\underset{i\in F}{\oplus} H_i$ is *quasi-discrete*, and hence has (D_1). Thus C can be written as $C = N \oplus S$, where $N \leq^{\oplus} \underset{i\in F}{\oplus} H_i$ (hence $N \leq^{\oplus} M$) and $S \ll \underset{i\in F}{\oplus} H_i$. Therefore by Lemma 2.10 M has (PD_1). $\square$

Proposition 2.14. *Let M be a module with (PD_1). If $M = X + Y$ such that $Y \leq^{\oplus} M$ and $X \cap Y$ is cyclic, then Y contains a supplement of X in M.*

Proof. Since M has (PD_1), and $X \cap Y$ is cyclic, we have by Lemma 2.10 $X \cap Y = N \oplus S$, where $N \leq^{\oplus} M$ and $S \ll M$. Since $Y \leq^{\oplus} M$, we have $S \ll Y$. Write $Y = N \oplus N$. It follows that $X \cap Y = N \oplus (X \cap Y \cap \; N_1) = N \oplus (X \cap N_1)$. Let $\pi : N \oplus N_1 \longrightarrow N_1$ be the natural projection. It follows that $X \cap N_1 =$

$\pi(N \oplus (X \cap N_1)) = \pi(X \cap Y) = \pi(N \oplus S) = \pi(S)$ hence $X \cap N_1 \ll N_1$, and that $M = X + Y = X + N + N_1 = X + N_1$. Therefore N_1 is a supplement of X in M that is contained in Y. □

Corollary 2.15. *Let M be a module with (PD_1) over a principal ideal ring R. If $M = X + mR$, then mR contains a supplement of X in M.*

Proof. By Lemma 2.10, we have $mR = N \oplus S$, where $N \leq^{\oplus} M$ and $S \ll M$. It follows that $M = X + N$, where N is a cyclic summand of M, hence $X \cap N$ is a cyclic submodule of M, and thus apply Proposition 2.14. □

Lemma 2.16. *Let M be a module with (PD_1). Then every indecomposable cyclic submodule C of M is either small in M or a summand of M.*

Proof. By Lemma 2.10, we have $C = N \oplus S$ with $N \leq^{\oplus} M$ and $S \ll M$. Since C is indecomposable, we have either $C = N$ or $C = S$. □

Lemma 2.17. *Let $M = A \oplus B$. Then the following are equivalent:*

(1) A is B-projective.

(2) If $M = N + B$, then $N \cap B \leq^{\oplus} N$ (hence $M = N_1 \oplus B$, where $N_1 \leq N$).

Proof. (1)⇒(2) It is given in ([5], 4.47).

(2)⇒(1) It is given in ([2], Proposition 3.2) □

Proposition 2.18. *Let $M = \overset{n}{\underset{i=1}{\oplus}} P_i$, where the P_i are local modules for all i. If M has (D_3), then the following are equivalent:*

(1) M has (PD_1),

(2) M is a quasi-discrete module.

Proof. (1)⇒(2) Since (PD_1) *and* (D_3) are inherited by summands, we have $P_i \oplus P_j$ has (PD_1) *and* (D_3) for all i, j $(i \neq j)$. Now if $P_i \oplus P_j = Y + P_j$, then $P_i \cong (P_i \oplus P_j)/P_j = (Y + P_j)/P_j \cong Y/(Y \cap P_j)$ is a cyclic module. Thus form some $m \in P_i \oplus P_j$, $Y = mR + (Y \cap P_j)$. By (PD_1) for $P_i \oplus P_j$ and by Lemma 2.10, $mR = N \oplus S$ with $N \leq^{\oplus} P_i \oplus P_j$ and $S \ll P_i \oplus P_j$ hence $P_i \oplus P_j = Y + P_j = (N \oplus S) + (Y \cap P_j) + P_j = N + P_j$ and by (D_3) for $P_i \oplus P_j$, we have $P_i \oplus P_j = N \oplus P_j$ with $N \leq Y$. Hence by Lemma 2.17, P_i is P_j-projective for all $i \neq j$. Therefore by Lemma 2.12 M is a *quasi-discrete* module.

(2)⇒(1) It is obvious. □

Remark 2.19. *In general, modules which satisfy both of the conditions (PD_1) and (D_3) need not be quasi-discrete modules, for instance the set of rational number $\mathbb{Q}$*

as a $\mathbb{Z}$-module is not a quasi-discrete module but it satisfies both of the conditions (PD_1) and (D_3).

Proposition 2.20. *Every finite uniform dimensional module with (PD_1) is a direct sum of P-hollow submodules.*

Proof. The proof will be by induction on the uniform dimension of M. If $U.dim(M) = 1$, then M is an indecomposable module; and hence M is a *P-hollow* module, by Lemma 2.8. Now let $2 \leq U.dim(M) = n < \infty$. Since M is not a *P-hollow* module, then there is $m \in M$ such that mR is not small and proper submodule of M. By (PD_1), we have $M = N \oplus N^{'}$, where $N \leq mR$ and $N^{'} \cap mR \ll M$. It is clear that N and $N^{'}$ have uniform dimensions less than n, and hence by induction, each of N and $N^{'}$ is a direct sum of *P-hollow* submodules. □

3. Pprojectivity and the condition $(\mathbf{PD}_1)$

Definition 3.1. Let A and B be R- modules. A is said to be *Pprojective* relative to B (or A is *B-Pprojective*) if for each $b \in B$, each epimorphism $g : bR \longrightarrow bR/K$, and each homomorphism $\varphi : A \to bR/K$, there exists a homomorphism $\mu : A \to bR$ such that $g\mu = \varphi$.

Lemma 3.2. *Let $M = A \oplus B$ be an R-module. Then the following are equivalent:*

(1) A is B-Pprojective;

(2) A is bR-projective for all $b \in B$;

(3) For each $b \in B$, if $A \oplus bR = Y + bR$, then there is $L \leq Y$ such that $A \oplus bR = L \oplus bR$.

Proof. Follows from the definition of relative *Pprojectivity*, and Lemma 2.17. □

Remark 3.3. *Clearly every B-projective module is B-Pprojective, and if B is a cyclic module then every B-Pprojective module is B-projective. There are R-modules A and B, where A is B-Pprojective while A is not B-projective. For example take $A = \mathbb{Q}$ (the set of all rational numbers), $R = \mathbb{Z}$, and $B = \underset{i \in I}{\oplus} \mathbb{Z}$, where $f : \underset{i \in I}{\oplus} \mathbb{Z} \to \mathbb{Q}$ is an epimorphism (as $\mathbb{Q}$ is a homomorphic image of a free $\mathbb{Z}$-module). Clearly $\mathbb{Q}$ is $\underset{i \in F}{\oplus} \mathbb{Z}$-projective for each finite subset F of I, hence $\mathbb{Q}$ is $(\underset{i \in I}{\oplus} \mathbb{Z})$-Pprojective, while $\mathbb{Q}$ is not $(\underset{i \in I}{\oplus} \mathbb{Z})$-projective, since f does not split (due to $\mathbb{Q}$ not a projective $\mathbb{Z}$-module).*

Proposition 3.4. *Let $M = (\underset{i\in I}{\oplus} A_i) \oplus (\underset{\alpha\in\Lambda}{\oplus} B_\alpha)$, where all the A_i are cyclic P-hollow modules, and all the B_α are non cyclic P-hollow modules. If the A_i is A_j-Pprojective (for all $i \neq j \in I$), and is B_α-Pprojective (for all $i \in I$, $\alpha \in \Lambda$), then M has (PD_1).*

Proof. Write $A = (\underset{i\in I}{\oplus} A_i)$ and $B = (\underset{\alpha\in\Lambda}{\oplus} B_\alpha)$. Then $M = A \oplus B$. We need to consider a cyclic submodule $(a+b)R$ in M ($a \in A$, and $b \in B$), and prove that $(a+b)R = P \oplus S$, with P a summand of M and $S \ll M$ (Lemma 2.10). Now $(a+b)R \leq aR \oplus bR \leq \underset{i\in F}{\oplus} A_i \oplus \underset{\alpha\in K}{\oplus} B_\alpha$ for finite subsets F and K of I and Λ respectively. So, there is no loss of generality if we assume that I and Λ are finite.

As A_i is A_j- *projective* for all $i \neq j$, A is *quasi-discrete* (Lemma 2.12). Hence $A = N \oplus \check{N}$, with $aR = N \oplus T$ where $T := aR \cap \check{N}$. Since N is $\check{N}$-*projective* ([5], 4.23), we get N is T-*projective*. Write $b = \sum_{\alpha\in\Lambda} b_\alpha$. As A_i is $b_\alpha R$-*projective*, A_i is $\underset{\alpha\in\Lambda}{\oplus} b_\alpha R$-*projective*; and consequently is bR-*projective*. It then follows that A, hence N is bR-*projective*. Therefore N is $(T \oplus bR)$-*projective*. Now

$$N \oplus (T \oplus bR) = aR \oplus bR = (a+b)R + bR = (a+b)R + (T \oplus bR).\ (*)$$

Hence $N \oplus (T \oplus bR) = P \oplus (T \oplus bR)$, with $P \leq (a+b)R$ (Lemma 2.17). Clearly $T \oplus bR \ll M$. It remains to show that P is a summand of M. To this end, add $\check{N}+B$ to both sides of (*), we get $M = P+\check{N}+B$. Now $P \cap (\check{N}+B) \leq (P+T+bR)\cap(\check{N}+B) = (N+T+bR)\cap(\check{N}+B) \leq (\check{N}+B+T+bR)\cap N + (N+\check{N}+B)\cap(T+bR) = T+bR$. Hence $P \cap (\check{N}+B) \leq P \cap (T \oplus bR) = 0$. □

Proposition 3.5. *Let M be a module over a local ring R. If M has (PD_1), then every cyclic submodule is either small in M, or a summand of M.*

Proof. The proof follows from Lemma 2.16, and the fact that every cyclic module over a local ring is a local module. □

Corollary 3.6. *Let $M = A \oplus B$ be a module over a local ring R, where A and B are relatively Pprojective. Then M has (PD_1) if and only if both of A and B have (PD_1).*

Proof. Let C be an arbitrary cyclic submodule of M. Then $C = (a+b)R$, where $a \in A$ and $b \in B$. Since A and B have (PD_1), then we have nothing to prove whenever $a = 0$ or $b = 0$. Now to avoid triviality we may consider C is not a small submodule of M. Since $(a+b)R \leq aR + bR$, we have aR or bR is not small in M. Without loss of generality we may assume aR is not small in M; hence it is not small in A. By Proposition 3.5, aR is a summand of A, and hence aR is

B-Pprojective, hence aR is bR-projective. Since $aR \oplus bR = (a+b)R + bR$, we have by Lemma 3.2 that there is $N \leq (a+b)R$ such that $aR \oplus bR = N \oplus bR$. It follows that $(a+b)R = N \oplus [(a+b)R \cap bR]$. Since C is a local module, and bR is not contained in C, we have that $C = N$. The proof is ended once we show that N is a summand of M. It is clear that $aR \oplus B = N + B$; and hence $N \cap B = N \cap (N \oplus bR) \cap B = (aR \oplus bR) \cap B \cap N = bR \cap N = 0$. As $aR \leq^{\oplus} A$, we have $N \oplus B = aR \oplus B \leq^{\oplus} M$; i. e. $C = N \leq^{\oplus} M$.

The converse follows from Proposition 2.7. □

Proposition 3.7. *If $M = A \oplus B$ is an R-module, where A is simple and B has a composition series $0 \leq K \leq B$, then M has (PD_1).*

Proof. Let C be a nonzero proper cyclic submodule of M. Without loss of generality we may assume that $C = (a+b)R$, where $0 \neq a \in A$ and $0 \neq b \in B$. It is clear that $C + A = A \oplus bR$, and that bR is either K or B. Since A is simple, we have that either $A \leq C$ or $A \cap C = 0$. If $A \leq C$, then $bR = K$ and $C = A \oplus K$ (due to C a proper submodule of M), where $A \leq^{\oplus} M$ and $K \ll M$. On the other hand, if $A \cap C = 0$ then $C \oplus A = A \oplus bR$. If $bR = B$, then $C \leq^{\oplus}$M, and if $bR = K$, then $C \cong K$ is a simple module. But $M = C + B$ with $C \nleq B$; and hence $C \cap B = 0$, which yields $C \leq^{\oplus} M$. Therefore M has (PD_1). □

It is known that if a module $M = A \oplus B$ is *quasi-discrete*, then A and B are relatively *projective*. We were aiming to show that if we replace *quasi-discreteness* by the condition (PD_1) for $M = A \oplus B$, then A and B are relatively *Pprojective*. This goal could not be achieved as it is shown in the following example.

Example 3.8. *We show that if a module $M = A \oplus B$ has (PD_1), then A need not be B-Pprojective. In fact if $M = C(p) \oplus C(p^2)$, then M as a $\mathbb{Z}$-module has (PD_1), by Proposition 3.7, while $C(p)$ is not $C(p^2)$-Pprojective.*

Recall from [2] that: a module A is called *B-Sprojective* if for every epimorphism $g : B \to N$ and every homomorphism $\varphi : A \to N$, with *kerg* has a *supplement* submodule in $ker(\varphi \oplus g)$, there exists a homomorphism ψ: A → B such that $\psi g = \varphi$.

Lemma 3.9. ([2], Proposition 3.2) *The following are equivalent for a module $M = A \oplus B$.*

(1) A is B-Sprojective.

(2) $M = B \oplus Y$ for every supplement Y of B in M.

Definition 3.10. An R-module M is called *principally supplemented* (for short *P-supplemented*) in case of, for each $m \in M$, if $M = mR + N$ then mR contains a supplement of N in M. For a submodule B of M, M is *B-supplemented* if A contains a supplement of B in M whenever $M = B + A$.

From Corollary 2.15, every module with (PD_1) over a principal ideal ring R is *P-supplemented.*

Lemma 3.11. ([2], Proposition 3.3) *The following are equivalent for a module $M = A \oplus B$.*

(1) A is B-projective;

(2) A is B-Sprojective, and M is B-supplemented.

Lemma 3.12. *Let $M = A \oplus B$ be a module such that B has a finite uniform dimension. If A is a simple module, then A is B-Sprojective.*

Proof. Let $U.dim(B) = n$, then $U.dim(M) = n + 1$. Now let L be a supplement of B in M. It follows that $L/(L \cap B) \cong A$ is a simple module; and hence $L \cap B$ is a maximal submodule of L. But since $L \cap B \ll L$, it follows that L is a local module. Now if $L \cap B \neq 0$, then $U.dim(L) = U.dim(L \cap B)$ (due to the essentiality of L over $L \cap B$. In this case we have $n + 1 = U.dim(M) = U.dim(L + B) = U.dim(L) + U.dim(B) - U.dim(L \cap B) = n$, which is a contradiction. Hence $L \cap B = 0$, and $M = L \oplus B$. Therefore by Lemma 3.9 A is *B-Sprojective.* □

The following example shows that a module M with (PD_1) need not be mR-supplemented for all $m \in M$.

Example 3.13. *Consider the $\mathbb{Z}$-module $M = C(p) \oplus C(p^2)$. In Example 3.8, we have shown that M has (PD_1), and from Lemma 3.12 $C(p)$ is $C(p^2)$-Sprojective. By Lemma 3.11 and since $C(p)$ is not $C(p^2)$-projective, then M is not $C(p^2)$-supplemented.*

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