International Electronic Journal of Algebra
Volume 1 (2007) 1–10

ON n-SEMIHEREDITARY AND n-COHERENT RINGS

Xiaoxiang Zhang and Jianlong Chen

Received: 6 June 2006; Revised: 5 August 2006

Communicated by N. Ding

Abstract. Let R be a ring. For a fixed positive integer n, R is said to be left n-semihereditary in case every n-generated left ideal is projective. R is said to be weakly n-semihereditary if each n-generated left (and/or right) ideal is flat. Some properties of n-semihereditary rings, respectively, weakly n-semihereditary rings and n-coherent rings are investigated. It is also proved that R is left n-semihereditary if and only if it is left n-coherent and weakly n-semihereditary, if and only if the ring of $n \times n$ matrices over R is left 1-semihereditary if and only if the class of all n-flat right R-modules form the torsion-free class of a torsion theory. Some known results are extended or obtained as corollaries.

Mathematics Subject Classification (2000): 16D50, 16P70
Keywords: (weakly) n-semihereditary ring, n-coherent ring

1. Introduction

The motivation of this paper comes mainly from Dauns and Fuchs [8] and Samei [14]. By virtue of some results of [15,16,17,18], we investigate some classes of rings which can be regarded as generalizations of hereditary rings. Thus some known results are extended or obtained as corollaries.

There are many generalizations of hereditary rings such as, in literatures, semihereditary rings, p.p. rings and p.f. rings. A *left p.p. ring* R (i.e., every principal left ideal of R is projective as an R-module) is also called a *left Rickart ring* (see Lam [11]). There exists a left p.p. ring which is not right p.p. (see Chase [4] or Lam [11]). However the property that R is a *p.f. ring* (i.e., every principal left ideal of R is flat as an R-module) is left-right-symmetric (see Jøndrup [10] or Dauns and Fuchs [8] where a p.f. ring is said to be *torsion-free*). Indeed, for a ring R and a fixed positive integer n, Jøndrup [10] has proved that if all n-generated left ideals

This work is supported in part by the NSFC of China (No.10571026) and the Foundation of Graduate Creative Program of JiangSu (xm04-10) and the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE, P.R.China.

are flat, then all n-generated right ideals are flat, too. Such a ring R was also discussed by Shamsuddin [15] under the terminology *n-semihereditary*. But we shall call it a *weakly n-semihereditary* ring since Zhu and Tan define a *left* (resp. *right*) *n-semihereditary ring* to be a ring R whose n-generated left (resp. right) ideals are all projective. Thus "left p.p." = "left Rickart" = "left 1-semihereditary" and "p.f." = "torsion-free" = "weakly 1-semihereditary".

Let X be a completely regular space and $C(X)$ the ring of all continuous real-valued functions defined on X. We refer the reader to [9] for details on $C(X)$. It is interesting that $C(X)$ is semihereditary if and only if $C(X)$ is a p.p. ring if and only if $C(X)$ is coherent if and only if X is basically disconnected (see [13, Theorem 1.1, Corollary 1.4 and Theorem 2.2]) while $C(X)$ is p.f. if and only if X is an F-space (see [1,12]).

Following Shamsuddin [15], a ring R is said to be *left n-coherent* if every n-generated left ideal of R is finitely presented. Note that this definition is at odds with another definition on n-coherence (see [5] and [7]) and R is a left n-coherent if and only if it is left $(1, n)$-coherent in the sense of [16] where R is *left (m, n)-coherent* means that every n-generated left submodule of the free left R-module R^m is finitely presented. It is easy to see that 1-coherence is nothing other than p-coherence in the sense of [6]. Moreover, "left semihereditary" = "left n-semihereditary for all positive integers n" and "left coherent" = "left n-coherent for all positive integers n" = "left (m, n)-coherent for all positive integers m and n".

For two fixed positive integers m and n, a right R-module M is called *(m, n)-flat* in [16,17] in case the canonical map $M \otimes_R I \to M \otimes_R R^m$ is a monomorphism for all n-generated submodule I of the left R-module R^m. A submodule K of a right R-module F is said to be *(m, n)-pure in F* if the canonical map $K \otimes_R (R^m/I) \to F \otimes_R (R^m/I)$ is a monomorphism for all n-generated submodule I of the left R-module R^m, or equivalently, if $K^m \cap F^n C = K^n C$ for every $n \times m$ matrix $C = (c_{ij})_{n\times m}$ over R, where $F^n C = \{(\sum_{i=1}^n x_i c_{i1}, \cdots, \sum_{i=1}^n x_i c_{im}) \mid (x_1, \cdots, x_n) \in F^n\}$ and the meaning of $K^n C$ is similar to $F^n C$ (see [17] for details). (m, n)-flatness and (m, n)-purity of left modules are defined similarly. Obviously, $(1, n)$-flatness coincides with n-flatness in the sense of [2]. In particular, 1-flatness coincides with torsion-freeness in the sense of [8].

According to Samei [14], a module A over a commutative ring R is *quasi-torsion-free* relative to an exact sequence of R-modules

$$0 \to K \to F \xrightarrow{\phi} A \to 0,$$

where F is a flat submodule of a free R-module, if for all $r \in R$ and $x \in A$ such that $rx = 0$ there are $x' \in F$ and $k \in K$ with $\phi(x') = x$ and $rx' = rk$. It is proved that A is quasi-torsion-free relative to every exact sequence if it is quasi-torsion-free relative to one exact sequence (see [14, Lemma 2.2]). Thus the notion of quasi-torsion-freeness is independent from the choice of the exact sequence. We shall transfer this notion as well as some results of Samei [14] to modules over any ring (which need not be commutative) and show that quasi-torsion-freeness coincides with 1-flatness.

The ring of all 2×2 matrices over a p.f. ring may fail to be p.f. (see [8]). There are analogous phenomena for p.p. rings and 1-coherent rings. Therefore it seems interesting to investigate a ring R such that the ring of all $n \times n$ matrices over R is, respectively, p.f., left 1-coherent and left p.p. for some fixed positive integer $n > 1$.

Throughout R is an associative ring with identity, all modules are unitary and $C(X)$ is the ring of all continuous real-valued functions on a completely regular space X. For two fixed positive integers m and n, we write $R^{m\times n}$ for the set of all $m \times n$ matrices over R and write $R^n = R^{1\times n}$ (resp., $R_n = R^{n\times 1}$). In general, for a right R-module M, we write M^n for the set of all formal $1 \times n$ matrices whose entries are elements of M.

2. Main Results

Let us start with the following characterizations of weakly n-semihereditary rings.

Proposition 2.1. *Let R be ring and n a fixed positive integer. Then the following are equivalent:*

(1) R is weakly n-semihereditary.

(2) Every n-generated submodule of an n-flat R-module is flat.

(3) Every submodule of a flat R-module is n-flat.

(4) Every n-generated submodule of a flat R-module is flat.

(5) Every submodule of a projective R-module is n-flat.

(6) Every n-generated submodule of a projective R-module is flat.

(7) Every submodule of a free R-module is n-flat.

(8) Every n-generated submodule of a free R-module is flat.

(9) For each $1 \leq m \in \mathbb{N}$, every submodule of an (m, n)-flat R-module is (m, n)-flat.

(10) For each $1 \leq m \in \mathbb{N}$, every n-generated submodule of an (m, n)-flat R-module is flat.

Proof. (1)⇒(2). Suppose R is weakly n-semihereditary, then every n-generated submodule of an n-flat module is n-flat [15, §5.(f)]. But an n-generated n-flat module is flat [15, §5.(a)].

(1)⇒(3) follows from [15, §5.(f)].

(8)⇒(9). Let I be an n-generated submodule of the free module R^m and K a submodule of an (m,n)-flat right R-module M. We have $\mathrm{Tor}_2^R(M/K, R^m/I) \cong \mathrm{Tor}_1^R(M/K, I) = 0$ by (8) and $\mathrm{Tor}_1^R(M, R^m/I) = 0$ by the characterization of (m,n)-flatness [17, Theorem 4.3(3)]. Thus the following long exact sequence yields that $\mathrm{Tor}_1^R(K, R/I) = 0$

$$\cdots \to \mathrm{Tor}_2^R(M/K, R^m/I) \to \mathrm{Tor}_1^R(K, R^m/I) \to \mathrm{Tor}_1^R(M, R^m/I) \to \cdots$$

(2)⇒(4)⇒(6)⇒(8)⇒(1), (3)⇒(5)⇒(7)⇒(1) and (9)⇒(10)⇒(2)are obvious. □

Recall that a ring R is *left* (resp., *right*) *Bezout* if every finitely generated left (resp., right) ideal is principal. It is easy to see that a module is flat if and only if it is n-flat for all positive integers n. On the other hand, it is well known that a module M is flat if and only if every finitely generated submodule of M is flat since M is a direct limit of its finitely generated submodules and every direct limit of a direct system of flat modules is flat. So we have the following corollary which transfers [14, Theorem 2.5 and Corollary 2.6] to a general case.

Corollary 2.2. *The following are equivalent for any ring R.*

(1) Every left ideal of R is flat.

(2) Every finitely generated left ideal of R is flat.

(3) Every submodule of a free left R-module is flat.

(4) Every finitely generated submodule of a free left R-module is flat.

All these conditions are equivalent to the corresponding ones on the opposite sides. Moreover, if R be a right or left Bezout ring then the above conditions are equivalent to

(5) R is p.f..

The equivalence (1)⇔(2)⇔(5) of the following theorem has been established by Zhu and Tan [18, Theorem 2].

Theorem 2.3. *Let R be ring and n a fixed positive integer. Then the following are equivalent.*

(1) R is left n-semihereditary.

(2) R is weakly n-semihereditary and left n-coherent.

(3) R is weakly n-semihereditary and left (m,n)-coherent for each $1 \le m \in \mathbb{N}$.

(4) R is weakly n-semihereditary and left (n,n)-coherent.

(5) Every torsion-less right R-module is n-flat.

(6) Every torsion-less right R-module is (m,n)-flat for each $1 \leq m \in \mathbb{N}$.

(7) Every torsion-less right R-module is (n,n)-flat.

Proof. (1)⇒(3) follows from the fact that R is left n-semihereditary if and only if every n-generated submodule of a projective left R-module is projective [18, Theorem 1].

(3)⇒(4)⇒(2) is obvious.

(2)⇒(1). Note that an n-generated module is projective if and only if is flat and finitely presented.

(2)⇔(5), (3)⇔(6) and (4)⇔(7) are immediate consequence of Proposition 2.1 and [16, Theorem 5.7]. □

Corollary 2.4. *(1)* ([8, Lemma 4.2]) *Every right* (*or left*) *p.p. ring is a torsion-free ring.*

(2) ([8, Theorem 4.5]) *R is a right p.p. ring if and only if R a is torsion-free ring in which the right annihilators of elements are finitely generated.*

(3) ([8, Theorem 4.6]) *The following are equivalent for a torsion-free ring R.*

(i) R is a right p.p. ring.

(ii) Direct products of torsion-free left R-modules are again torsion-free.

(iii) R^I is a torsion-free left R-module for any set I.

(4) The following are equivalent for a p.f. ring R.

(i) R is a right p.p. ring.

(ii) R is right 1-coherent.

(iii) R is right pseudo-coherent.

The proof of [13, Theorem 2.2] virtually shows that X is basically disconnected if $C(X)$ is 1-coherent. According to [9, 1H.2 and Theorem 14.25], $C(X)$ is Bezout and (equivalently) p.f. if X is basically disconnected. Therefore, the following result (essentially due to Neville [13]) is an immediate consequence of Corollary 2.4(4).

Corollary 2.5. *The following are equivalent for a Hausdorff completely regular space X.*

(1) $C(X)$ is a p.p. ring.

(2) $C(X)$ is a semihereditary ring.

(3) $C(X)$ is a 1-coherent ring.

(4) $C(X)$ is a pseudo-coherent ring.

(5) $C(X)$ is a coherent ring.

The following theorem and the proof of it are motivated by [8, Theorem 6.1].

Theorem 2.6. *Let n be a positive integer and n-$\mathcal{F}$ the class of all n-flat left R-modules. Then n-$\mathcal{F}$ is a torsion-free class of a torsion theory in R-Mod if and only if R is right n-semihereditary.*

Proof. Note that n-$\mathcal{F}$ is closed under isomorphism and extensions. By [15, §5.(f)], n-$\mathcal{F}$ is closed under submodules if and only if R is weakly n-semihereditary. Furthermore, by [15, §5.(h)], n-$\mathcal{F}$ is closed under direct products if and only if R is right n-coherent. Thus the result follows from Theorem 2.3. □

Corollary 2.7. *(1) The torsion-free left R-modules form the torsion-free class of a torsion theory in R-Mod if and only if R is right p.p..*

(2) The flat left R-modules form the torsion-free class of a torsion theory in R-Mod if and only if R is right semihereditary.

Note that (1) of Corollary 2.7 removes the hypotheses that R is a torsion-free ring in [8, Theorem 6.1].

Given a right R-module M and $n \in \mathbb{N}$, we shall define the *n-flat dimension* n-$\mathrm{fd}_R M$ of M to be the smallest integer $d \geq 0$ such that

$$\mathrm{Tor}_{d+k}^R(M, R/I) = 0 \text{ for all } n\text{-generated left ideal } I \text{ and } k \geq 1.$$

The *right global n-flat dimension* r-n-fd(R) of R is defined to be the supremum of the n-flat dimensions of all right R-modules. n-flat dimension of a left R-module and the left global n-flat dimension l-n-fd(R) of R can be defined similarly. Obviously, m-$\mathrm{fd}_R M \leq n$-$\mathrm{fd}_R M$ whenever $m \leq n$, and the flat dimension of M $\mathrm{fd}_R M = \sup\{ n\text{-}\mathrm{fd}_R M \mid n \in \mathbb{N} \}$. Moreover, for each fixed $1 \leq n \in \mathbb{N}$, we have r-n-fd$(R) = 0$ $\Leftrightarrow$ every right R-module is n-flat $\Leftrightarrow$ every right R-module is 1-flat $\Leftrightarrow$ R is von Neumann regular $\Leftrightarrow$ every left R-module is 1-flat $\Leftrightarrow$ every left R-module is n-flat $\Leftrightarrow$ l-n-fd$(R) = 0$.

Theorem 2.8. *Let R be ring and n a fixed positive integer. Then the following are equivalent.*

(1) R is weakly n-semihereditary.

(2) The right global n-flat dimension r-n-fd(R) of R is at most 1.

(3) The left global n-flat dimension l-n-fd(R) of R is at most 1.

Proof. We need only to show that (1) $\Leftrightarrow$ (2). (1) $\Leftrightarrow$ (3) follows similarly.

(1) $\Rightarrow$ (2). Suppose that R is weakly n-semihereditary. For any right R-module M and any n-generated left ideal I of R, we have $\mathrm{Tor}_{1+k}^R(M, R/I) \cong \mathrm{Tor}_k^R(M, I) = 0$ for every $1 \leq k \in \mathbb{N}$. Thus the global n-flat dimension of R is at most 1.

(2) $\Rightarrow$ (1). Let I an n-generated left ideal I of R. We have $\mathrm{Tor}_1^R(M,I) \cong \mathrm{Tor}_2^R(M,R/I) = 0$ for every left R-module M. That is, I is flat. $\square$

Corollary 2.9. ([3, Theorem 4.1]) *For any ring R, the following statements are equivalent.*

(1) R is left semihereditary.

(2) R is left coherent and the weak dimension of R is at most 1.

(3) Every torsion-less right R-module is flat.

Theorem 2.10. *Let R be ring, n a fixed positive integer and $S = R^{n\times n}$. Then*

(1) R is weakly n-semihereditary if and only if S is p.f..

(2) R is left (n,n)-coherent if and only if S is left 1-coherent.

(3) R is left n-semihereditary if and only if S is left p.p..

Proof. (1) It is easy to see that R is weakly n-semihereditary if and only if every n-generated submodule K of the right R-module R_n is flat. Now suppose that R is weakly n-semihereditary and A is an arbitrary matrix in S. Given any equation $AB = 0$ in S with $A = (\alpha_1, \cdots, \alpha_n)$. The n-generated submodule $K = \alpha_1 R + \cdots + \alpha_n R$ of the right R-module R_n is flat and hence (n,n)-flat. By the characterization of (n,n)-flat module [16,17] we have $A = (\alpha_1, \cdots, \alpha_n) = YC$ and $CB = 0$ for some $l \in \mathbb{N}$, $Y \in K^l$ and $C \in R^{l\times n}$. But $Y \in K^l$ implies $Y = (\alpha_1, \cdots, \alpha_n)D = AD$ for some $D \in R^{n\times l}$. Thus, $A = YC = ADC$ and $DCB = 0$. By [8, Proposition 3.2], S is p.f..

Conversely, let $K = \alpha_1 R + \cdots + \alpha_n R$ be an n-generated submodule of the right R-module R_n. For all $A \in K^n$, $B \in R^{n\times n}$ with $AB = 0$, by regarding $AB = 0$ as an equation in S we have $A = AC$ and $CB = 0$ for some $C \in S$ since S is p.f.. Therefore K is (n,n)-flat.

(2) Suppose R is left (n,n)-coherent and $A \in S$. Then the left annihilator of A in R^n $\mathbf{l}_{R^n}(A) = R^k B$ for some $1 \le k \in \mathbb{N}$ and $B \in R^{k\times n}$. We may assume that $k = tn$ for some $t \in \mathbb{N}$. Thus it is easy to see that $\mathbf{l}_S(A)$ is t-generated. Therefore S is left 1-coherent. The converse implication is easy.

(3) follows from (1), (2) and Theorem 2.3(4). $\square$

Corollary 2.11. *Let R be ring. Then*

(1) The weak dimension $WD(R) \le 1$ if and only if $R^{n\times n}$ is p.f. for all positive integers n if and only if $R^{n\times n}$ has weak dimension ≤ 1 for all positive integers n.

(2) R is left coherent if and only if $R^{n\times n}$ is left 1-coherent for all positive integers n if and only if $R^{n\times n}$ is left coherent for all positive integers n.

(3) R is left semihereditary if and only if $R^{n\times n}$ is left p.p. for all positive integers n if and only if $R^{n\times n}$ is left semihereditary for all positive integers n.

The following result uncovers the relationship between quasi-torsion-freeness and 1-flatness by taking $m = n = 1$.

Proposition 2.12. *Given two fixed positive integers m and n and an epimorphism of right R-modules $\phi : F \to M$ with kernel K, the following are equivalent.*

(1) K is (m, n)-pure in F, i.e. $K^m \cap F^n C = K^n C$ for every $C \in R^{n\times m}$.

(2) For every matrix $C \in R^{n\times m}$ and $x \in M^n$ such that $xC = 0 \in M^m$, there are $x' = (x'_1, \cdots, x'_n) \in F^n$ and $k \in K^n$ such that $(\phi(x'_1), \cdots, \phi(x'_n)) = x$ and $x'C = kC$.

Furthermore, if F is (m, n)-flat then the above conditions are equivalent to

(3) M is (m, n)-flat.

Proof. (1)⇔(2) follows from a slight modification the proof of [14, Theorem 2.3]. (1)⇔(3) is an immediate consequence of [17, Theorem 3.6]. □

It is easy to see that every flat right R-module is 1-flat and the converse holds if R is left Bezout. Therefore [14, Lemma 2.2 and Theorem 2.3] can be obtained by taking $m = n = 1$ in Proposition 2.12.

We complete this note with the following proposition which is motivated by [8, Theorem 6.2]. Recall that a right R-module M is *n-injective* if every R-linear map from every n-generated right ideal of R to M extends to one from R_R to M.

Proposition 2.13. *Given a positive integer n, the following conditions are equivalent for a right n-coherent ring R.*

(1) n-flat left R-modules are flat.

(2) n-injective right R-modules are FP-injective.

(3) n-injective pure-injective right R-modules are injective.

Moreover, R is right coherent if any one of the above equivalent conditions holds.

Proof. (2)⇒(3)⇒(1) follows for any ring R. Indeed, an FP-injective and pure-injective right R-module is injective. So (2)⇒(3) follows. To see (3)⇒(1), we need only note that the character module, $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, of any left R-module M is a pure-injective right R-module and that ${}_RM$ is n-flat (flat) if and only if M^+ is n-injective (injective).

(1)⇒(2) By the n-coherence of R and [16, Theorem 5.7], we have that a right R-module is n-injective (FP-injective) if and only if its character module is n-flat (flat) as a left R-module. Therefore the result follows.

Suppose (1) holds, then every direct product of copies of ${}_RR$ is flat by hypothesis. Therefore R is right coherent. □

Acknowledgements. The authors are grateful to the referee for a very careful reading of the manuscript and several helpful comments that improved the paper.

References

[1] H. Al-Ezeh, M. A. Natsheh and D. Hussein, *Some properties of the ring of continuous functions*, Arch. Math., 51 (1988), 60-64.

[2] D. D. Anderson and D. E. Dobbs, *Flatness, LCM-stability, and related module-theoretic properties*, J. Algebra, 112 (1988), 139-150.

[3] S. U. Chase, *Direct Products of Modules*, Trans. Amer. Math. Soc., 97 (1960), 457-473.

[4] S. U. Chase, *A generalization of the ring of triangular matrices*, Nagoya Math. J., 18 (1961), 13-25.

[5] J. L. Chen and N. Q. Ding, *On n-coherent rings*, Comm. Algebra, 24 (1996), 3211-3216.

[6] J. L. Chen and Y. Q. Zhou, *Extentions of injectivity and coherent rings*, Comm. Algebra, 34 (2006), 275-288.

[7] D. L. Costa, *Parameterizing families of non-notherian rings*, Comm. Algebra, 22 (1994), 3997-4011.

[8] J. Dauns and L. Fuchs, *Torsion-freeness in rings with zero-divisors*, J. Algebra Appl., 3 (2004), 221-237.

[9] L. Gillman and M. Jerison, Rings of Continuous Functions, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

[10] S. Jøndrup, *p.p. rings and finitely generated flat ideals*, Proc. Amer. Math. Soc., 28 (1971), 431-435.

[11] T. Y. Lam, Lectures on Modules and Rings, Springer-Verlag, New York, 1998.

[12] C. W. Neville, *Flat $C(X)$-modules and F spaces*, Math. Proc. Cambridge Philos. Soc., 106 (1989), 237-244.

[13] C. W. Neville, *When is $C(X)$ a coherent ring*? Proc. Amer. Math. Soc., 110 (1990), 505-508.

[14] K. Samei, *Flat submodules of free modules over commutative Bezout rings*, Bull. Austral. Math. Soc., 71 (2005), 113-119.

[15] A. Shamsuddin, *n-injective and n-flat modules*, Comm. Algebra, 29 (2001), 2039-2050.

[16] X. X. Zhang, J. L. Chen and J. Zhang, *On (m,n)-injective modules and (m,n)-coherent rings*, Algebra Colloq., 12 (2005), 149-160.

[17] Z. M. Zhu, J. L. Chen and X. X. Zhang, *On (m,n)-purity of modules*, East-west J. Math., 5 (2003), 35-44.

[18] Z. M. Zhu and Z. S. Tan, *On n-semihereditary rings*, Scientiae Mathematicae Japonicae, 62 (2005), 455-459.

Xiaoxiang Zhang and Jianlong Chen
Department of Mathematics, Southeast University
Nanjing 210096, P. R. China
e-mail: z990303@seu.edu.cn (X. Zhang), jlchen@seu.edu.cn (J. Chen)