International Electronic Journal of Algebra
Volume 2 (2007) 208-225

D-NICE SYMMETRIC POLYNOMIALS WITH FOUR ROOTS OVER INTEGRAL DOMAINS D OF ANY CHARACTERISTIC

Jonathan Groves

Received: 6 April 2007; Revised: 11 June 2007
Communicated by Derya Keskin Tütüncü

Abstract. Let D be any integral domain of any characteristic. A polynomial $p(x) \in D[x]$ is D-nice if $p(x)$ and its derivative $p'(x)$ split in $D[x]$. We give a complete description of all D-nice symmetric polynomials with four roots over integral domains D of any characteristic not equal to 2 by giving an explicit formula for constructing these polynomials and by counting equivalence classes of such D-nice polynomials. To illustrate our results, we give several examples we have found using our formula. We conclude by stating the open problem of finding all D-nice symmetric polynomials with four roots over integral domains D of characteristic 2 and all D-nice polynomials with four roots over all integral domains D of any characteristic.

Mathematics Subject Classification (2000): Primary: 13F20; Secondary: 11C08, 26C10
Keywords: Critical point, D-nice polynomial, integral domain, nice polynomial, polynomial, root, symmetric polynomial

1. Introduction

Let D be any integral domain of any characteristic. We say that a polynomial $p(x) \in D[x]$ *splits* in $D[x]$ if, for some $n \geq 0$, $p(x)$ can be written in the form $p(x) = a(x-r_1)(x-r_2)\cdots(x-r_n)$ where $a \neq 0$ and $a, r_1, \ldots, r_n \in D$. We say that $p(x) \in D[x]$ is *D-nice* or (*nice in D*) if $p(x)$ and its derivative $p'(x)$ split in $D[x]$. By our definition of splitting, if $p(x)$ is D-nice, then $p'(x) \neq 0$. Most mathematicians who had researched $\mathbb{Z}$-nice polynomials were interested in constructing polynomials with integer coefficients, roots, and critical points—polynomials that are "nice" for calculus students to sketch (see [1], for example). We found this problem of extending these earlier results in $\mathbb{Z}$ to all integral domains D of any characteristic worth further study.

Since many earlier papers use the term *nice* instead of $\mathbb{Z}$-*nice*, we too will often use the term *nice* instead of $\mathbb{Z}$-*nice*. The first paper to consider nice polynomials [4] was published in 1960. This paper, along with the papers [1] and [12], give a formula for constructing all nice cubics. The paper [3] gives the first five known examples of nice nonsymmetric quartics and a formula for all nice symmetric quartics. In 1989 the problem of finding all nice quartics using elliptic curves was added to the list of unsolved problems [10] in the *American Mathematical Monthly.* Ten years later, the problem of constructing, describing, and classifying nice polynomials was added to the the list of unsolved problems [11] in the *American Mathematical Monthly.* Other papers soon followed, including the main paper on nice polynomials [2], the submitted paper [5] with a different approach to D-nice polynomials, the accepted paper [6] with several new results on nice symmetric and antisymmetric polynomials, the accepted paper [7] with a complete description of all nice polynomials with three distinct roots (which generalizes the known results on nice cubics), and the accepted paper [8] with a complete description of all nice symmetric polynomials with four distinct roots (which generalizes the known results on nice symmetric quartics). The paper [5] takes a new approach to D-nice polynomials by considering the relations between the roots and critical points of all polynomials over integral domains D of characteristic 0. The main result in [5] gives several properties of D-nice polynomials $p(x)$ where D is an integral domain of characteristic 0 satisfying $D \cap \mathbb{Q} = \mathbb{Z}$ and where the degree of $p(x)$ is a prime power in D [5, Theorem 6.3]. All the results in [5], including the main one, and all the results in [6]-[8] follow from these relations. The relations as stated in [5] and [6] depend on the degree of $p(x)$, not on the maximal number of distinct roots of $p(x)$. The relations in [7] and [8] given for polynomials with up to three and four distinct roots, respectively, do not depend on the degree of the polynomials. The latest paper [9] gives the relations between the roots and critical points of all polynomials $p(x) \in D[x]$ where D has any characteristic and where the relations depend on the maximal number of distinct roots of $p(x)$, not on the degree of $p(x)$.

The only results on nice polynomials of degree higher than 5 that has been published so far are [2, Theorem 2] and the formula for nice symmetric quartics in [3]. But most of the results in [5]-[9] apply to nice polynomials of degrees higher than 5. Furthermore, [5] and [9] are the only papers besides this one that consider integral domains other than $\mathbb{Z}$ or $\mathbb{Q}$.

We now solve the problem of constructing all D-nice symmetric polynomials with four distinct roots (where D has any characteristic other than 2). This is a natural generalization of the problem in [8] of finding all nice symmetric polynomials with four distinct roots. We explain later why this problem remains unsolved for integral domains of characteristic 2. We find an explicit formula for constructing D-nice symmetric polynomials with four roots, and we count equivalence classes of such D-nice polynomials. We begin by defining the type of a polynomial and what it means for two D-nice polynomials to be equivalent.

2. Preliminaries

The *type* of a polynomial is a list of the multiplicities of its distinct roots. For example, all polynomials of the type (6,5,5,3) are of the form $p(x) = a(x-r_1)^6(x-r_2)^5(x-r_3)^5(x-r_4)^3$ where r_1, r_2, r_3, and r_4 are all distinct and $a \neq 0$.

Most of the earlier papers on nice polynomials note that horizontal translations by integers, horizontal or vertical stretches by integer factors, and reflections over the coordinate axes transform a nice polynomial $p_1(x)$ into another nice polynomial $p_2(x)$. Each of these transformations has an inverse transformation that transforms $p_2(x)$ into $p_1(x)$. The paper [5] extends these transformations and their inverses to all integral domains D of characteristic 0 [5, Proposition 2.1 and Corollary 2.2]. But it is easy to extend these to integral domains of characteristic $p > 0$. For convenience, we will refer to these transformations by using the same geometric descriptions we use in $\mathbb{Z}$ or in $\mathbb{Q}$, even if D is not ordered. Thus, we define the horizontal translation of $p(x)$ by $a \in D$ units to be $p(x-a)$. The horizontal stretch and compression of $p(x)$ by a nonzero factor of $a \in D$ are defined by $p(x/a)$ and $p(ax)$, respectively. If necessary, the division occurs in the field of fractions of D. The vertical stretch and compression of $p(x)$ by a nonzero factor of $a \in D$ are defined by $ap(x)$ and $\frac{1}{a}p(x)$, respectively. Reflections of $p(x)$ over the x- and y-axes are defined by $-p(x)$ and $p(-x)$, respectively. All these transformations preserve the type of a polynomial.

Two newly discovered transformations that behave similarly are the power transformation and its inverse, the root transformation [6, Theorem 2.1]: For any natural number n, a polynomial $p(x)$ is nice iff $[p(x)]^n$ is nice. This result clearly holds in any integral domain D of characteristic 0 and holds in integral domains of characteristic $p > 0$ as long as n is not a multiple of p. If n is a multiple of p, then $\frac{d}{dx}[p(x)]^n = 0$; so, by our convention, $[p(x)]^n$ is not D-nice. It is obvious that the root transformation transforms a D-nice polynomial $p_1(x)$ into another D-nice polynomial $p_2(x)$ iff $p_1(x) = [p_2(x)]^n$ for some natural number n and some D-nice polynomial $p_2(x)$. The power transformation and the root transformation do not preserve the type of a polynomial. More precisely, if $p(x)$ is of the type $(m_1, \ldots, m_s)$, then $[p(x)]^n$ is of the type $(nm_1, \ldots, nm_s)$.

Translating horizontally $a \in D$ units, stretching horizontally or vertically by factors of $a \in D$, reflecting over the coordinate axes, and taking powers are transformations we call *equivalence tranformations*. The corresponding *inverse transformations* are translating horizontally $-a \in D$ units, compressing horizontally or vertically by factors of $a \in D$, reflecting over the coordinate axes, and taking roots. Although equivalence transformations do transform a D-nice polynomial into a D-nice polynomial, these inverse transformations do not necessarily transform a D-nice polynomial into another D-nice polynomial. For example, a horizontal or vertical compression may transform a D-nice polynomial $p_1(x)$ into a polynomial $p_2(x)$ where $p_2(x)$ is nice in the field of fractions of D rather than in D. The root transformation applied to arbitrary D-nice polynomials may result in nonpolynomials.

Since any finite composition of equivalence tranformations transform a D-nice polynomial $p_1(x)$ into another D-nice polynomial $p_2(x)$, we say that the two D-nice polynomials $p_1(x)$ and $p_2(x)$ are *equivalent* whenever $p_1(x)$ can be transformed into $p_2(x)$ and vice-versa by a finite composition of equivalence transformations or their inverse transformations. Since all equivalence transformations and their inverses, except the power transformation and its inverse, preserve the type of a polynomial, when we count equivalence classes of D-nice polynomials, we count the number of equivalence classes of D-nice polynomials of the same type. We note that, by the power transformation and its inverse the root transformation, the number of equivalence classes of D-nice polynomials of the type $(m_1, \ldots, m_s)$ is the same as the number of equivalence classes of D-nice polynomials of the type $(nm_1, \ldots, nm_s)$ for any $n > 1$ if D has characteristic 0 and any $n > 1$ that is not a multiple of p if D has characteristic $p > 0$. In fact, we may consider these types to be the "same" type.

A polynomial $p(x) \in D[x]$ is *symmetric* if there exists a unique c in some extension field of D, called the *center*, such that $p(c - x) = p(c + x)$ for all x, and $p(x)$ *antisymmetric* if $p(c - x) = -p(c + x)$ for all x. We first prove that the center c of a D-nice symmetric or antisymmetric polynomial $p(x)$ lies in D. We then use this result to make an assumption that greatly simplifies our work.

Proposition 2.1. *Let D be any integral domain of any characteristic, and let $p(x)$ be a D-nice symmetric or antisymmetric polynomial with* (*unique*) *center c. Then $c \in D$.*

Proof. Suppose $p(x)$ is a D-nice antisymmetric polynomial with center c, so $p(c-x) = -p(c+x)$ for all x. Let $x = 0$, so $p(c) = -p(c)$. If D has characteristic not equal to 2, then $p(c) = 0$, so $c \in D$.

Now suppose $p(x)$ is a D-nice symmetric polynomial with center c. Then $p(c+x) = p(c-x)$ for all x. Differentiating both sides of this equation gives $-p'(c+x) = p'(c-x)$, so $p'(x)$ is antisymmetric with center c. As shown above, if D has characteristic not equal to 2, then $p'(c) = 0$, so $c \in D$.

Finally, if D has characteristic 2, then $p(x) = p(-x) = -p(x)$ for all $p(x)$, so any polynomial (whether D-nice or not) is symmetric or antisymmetric with center 0. Since $0 \in D$, the proof is complete. □

By Proposition 2.1, we may center any D-nice symmetric polynomial at 0. Since the translated polynomial is equivalent to the original one, we may assume that all D-nice symmetric polynomials have center 0. Multiplying $p(x)$ by a nonzero element of D (which is a vertical stretch or compression) results in an equivalent D-nice polynomial, so we may assume $p(x)$ is monic. These two assumptions greatly simplify the problem of constructing and counting equivalence classes of D-nice symmetric polynomials with four distinct roots. Thus, with these assumptions, such polynomials have the form $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ if the characteristic of D is not equal to 2.

3. The Case of Integral Domains D of Characteristic 0

We first consider the problem of constructing and counting equivalence classes of all D-nice symmetric polynomials with four distinct roots over integral domains D of characteristic 0. To construct these, we use Lemma 3.1 below, which gives the relations between the roots and critical points of all symmetric polynomials in $\mathrm{QF}(D)[x]$ with four roots where $\mathrm{QF}(D)$ is the field of fractions of D. As we will see below, this set of relations is the key tool in finding our formula for constructing these D-nice symmetric polynomials. Since it is easier to find a formula that gives $\mathrm{QF}(D)$-nice symmetric polynomials than one that gives D-nice symmetric polynomials, we state Lemma 3.1 in terms of $\mathrm{QF}(D)$-nice symmetric polynomials. When we find a $\mathrm{QF}(D)$-nice symmetric polynomial with our formula, we may stretch it horizontally to obtain an equivalent D-nice symmetric polynomial. Thus, our formula may be used to construct and count equivalence classes of D-nice symmetric polynomials of the same type with four distinct roots. The paper [8] takes the same approach in constructing and counting equivalence classes of nice symmetric polynomials with four roots using the formula for $\mathbb{Q}$-nice symmetric polynomials with four roots. Lemma 3.1 below follows directly from [9, Corollary 4.3], which gives the relations between the roots and critical points of all symmetric polynomials $p(x) \in D[x]$ with an even number of roots. Lemma 3.1 in [8] gives the relations between the roots and critical points of all polynomials in $\mathbb{Q}[x]$ with four roots. Later, in the same paper, these relations are used to derive the relations for symmetric polynomials in $\mathbb{Q}[x]$ with four roots [8, Lemma 4.1]. We mention this because Lemma 3.1 below may be proven without using [9, Corollary 4.3] by using arguments similar to the ones used in [8].

Lemma 3.1. [*Key lemma.*] *Let D be an integral domain of characteristic* 0, *and let $QF(D)$ be the field of fractions of D. Let $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2} \in QF(D)[x]$ be a symmetric polynomial of degree $d = 2m_1 + 2m_2$ with derivative $p'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c^2)$ and with four roots in $QF(D)$. Then $p(x)$ is $QF(D)$-nice iff there exists a number c in $QF(D)$ such that*

$$m_2 r_1^2 + m_1 r_2^2 = (m_1 + m_2)c^2. \tag{3.1}$$

Remark. By Lemma 3.1, all solutions to (3.1) in QF(D) give us all QF(D)-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2). Thus, to find our formula, we solve (3.1) in QF(D).

Our formula is stated in the following theorem.

Theorem 3.2. [*Main theorem: The formula for constructing D-nice symmetric polynomials with four roots.*] *Let D be an integral domain of characteristic* 0, *and let $QF(D)$ be its field of fractions. The polynomial $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2} \in QF(D)[x]$ of degree $d = 2m_1 + 2m_2$ with derivative $p'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c^2)$ is $QF(D)$-nice iff*

$$c = \frac{m_2 a^2 + m_1 b^2}{2m_2 a + 2m_1 b}, \tag{3.2}$$

$$r_1 = c - a, \tag{3.3}$$

$$r_2 = c - b \tag{3.4}$$

where a and b are nonzero elements of $QF(D)$ such that $2m_2 a + 2m_1 b \neq 0$ or, equivalently, $a \neq (-\frac{m_1}{m_2})b$.

Proof. Set $r_1 = c - a$ and $r_2 = c - b$ where $a, b \in D$ are nonzero. Since D has characteristic 0, $r_1 \neq c$ and $r_2 \neq c$, so a and b are nonzero. Substituting these expressions for r_1 and r_2 into (3.1) and simplifying the equation gives the equation $m_2a^2 + m_1b^2 = (2m_2a + 2m_1b)c$. Solving this for c gives the formula above.

To complete the proof, we prove that if a and b are nonzero and if $a = (-m_1/m_2)b$, then the relation (3.1) has no solution. Again let $r_1 = c - a$ and $r_2 = c - b$ to obtain the equation $m_2a^2 + m_1b^2 = (2m_2a + 2m_1b)c$. Now assume $a = (-m_1/m_2)b$. Substituting into our equation, we have $m_2[(-m_1/m_2)b]^2 + m_1b^2 = 0$, or $(m_1^2/m_2 + m_1)b^2 = 0$, which is a contradiction since $b \neq 0$ and $m_1^2/m_2 + m_1 \neq 0$.

The converse follows from Lemma 3.1. □

Remarks. (1). The formulas in [8] for nice symmetric polynomials with four roots are similar to formulas (3.2)-(3.4) but are not exactly the same as these. The proof in [8] assumes $r_1 < r_2$ and then uses Rolle's theorem for $\mathbb{R}$ to conclude $r_1 < c < r_2$. Thus, Rolle's theorem guarantees the existence of $a, b > 0$ such that $r_1 = c - a$ and $r_2 = c + b$; but, for arbitrary integral domains D, we cannot use Rolle's theorem to establish the existence of such an a and b since D is not necessarily ordered. However, we may conclude such a nonzero a and b exist as we did in the proof above, but all we may say now is that r_1, r_2, and c are all distinct. Thus, formulas (3.2)-(3.4) are not exactly the same as the formulas in [8].

(2). Since Theorem 3.2 is an equivalence, we may find representatives of all the equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) by choosing nonzero a and b so that $a \neq (-\frac{m_1}{m_2})b$ and using formulas (3.2)-(3.4) above. In this sense, our formulas give all examples of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2).

We now give two examples of $\mathbb{Z}[i]$-nice symmetric polynomials we have found using formulas (3.2)-(3.4).

Example 3.3. Suppose $m_1 = 2$ and $m_2 = 5$. Let $a = 2i$ and $b = 1 + i$. Then the formulas give $c = \frac{1}{37} + \frac{31}{37}i$, $r_1 = \frac{1}{37} - \frac{43}{37}i$, $r_2 = -\frac{36}{37} - \frac{6}{37}i$. Stretching our polynomial horizontally by a factor of 37, we see that our polynomial is equivalent to the following $\mathbb{Z}[i]$-nice polynomial:

$$\begin{aligned} p(x) &= [x^2 - (1 - 43i)^2]^2[x^2 - (36 + 6i)^2]^5, \\ p'(x) &= 14x[x^2 - (1 - 43i)^2][x^2 - (36 + 6i)^2]^4[x^2 - (1 + 31i)^2]. \end{aligned} \tag{3.5}$$

Example 3.4. Suppose $m_1 = 6$ and $m_2 = 5$. Let $a = 3+i$ and $b = 6-i$. Then the formulas give $c = \frac{3198}{1301} - \frac{473}{1301}i$, $r_1 = -\frac{705}{1301} - \frac{1774}{1301}i$, $r_2 = -\frac{4608}{1301} + \frac{828}{1301}i$. Stretching our polynomial horizontally by a factor of 1301, we see that our polynomial is equivalent to the following $\mathbb{Z}[i]$-nice polynomial:

$$\begin{aligned} p(x) &= [x^2 - (705 + 1774i)^2]^6[x^2 - (4608 - 828i)^2]^5, \qquad (3.6)\\ p'(x) &= 22x[x^2 - (705 + 1774i)^2]^5[x^2 - (4608 - 828i)^2]^4 \\ &\quad \cdot [x^2 - (3198 - 473i)^2]. \end{aligned}$$

These two examples are not equivalent to nice polynomials (i.e., to $\mathbb{Z}$-nice polynomials). To see this, first note that any two D-nice symmetric polynomials with four roots centered at 0 are equivalent using only horizontal stretches or compressions. Because horizontal stretches and compressions preserve the ratio a/b where $a = c - r_1$ and $b = c - r_2$ and because nice symmetric polynomials with four distinct roots centered at the origin satisfy $a/b \in \mathbb{Q}$, we see that these two examples are not equivalent to nice polynomials because the ratios $a/b \notin \mathbb{Q}$.

We now determine conditions on a and b so that our formulas (3.2)-(3.4) give a D-nice symmetric polynomial with exactly four roots. To do so, we find all ratios a/b so that the formulas (3.2)-(3.4) fail to give a D-nice symmetric polynomial with exactly four roots. We consider the three cases $r_1 = \pm r_2$, $r_1 = 0$, and $r_2 = 0$. By the proof of Theorem 3.2, c is undefined iff $a/b = -m_1/m_2$, and no D-nice symmetric polynomials exist when $a/b = -m_1/m_2$; so the case where c is undefined has already been considered.

Case 1. Suppose a and b are chosen so that $r_1 = \pm r_2$. We solve the two equations $r_1 = \pm r_2$ for a in terms of b, which is equivalent to solving the equations $c - a = \pm(c - b)$. Solving $c - a = c - b$ for a gives us $a = b$. We now solve $c - a = -(c - b)$ for a, which is equivalent to the equation $2c = a + b$. Using (3.2), we now have $\frac{m_2a^2+m_1b^2}{m_2a+m_1b} = a+b$. By clearing fractions, we obtain the equation $m_2ab+m_1ab = 0$. If a and b are both nonzero (so ab is nonzero), then this last equation is equivalent to $m_1 = -m_2$, which is a contradiction since D has characteristic 0. Thus, $a = 0$ or $b = 0$ but not both. Furthermore, it is easy to see that if $a = b \neq 0$, then $r_1 = r_2$ and if $a = 0$ or $b = 0$ (but not both), then $r_1 = -r_2$. Thus, $r_1 = \pm r_2$ iff $a = 0$ or $b = 0$ (but not both) or $a = b \neq 0$. Thus, we require that a and b be distinct nonzero elements of $\mathrm{QF}(D)$.

Case 2. Suppose a and b are chosen so that $r_1 = 0$. By (3.3), $r_1 = 0$ iff $c = a$, so we solve the equation $c = a$ for the ratio a/b. Replacing c by the expression in (3.2) and clearing fractions gives us the equation $m_1b^2 - 2m_1ab - m_2a^2 = 0$. We now divide both sides by $b^2 \neq 0$ and rewrite the equation as $m_2(\frac{a}{b})^2 + 2m_1\frac{a}{b} - m_1 = 0$, where a/b is regarded as a variable. By the quadratic formula,

$$\frac{a}{b} = \frac{-m_1 \pm \sqrt{m_1(m_1+m_2)}}{m_2}. \tag{3.7}$$

These steps are reversible, so choosing a/b equal to one of these numbers gives $r_1 = 0$. Thus, $r_1 = 0$ iff a/b is one of the numbers in (3.7). However, r_1 will never be 0 if $m_1(m_1 + m_2) = \frac{dm_1}{2}$ is not a square in D.

Case 3. Suppose a and b are chosen so that $r_2 = 0$. By (3.4), $r_2 = 0$ iff $c = b$, so we solve the equation $c = b$ for the ratio a/b. Proceeding as in Case 2, we obtain the equation $m_2(\frac{a}{b})^2 - 2m_2\frac{a}{b} - m_1 = 0$. By the quadratic formula,

$$\frac{a}{b} = \frac{m_2 \pm \sqrt{m_2(m_1+m_2)}}{m_2}. \tag{3.8}$$

These steps are reversible, so choosing a/b equal to one of these numbers gives $r_2 = 0$. Thus, $r_2 = 0$ iff a/b is one of the numbers in (3.8). However, r_2 will never be 0 if $m_2(m_1 + m_2) = \frac{dm_2}{2}$ is not a square in D.

Cases 1-3 lead to the following proposition.

Proposition 3.5. [*Conditions on a/b so that the formulas give a D-nice symmetric polynomial with exactly four roots.*] *Assume the hypotheses of Theorem* 3.2*, and suppose nonzero a and b are chosen. Then the formulas* (3.2)-(3.4) *give exactly four roots iff the ratio a/b is not an element of the set $\{1, -\frac{m_1}{m_2}, R_1, R_2, R_3, R_4\}$ where R_1 and R_2 are the numbers defined in* (3.7) *and R_3 and R_4 are the numbers defined in* (3.8)*.*

We now count equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) over integral domains D of characteristic 0. We first note that any two D-nice symmetric polynomials obtained from formulas (3.2)-(3.4) are equivalent iff their ratios a/b are equal. We state this result more precisely and prove it below.

Proposition 3.6. [*Necessary and sufficient conditions for two D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) to be equivalent.*] *Let D be an integral domain of characteristic* 0*. Suppose $p_1(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ where $p_1'(x) = dx(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}(x^2 - c_1^2)$ and $p_2(x) = (x^2 - s_1^2)^{m_1}(x^2 - s_2^2)^{m_2}$ where $p_2'(x) = dx(x^2 - s_1^2)^{m_1-1}(x^2 - s_2^2)^{m_2-1}(x^2 - c_2^2)$ are two D-nice symmetric polynomials with four distinct roots. Let $a_1 = c_1 - r_1$, $b_1 = c_1 - r_2$, $a_2 = c_2 - s_1$, and $b_2 = c_2 - s_2$. Then $p_1(x)$ and $p_2(x)$ are equivalent iff $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.*

Proof. Since $p_1(x)$ and $p_2(x)$ are symmetric, monic, and centered at 0, $p_1(x)$ and $p_2(x)$ are equivalent using only horizontal stretches and compressions. Since horizontal stretches and compressions preserve the ratio a/b defined by formulas (3.3)-(3.4) above, the result follows. □

Corollary 3.7. [*Number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) where D has characteristic* 0.] *Let D be an integral domain of characteristic* 0*, and let m_1 and m_2 be any two positive integers. Then there exist infinitely many equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2).*

Proof. Fix m_1 and m_2. Define the set $A = \{1, -\frac{m_1}{m_2}, R_1, R_2, R_3, R_4\}$ as in Proposition 3.5. Choose all nonzero ratios $a/b \in \mathrm{QF}(D) - A$, and use formulas (3.2)-(3.4) to find r_1, r_2, and c. Then each ratio a/b gives a QF(D)-nice symmetric polynomial with four distinct roots, which is equivalent to a D-nice symmetric polynomial with four distinct roots. Using Proposition 3.6 and that $\mathrm{QF}(D) - A$ is infinite, we see that any two distinct examples we have constructed are not equivalent, so the result follows. □

4. The Case of Integral Domains D of Characteristic $p > 2$

We now consider the analogous problem over integral domains D of characteristic $p > 2$. First note that if D has characteristic 2, then $p(x) = p(-x) = -p(x)$, so any $p(x) \in D[x]$ is both symmetric and antisymmetric with center 0. Thus, solving the problem of constructing all D-nice symmetric polynomials with four roots where D has characteristic 2 is the same as solving the problem of constructing all D-nice polynomials with four roots where D has characteristic 2. However, the problem of constructing all D-nice polynomials with four distinct roots over any integral domain D, including those of characteristic 2, has not been solved, so we assume D has characteristic $p > 2$. We give more details in Section 5.

The proof of Lemma 3.1 does not use the assumption that D has characteristic 0, only the assumption that the degree of $p'(x)$ is $d - 1$ if $p(x)$ has degree d. Thus, Lemma 3.1 does apply to $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ if the degree d of $p(x)$ is not a multiple of p since $p'(x)$ has degree $d - 1$, so formulas (3.2)-(3.4) may be used to construct these types of D-nice symmetric polynomials. But we will show in Theorem 4.1 below that if the degree of $p(x)$ is not a multiple of p but one of the multiplicities m_1 and m_2 is a multiple of p, then we need not use formulas (3.2)-(3.4) to construct these types of D-nice symmetric polynomials. However, Lemma 3.1 does not apply to $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ if the degree d of $p(x)$ is a multiple of p since $p'(x)$ has degree $d - 2$ or less. Therefore, D-nice symmetric polynomials of these types need to be considered separately since formulas (3.2)-(3.4) cannot be used to construct these. But we do not need to derive a formula for constructing D-nice symmetric polynomials of these types, as we will show in Theorem 4.1 below. Then we will show formulas are required for the construction of D-nice symmetric polynomials with four roots only when none of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p.

Theorem 4.1. [*Results for the cases where D has characteristic $p > 2$ and exactly one of m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p.*] *Let D be an integral domain of characteristic $p > 2$. Suppose $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2} \in D[x]$ has degree $d = 2m_1 + 2m_2$, and suppose exactly one of the numbers d, m_1, and m_2 is a multiple of p. Then $p(x)$ is D-nice iff $p(x)$ splits in $D[x]$ and $p'(x) \neq 0$.*

Proof. To prove this result, we consider three cases. In each case, we differentiate $p(x)$ by the product rule and simplify the derivative. In all cases, it is easy to see that $p'(x)$ splits in $D[x]$ if $p'(x) \neq 0$.

Case 1. m_1 is a multiple of p, and m_2 and d are not. Then $p'(x) = 2m_2x(x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2-1}$, which always splits in $D[x]$ since $p'(x)$ is never 0.

Case 2. m_2 is a multiple of p, and m_1 and d are not. Then $p'(x) = 2m_1x(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2}$, which always splits in $D[x]$ since $p'(x)$ is never 0.

Case 3. d is a multiple of p, but m_1 and m_2 are not. Then $p'(x) = (-2m_1r_2^2 - 2m_2r_1^2)x(x^2 - r_1^2)^{m_1-1}(x^2 - r_2^2)^{m_2-1}$, splits in $D[x]$ if $m_1r_2^2 \neq m_2r_1^2$. Otherwise, $p'(x) = 0$. □

Remarks. (1). If more than one of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p, then $p'(x) = 0$. By our definition, any polynomial $p(x) \in D[x]$ where $p'(x) = 0$ is not D-nice.

(2). By Theorem 4.1 and its proof, if exactly one of the numbers m_1, m_2, $d = 2m_1 + 2m_2$ is a multiple of p, then we may construct D-nice symmetric polynomials of these types simply by choosing polynomials that split in $D[x]$ if d is not a multiple of p. If d is a multiple of p, we choose polynomials $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ that split in $D[x]$ where $m_1r_2^2 \neq m_2r_1^2$, which is equivalent to the condition that $r_2/r_1 \neq \pm\sqrt{m_2/m_1}$. It is easy to see that if m_2/m_1 is not a square in $\mathrm{QF}(D)$, then $p'(x)$ is never 0. Thus, the problem of $p'(x) = 0$ where exactly one of m_1, m_2, and

$d = 2m_1 + 2m_2$ is a multiple of p arises only when d is a multiple of p and m_2/m_1 is a square in $\mathrm{QF}(D)$.

A polynomial $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2} \in D[x]$ that splits where none of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p is not necessarily D-nice. For example, $p(x) = (x^2 - 1^2)(x^2 - 6^2)$ splits in $\mathbb{Z}/(7)[x]$, but $p(x)$ is not $\mathbb{Z}/(7)$-nice. So we need formulas (3.2)-(3.4) to construct D-nice symmetric polynomials of these types.

Theorem 4.2. [*Extension of Theorem* 3.2 *and Proposition* 3.5 *to the case where none of* m_1, m_2, $d = 2m_1 + 2m_2$ *are multiples of* p.] *Suppose D is an integral domain of characteristic $p > 2$ and $QF(D)$ is its field of fractions. Suppose $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2} \in QF(D)[x]$ has degree $d = 2m_1 + 2m_2$, and suppose none of m_1, m_2, and d are multiples of p. Then Theorem* 3.2 *and Proposition* 3.5 *apply to $p(x)$.*

Proof. The proof is the same as the proofs of Theorem 3.2 and Proposition 3.5. □

We now give three more examples we have found using formulas (3.2)-(3.4).

Example 4.3. Let $D = \mathbb{Z}/(11)$, $m_1 = 4$, and $m_2 = 3$. Choose $a = 2$ and $b = 3$. Then the formulas give $c = 5$, $r_1 = 3$, and $r_2 = 2$. So our D-nice symmetric polynomial is

$$\begin{aligned} p(x) &= (x^2 - 3^2)^4(x^2 - 2^2)^3, \\ p'(x) &= 3x(x^2 - 3^2)^3(x^2 - 2^2)^2(x^2 - 5^2). \end{aligned} \tag{4.1}$$

Example 4.4. Let $D = \mathbb{Z}/(7)$, $m_1 = 2$, and $m_2 = 1$. Choose $b = 1$, and $a = 2$. Then the formulas give $c = 6$, $r_1 = 4$, and $r_2 = 5$. So our D-nice symmetric polynomial is

$$\begin{aligned} p(x) &= (x^2 - 4^2)^2(x^2 - 5^2), \\ p'(x) &= 6x(x^2 - 4^2)(x^2 - 6^2). \end{aligned} \tag{4.2}$$

Example 4.5. Let $D = \mathbb{Z}/(5)[y]$, $m_1 = m_2 = 1$. Choose $a = 4y^4 + 3y^3 + y^2 + y + 2$ and $b = y^3 + 4y + 1$. Then the formulas give $c = \frac{(4y^4+3y^3+y^2+y+2)^2+(y^3+4y+1)^2}{2(4y^4+3y^3+y^2+y+2)+2(y^3+4y+1)}$, $r_1 = c - (4y^4 + 3y^3 + y^2 + y + 2)$, and $r_2 = c - (y^3 + 4y + 1)$. Stretching our polynomial horizontally by a factor of $2(4y^4 + 3y^3 + y^2 + y + 2) + 2(y^3 + 4y + 1)$, we see that our polynomial is equivalent to the D-nice symmetric polynomial

$$\begin{aligned} p(x) &= [x^2 - (4y^8 + 3y^7 + 3y^6 + 2y^5 + y^4 + y^2 + y + 3)^2] \\ &\quad \cdot [x^2 - (y^8 + y^7 + y^4 + 4y^3 + 4y^2 + 3y + 4)^2], \\ p'(x) &= 4x[x^2 - (y^8 + 4y^7 + 3y^6 + 4y^5 + y^4 + y^3 + y^2 + 2y)^2]. \end{aligned} \tag{4.3}$$

We now count equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) over integral domains D of characteristic $p > 2$. We first assume D is infinite.

Corollary 4.6. [*Number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) where D has characteristic $p > 2$ and is infinite.*] *Suppose D is an infinite integral domain of characteristic $p > 2$, and suppose not all of m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p. Then there exist infinitely many equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2).*

Proof. Assume the hypotheses in the statement above, but first assume none of m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p. Then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is infinite. To see this, we use the same argument in the proof of Corollary 3.7.

Now assume exactly one of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p. Then, by Theorem 4.1, $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ is D-nice iff $p(x)$ splits in $D[x]$ and $p'(x) \neq 0$. By Remark 2 after Theorem 4.1, $p'(x) = 0$ iff d is a multiple of p and $r_2/r_1 = \pm\sqrt{m_2/m_1}$. Since any two D-nice polynomials $p_1(x) = (x^2 - s_1^2)^{m_1}(x^2 - s_2^2)^{m_2}$ and $p_2(x) = (x^2 - t_1^2)^{m_1}(x^2 - t_2^2)^{m_2}$ with nonzero roots are equivalent using only horizontal stretches or compressions, $p_1(x)$ and $p_2(x)$ are equivalent iff $s_2/s_1 = t_2/t_1$. Since D is infinite and the ratio r_2/r_1 for $p(x)$ above may assume any value in $\mathrm{QF}(D)$ except 0, ± 1, and possibly two other values—the number of equivalence classes of such D-nice symmetric polynomials is infinite. $\square$

We now consider those types of D-nice symmetric polynomials with four roots in which exactly one of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p and D is finite. We need to consider several cases, as the proof of Theorem 4.1 and Remark 2 following the proof suggest.

Corollary 4.7. [*Number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) where D has characteristic $p > 2$, D is finite, and exactly one of m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p.*] *Suppose D is an* *integral domain with p^n elements where $p > 2$. Suppose exactly one of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ is a multiple of p.*

(a) *If d is not a multiple of p, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 3$.*

(b) *If d is a multiple of p and $m_1 = m_2$ in D or if d is a multiple of p and m_2/m_1 is not a square in D, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 3$.*

(c) *If d is a multiple of p and if $m_1 \neq m_2$ in D and if m_2/m_1 is a square in D, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 5$.*

Proof. (a). We use a similar argument as in the second part of the proof of Corollary 4.6 above. Since D is finite, $\mathrm{QF}(D) = D$. However, note here that, for any such D-nice symmetric polynomial $p(x) = (x^2 - r_1^2)^{m_1}(x^2 - r_2^2)^{m_2}$ with exactly four roots, r_2/r_1 may assume any value in D except $0, \pm 1$ since d is not a multiple of p. Thus, the number of equivalence classes of such D-nice symmetric polynomials is $p^n - 3$ if D has p^n elements.

(b). The proof is similar to the proof of part (a) above; but now note that if $m_1 = m_2$ in D, $\pm\sqrt{m_2/m_1} = \pm 1$, so r_2/r_1 may assume any value in D except $0, \pm 1$. If m_2/m_1 is not a square in D, then $p'(x) \neq 0$ for any $p(x) \in D[x]$ that splits. In either case, r_2/r_1 may assume any value in D except $0, \pm 1$.

(c). Again, use an argument similar to the ones in (a) and (b) above, but now note that r_2/r_1 may assume any value in D except $0, \pm 1, \pm\sqrt{r_2/r_1}$. By the hypotheses in (c), all five of these values are distinct, so the number of equivalence classes of such D-nice symmetric polynomials is $p^n - 5$. □

If D is finite, then counting equivalence classes of D-nice symmetric polynomials with four roots of those types in which the degree d and the multiplicities m_1 and m_2 are not multiples of p requires several cases because the problem of $r_1 = 0$ or $r_2 = 0$ for appropriate choices of a and b in formulas (3.2)-(3.4) may or may not arise. As mentioned in Section 3, at least one of these two problems arises if at least one of the two numbers $\frac{dm_1}{2}$ or $\frac{dm_2}{2}$ is a square in D. So we need to consider the case where neither of these two numbers is a square in D, the two cases where exactly one of these numbers is a square in D, and the case where both numbers are squares in D. Earlier, when we had counted equivalence classes of D-nice symmetric polynomials where D is infinite and formulas (3.2)-(3.4) apply, we did not need to consider these cases separately because the set $\mathrm{QF}(D) - A$ is infinite in all cases. However, if D is finite, the number of elements in $\mathrm{QF}(D) - A = D - A$ varies as m_1 and m_2 vary, even if D is fixed. Thus, if D is finite, we do need to consider these cases separately.

Corollary 4.8. [*Number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) where D has characteristic $p > 2$, D is finite, and none of m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p.*] *Let D be an integral domain with p^n elements where $p > 2$. Furthermore, suppose none of the numbers m_1, m_2, and $d = 2m_1 + 2m_2$ are multiples of p.*

(a) *If neither of the two numbers $\frac{dm_1}{2}$ and $\frac{dm_2}{2}$ are squares in D, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 3$.*
(b) *If exactly one of the two numbers $\frac{dm_1}{2}$ and $\frac{dm_2}{2}$ is a square in D, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 5$.*
(c) *If both of the numbers $\frac{dm_1}{2}$ and $\frac{dm_2}{2}$ are squares in D, then the number of equivalence classes of D-nice symmetric polynomials of the type (m_1, m_1, m_2, m_2) is $p^n - 7$.*

Proof. (a). We follow the argument in the proof of Corollary 3.7, but we now note that the ratio a/b may assume any value in D except 0, 1, and $-m_1/m_2$ since none of R_1, R_2, R_3, and R_4 defined in Proposition 3.5 lie in D. Thus, by Proposition 3.6, the number of equivalence classes of these types of D-nice symmetric polynomials is $p^n - 3$.

(b). Suppose $\frac{dm_1}{2}$ is a square in D, but $\frac{dm_2}{2}$ is not a square in D. Hence, $R_1, R_2 \in D$, but $R_3, R_4 \notin D$. Thus, the ratio a/b may assume any value in D except 0, 1, $-m_1/m_2$, R_1, and R_2. It is not hard to check that all five of these are distinct, so the number of values a/b may assume is $p^n - 5$, which is the number of equivalence classes of these types of D-nice symmetric polynomials.

The argument of the case where $\frac{dm_2}{2}$ is a square in D but $\frac{dm_1}{2}$ is not is similar.

(c). Now assume both of the numbers $\frac{dm_1}{2}$ and $\frac{dm_2}{2}$ are squares in D, so $R_1, R_2, R_3, R_4 \in D$. Thus, the ratio a/b may assume any value in D except 0, 1, $-m_1/m_2$, R_1, R_2, R_3, and R_4. It is not hard to check that all seven of these are distinct, so the number of values a/b may assume is $p^n - 7$, which is the number of equivalence classes of these types of D-nice symmetric polynomials. $\square$

5. Open Problems

As we have seen in the beginning of Section 4, the problem of constructing all D-nice symmetric polynomials over integral domains D of characteristic 2 remains unsolved because all polynomials $p(x)$ over integral domains D of characteristic 2 are symmetric with center 0. We state the relations between the roots and critical points of all polynomials $p(x) \in D[x]$ with four roots over integral domains D of characteristic 2 to show why this problem remains unsolved. As before, we may assume $p(x)$ is monic. By the equivalence of the horizontal translation, we may assume $p(x)$ has a root at 0. If the degree d of $p(x)$ is odd, then the degree of its derivative is $d-1$, so $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}$ with derivative $p'(x) = x^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}(x-c_1)(x-c_2)(x-c_3)$. Since d is odd and D has characteristic 2, $d = 1$ in D. We now state these relations.

Proposition 5.1. *Let D be an integral domain of characteristic 2. Suppose $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \in D[x]$ with derivative $p'(x) = x^{m_0-1}(x-r_1)^{m_1-1}(x-r_2)^{m_2-1}(x-r_3)^{m_3-1}(x-c_1)(x-c_2)(x-c_3)$ is a polynomial of odd degree $d = m_0+m_1+m_2+m_3$ with four roots in D. Then $p(x)$ is D-nice iff there exist c_1, c_2, and c_3 in D such that*

$$m_1r_1 + m_2r_2 + m_3r_3 = c_1 + c_2 + c_3, \tag{5.1}$$

$$(m_1+m_2)r_1r_2 + (m_1+m_3)r_1r_3 + (m_2+m_3)r_2r_3 = c_1c_2 + c_1c_3 + c_2c_3, \tag{5.2}$$

$$m_0r_1r_2r_3 = c_1c_2c_3. \tag{5.3}$$

Proof. This is a direct consequence of [9, Theorem 3.2]. These relations may also be established directly by using an argument similar to the one used to establish [8, Lemma 3.1]. □

If the degree d of $p(x)$ is even, then the degree of its derivative is $d-2$ or less. We now state the relations between the roots and critical points of $p(x) \in D[x]$ where D has characteristic 2 and the degree of $p(x)$ is even. Note that $p(x)$ still has the same form as above, but $p'(x)$ does not.

Proposition 5.2. *Let D be an integral domain of characteristic 2. Suppose $p(x) = x^{m_0}(x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \in D[x]$ is a polynomial of even degree $d = m_0+m_1+m_2+m_3$.*

(a) *Suppose the derivative* $p'(x)$ *has degree* $d-2$. *Then* $p(x)$ *with four roots in* D *is* D*-nice iff there exist* c_1 *and* c_2 *in* D *such that*

$$(m_1+m_2)r_1r_2+(m_1+m_3)r_1r_3+(m_2+m_3)r_2r_3 = (m_1r_1+m_2r_2+m_3r_3)(c_1+c_2), \tag{5.4}$$

$$(m_1+m_2+m_3)r_1r_2r_3 = (m_1r_1+m_2r_2+m_3r_3)c_1c_2. \tag{5.5}$$

(b) *Suppose the derivative* $p'(x)$ *has degree* $d-3$. *Then* $p(x)$ *with four roots in* D *is* D*-nice iff there exists* c *in* D *such that*

$$(m_1+m_2+m_3)r_1r_2r_3 = [(m_1+m_2)r_1r_2+(m_1+m_3)r_1r_3+(m_2+m_3)r_2r_3]c. \tag{5.6}$$

(c) *If the derivative* $p'(x)$ *has degree* $d-4$, *then* $p(x)$ *is* D*-nice iff* $p(x)$ *splits in* $D[x]$ *and* $p'(x) \neq 0$.

Proof. This is a direct consequence of [9, Theorem 3.4]. □

No formula has been found for the solutions in D to relations (5.1)-(5.3), to relations (5.4)-(5.5), and to relation (5.6). Thus, the problem of constructing all D-nice symmetric polynomials over integral domains of characteristic 2 remains unsolved. We conclude our discussion by formally stating this open problem.

Problem 5.3. Find formulas for all D-nice (symmetric) polynomials with four roots over integral domains D of characteristic 2 by solving the relations stated in Propositions 5.1 and 5.2 in D.

Problem 5.4. Which types of D-nice (symmetric) polynomials with four roots over integral domains D of characteristic 2 exist? For each type that exists, how many equivalence classes of that type are there?

As mentioned earlier, the following problem remains unsolved. We repeat it here for convenience.

Problem 5.5. Find formulas for all D-nice polynomials with four roots over all integral domains D of any characteristic, and use these formulas to count equivalence classes of all types of such D-nice polynomials.

References

[1] T. Bruggeman and T. Gush, Nice cubic polynomials for curve sketching, Math Magazine, 53(4) (1980), 233-234.

[2] R.H. Buchholz and J.A. MacDougall, When Newton met Diophantus: A study of rational-derived polynomials and their extensions to quadratic fields, J. Number Theory, 81 (2000), 210-233.

[3] C.K. Caldwell, Nice polynomials of degree 4, Math. Spectrum, 23(2) (1990), 36-39.

[4] M. Chapple, A cubic equation with rational roots such that it and its derived equation also has rational roots, Bull. Math. Teachers Secondary Schools 11 (1960), 5-7 (Republished in Aust. Senior Math. J., 4(1) (1990), 57-60).

[5] J.-C. Evard, *Polynomials whose roots and critical points are integers*, Submitted and posted on the Website of Arxiv Organization at the address http://arxiv.org/abs/math.NT/0407256.

[6] J. Groves, Nice symmetric and antisymmetric polynomials, To appear in Math. Gazette.

[7] J. Groves, Nice polynomials with three roots, To appear in Math. Gazette.

[8] J. Groves, Nice polynomials with four roots, To appear in Far East J. Math. Sci.

[9] J. Groves, A new tool for the study of D-nice polynomials, Version of January 26, 2007.

[10] R.K. Guy, Unsolved problems come of age, Amer. Math. Monthly, 96(10) (1989), 903-909.

[11] R. Nowakowski, Unsolved problems, 1969-1999, Amer. Math. Monthly, 106(10) (1999), 959-962.

[12] Karl Zuser, Uber eine gewisse Klasse von ganzen rationalen Funktionen 3. Grades, Elem. Math, 18 (1963), 101-104.

Jonathan Groves
Department of Mathematics
Patterson Office Tower 713
University of Kentucky
Lexington, KY 40506-0027
E-mail: JGroves@ms.uky.edu, Jonny77889@yahoo.com