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EXTENSIONS OF GM-RINGS OVER GENERALIZED POWER SERIES RINGS

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Abstract. Let R be a reduced ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid, it is shown that ring $[[R^{S,\leq}]]$ is a $GM-$ ring if and only if R is a $GM-$ring. We also investigate $GM-$ rings for some special Morita Contexts and module extensions over generalized power series rings.

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1. Introduction

All rings considered here are associative with identity and R denotes such a ring. We use $U(R)$ to denote the group of units of R. Any concept and notation not defined here can be found in [6, 7].

A ring R is said to be a $GM-$ ring provided that for any $x, y \in R$, there exist idempotents $e, f \in R$ and $u \in U(R)$ such that $x - eu, y - fu^{-1} \in U(R)$. A ring R is called a clean ring if for any $x \in R$, there exists $e^2 = e \in R$ such that $x - e \in U(R)$. Clearly, all clean rings are $GM-$ rings. Many examples and results of $GM-$ rings are given in [1, 2].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let $(S, \leq)$ be a strictly ordered monoid and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow. With pointwise addition and the operation of convolution

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v)$$

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where $X_s(f,g) = \{(u,v) \in S \times S | s = u+v, f(u) \neq 0, g(v) \neq 0\}$ is a finite set by [8, Theorem 4.1] for every s $\in S$ and $f, g \in [[R^{S,\leq}]]$, $[[R^{S,\leq}]]$ becomes a ring, with unit element e^*, namely

$$e^*(0) = 1, \quad e^*(s) = 0 \text{ for every } s \in S, \quad s \neq 0.$$

The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S. For any a $\in R$, $C_a \in [[R^{S,\leq}]]$ is given by $C_a(0) = a, C_a(s) = 0$ for all $0 \neq s \in S$. Ordered monoid $(S, \leq)$ is said to satisfy condition $(S0)$ in case $s \geq 0$ for all $s \in S$. Henceforth, unless otherwise mentioned, in this paper, $(S, \leq)$ will always denote a strictly ordered monoid satisfying condition (S0).

In this paper, we show that if R is a reduced ring, then ring $[[R^{S,\leq}]]$ is a $GM-$ ring if and only if R is a $GM-$ ring. We also investigate $GM-$ rings for some special Morita Contexts and module extensions rings over generalized power series rings. These given generalizations of [3, Theorem], [2, Theorem 6] and [2, Theorem 11].

2. Main results

Lemma 2.1. [6] *Let R be a ring, $M_{n\times n}(R)$ the ring of all $n \times n$ matrices with entries in R. Then $[[M_{n\times n}(R)^{S,\leq}]] \cong M_{n\times n}([[R^{S,\leq}]])$.*

Lemma 2.2. [8] *Let $(S, \leq)$ be a cancellative torsion-free strictly ordered monoid and satisfy condition (S0), and let $f \in [[R^{S,\leq}]]$. Then $f \in U([[R^{S,\leq}]])$ if and only if $f(0) \in U(R)$.*

Lemma 2.3. *Let R be a ring, and $e_1^2 = e_1, e_2^2 = e_2 \in R$. Then $[[(e_1Re_2)^{S,\leq}]] = C_{e_1}[[R^{S,\leq}]]C_{e_2}$.*

Proof. For any $f \in C_{e_1}[[R^{S,\leq}]]C_{e_2}$, there exists $g \in [[R^{S,\leq}]]$ such that $f = C_{e_1}gC_{e_2}$. Thus for any $s \in S$, we have $f(s) = (C_{e_1}gC_{e_2})(s) = C_{e_1}(0)(gC_{e_2})(s) = C_{e_1}(0)g(s)c_{e_2}(0) = e_1g(s)e_2 \in e_1Re_2$. So $f \in [[(e_2Re_2)^{S,\leq}]]$. Hence $C_{e_1}[[R^{S,\leq}]]C_{e_2} \subseteq [[(e_1Re_2)^{S,\leq}]]$. Conversely, for any $f \in [[(e_1Re_2)^{S,\leq}]]$ and any $s \in supp(f)$, there exists $r_s \in R$ such that $0 \neq f(s) = e_1r_se_2 \in e_1Re_2$. Define a map $g : S \longrightarrow R$ via

$$g(s) = \begin{cases} r_s, & s \in supp(f) \\ 0, & s \in S \setminus supp(f) \end{cases}$$

Clearly, $supp(g) = supp(f)$. Thus $g \in [[R^{S,\leq}]]$. For any $s \in supp(f), (C_{e_1}gC_{e_2})(s) = e_1g(s)e_2 = e_1r_se_2 = f(s)$, for any $s \in S\setminus supp(f), (C_{e_1}gC_{e_2})(s) = 0 = f(s)$. Thus $f = C_{e_1}gC_{e_2} \in C_{e_1}[[R^{S,\leq}]]C_{e_2}$. This implies that $[[(e_1Re_2)^{S,\leq}]] \subseteq C_{e_1}[[R^{S,\leq}]]C_{e_2}$. Therefore we have $[[(e_1Re_2)^{S,\leq}]] = C_{e_1}[[R^{S,\leq}]]C_{e_2}$. □

Lemma 2.4. *If R is a $GM-$ ring. Then $[[R^{S,\leq}]]$ is a $GM-$ ring.*

Proof. Let $f, g \in [[R^{S,\leq}]]$, There exist $e^2 = e, f^2 = f \in R$ and $u \in U(R)$ such that $f(0) - eu, g(0) - fu^{-1} \in U(R)$ by R is a $GM-$ ring. Since $C_u C_{u^{-1}} = e^*$, and $(f - C_e C_u)(0), (g - C_f C_u^{-1})(0) \in U(R)$, it is easy to see that $f - C_e C_u, g - C_f C_u^{-1} \in U([[R^{S,\leq}]])$ and $C_e^2 = C_e, C_f^2 = C_f, C_u \in U([[R^{S,\leq}]])$. Thus $[[R^{S,\leq}]]$ is a $GM-$ ring. □

Example 1 Let $\mathbf{N} \cup \{\mathbf{0}\}$ denote the monoid which consists of natural numbers and zero. If $S = \mathbf{N} \cup \{\mathbf{0}\}$ with the usual order. Then $[[R^{S,\leq}]] \cong R[[X]]$ (rings of formal power series in one indeterminate and coefficients in R). So if R is a $GM-$ ring, then $R[[X]]$ is also a $GM-$ ring. [2, Theorem 14]

Example 2 Let $S = N^n \cup \{0\}$, with the usual order($\text{II} \leq_i$), or the lexicographic ($lex \leq_i$) order, or the reverse lexicographic($revlex \leq_i$) order. If R is a $GM-$ ring, then $[[R^{N^n \cup\{0\},\text{II}\leq_i}]]$, $[[R^{N^n \cup\{0\},lex\leq_i}]]$, $[[R^{N^n \cup\{0\},revlex\leq_i}]]$ are also $GM-$ rings. Since rational number field Q and real number field $\mathbb{R}$ are $GM-$ rings, then $[[Q^{N^n \cup\{0\},\text{II}\leq_i}]]$, $[[Q^{N^n \cup\{0\},lex\leq_i}]]$, $[[Q^{N^n \cup\{0\},revlex\leq_i}]]$ and $[[\mathbb{R}^{N^n \cup\{0\},\text{II}\leq_i}]]$, $[[\mathbb{R}^{N^n \cup\{0\},lex\leq_i}]]$, $[[\mathbb{R}^{N^n \cup\{0\},revlex\leq_i}]]$ are $GM-$ rings.

Let $(S_1, \leq_1), (S_2\ \leq_2), \cdots, (S_n, \leq_n)$ be cancellative torsion-free strictly ordered monoids satisfying the condition (S0). If R is a $GM-$ ring, then $[[R^{S_1\times S_2\times\cdots\times S_n,\text{II}\leq_i}]]$, $[[R^{S_1\times S_2\times\cdots\times S_n,(lex\leq_i)}]]$ $[[R^{S_1\times S_2\times\cdots\times S_n,(revlex\leq_i)}]]$ are $GM-$ rings.

A ring R is called reduced if it has no nonzero nilpotent element. It was proved in [5, Lemma 3.4] that if R is a reduced ring, and $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. Then for every idempotent $f^2 = f \in [[R^{S,\leq}]]$, there exists an idempotent $e \in R$ such that $f = C_e$.

Lemma 2.5. *Let R be a reduced ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. If $[[R^{S,\leq}]]$ is a $GM-$ ring, then R is a $GM-$ ring.*

Proof. Let $a, b \in R$, then $C_a, C_b \in [[R^{S,\leq}]]$. Since $[[R^{S,\leq}]]$ is a $GM-$ ring, there exist $C_e^2 = C_e, C_f^2 = C_f \in [[R^{S,\leq}]]$ where $e^2 = e \in R, f^2 = f \in R$, and $\tau \in U([[R^{S,\leq}]])$ such that $C_a - C_e\tau, C_b - C_f\tau^{-1} \in U(([[R^{S,\leq}]])$. Thus $(C_a - C_e\tau)(0) = a - e\tau(0) \in U(R)$ and $(C_b - C_f\tau^{-1})(0) = b - f\tau^{-1}(0) \in U(R)$. This implies that R is a $GM-$ ring. □

Example 3 Let R be a reduced ring. If the formal power series ring $R[[X]]$ is a $GM-$ ring, then so is R by Lemma 2.5. This can be proved in a directly simple manner. Given any $x, y \in R$, we have $x, y \in R[[X]]$ as well. Thus we can find idempotents $e(x), f(x) \in R[[X]]$ and a unit $u(x) \in R[[X]]$ such that $x-e(x)u(x), y-f(x)u(x)^{-1} \in U(R[[X]])$. It is well know that $h(x) \in R[[X]]$ is a unit if and only if $h(0) \in R$ is a unit, and if R is a reduced ring, then the set of all idempotents in $R[[X]]$ equal to the set of all idempotents in R. Thus we know $x - e(0)u(0), y - f(0)u(0)^{-1} \in U(R)$, One easily checks that $e(0) = e, f(0) = f$ are idempotents and $u(0) \in R$ is a unit. Thus R is a $GM-$ ring.

Let $e_1, e_2, \cdots, e_n \in R$ be idempotents. Clearly,

$$\begin{pmatrix} C_{e_1}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_1}[[R^{S,\leq}]]C_{e_n} \\ \vdots & \ddots & \vdots \\ C_{e_n}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_n}[[R^{S,\leq}]]C_{e_n} \end{pmatrix}$$
$$= \left\{ \begin{pmatrix} C_{e_1}r_{11}C_{e_1} & \cdots & C_{e_1}r_{1n}C_{e_n} \\ \vdots & \ddots & \vdots \\ C_{e_n}r_{n1}C_{e_1} & \cdots & C_{e_n}r_{nn}C_{e_n} \end{pmatrix} r_{ij} \in [[R^{S,\leq}]](0 \leqslant i, j \leqslant n) \right\}$$

form a ring with the identity $diag(C_{e_1}, \cdots, C_{e_n})$.

Theorem 2.6. *Let $e_1, e_2, \cdots, e_n$ be idempotents of a ring R. If all e_iRe_i are $GM-$rings, then so is the ring*

$$\begin{pmatrix} C_{e_1}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_1}[[R^{S,\leq}]]C_{e_n} \\ \vdots & \ddots & \vdots \\ C_{e_n}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_n}[[R^{S,\leq}]]C_{e_n} \end{pmatrix}.$$

Proof. Clearly, the ring $\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}$ is a $GM-$ring by virtue of [2, Lemma 1]. Since

$$\begin{aligned} & \left[\left[\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}^{S,\leq}\right]\right] \\ \cong\ & \big[\big[(diag(e_1, \cdots, e_n)M_n(R)diag(e_1, \cdots, e_n))^{S,\leq}\big]\big] \\ \cong\ & [[(diag(e_1, \cdots, e_n)^{S,\leq}]][[(M_n(R))^{S,\leq}]][[(diag(e_1, \cdots, e_n)^{S,\leq}]] \\ \cong\ & diag(C_{e_1}, \cdots, C_{e_n})M_n([[R^{S,\leq}]])diag(C_{e_1}, \cdots, C_{e_n}) \end{aligned}$$

$$\cong \begin{pmatrix} C_{e_1}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_1}[[R^{S,\leq}]]C_{e_n} \\ \vdots & \ddots & \vdots \\ C_{e_n}[[R^{S,\leq}]]C_{e_1} & \cdots & C_{e_n}[[R^{S,\leq}]]C_{e_n} \end{pmatrix}.$$

Apply Lemma 2.4, we get the result. □

Let M be an $R-$ module. $[[M^{S,\leq}]]$ denotes the set of all maps $\phi : S \to M$ such that $supp(\phi) = \{s \in S | \phi(s) \neq 0\}$ is artinian and narrow. From [9], it is immediate that $[[M^{S,\leq}]]$ is an $[[R^{S,\leq}]]-$ module. For any $f \in [[R^{S,\leq}]], \phi \in [[M^{S,\leq}]]$ and $s \in S$, the scalar multiplication is defined as follow:

$$(f\phi)(s) = \sum_{(u,v)\in X_s(f,\phi)} f(u)\phi(v).$$

Let A_1, A_2, A_3 be associative rings with identity. Let M_{21}, M_{31}, M_{32} be $(A_2, A_1)-$, $(A_3, A_1)-, (A_3, A_2)-$bimodule, respectively. Let $\psi : M_{32} \bigotimes_{A_2} M_{21} \longrightarrow M_{31}$ be an $(A_3, A_1)-$homomorphism, and let

$$T = \begin{pmatrix} A_1 & 0 & 0 \\ M_{21} & A_2 & 0 \\ M_{31} & M_{32} & A_3 \end{pmatrix}, T^S = \begin{pmatrix} [[{A_1}^{S,\leq}]] & 0 & 0 \\ [[{M_{21}}^{S,\leq}]] & [[{A_2}^{S,\leq}]] & 0 \\ [[{M_{31}}^{S,\leq}]] & [[{M_{32}}^{S,\leq}]] & [[{A_3}^{S,\leq}]] \end{pmatrix},$$

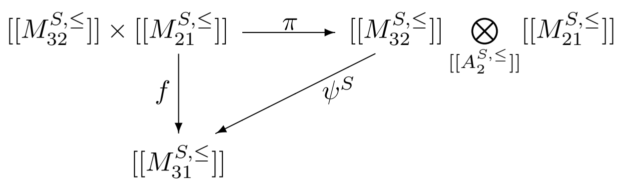
with the usual matrix operations (see[4]), T is a ring. Now we show that T^S is also a ring.

Theorem 2.7. *There exists a* $([[{A_3}^{S,\leq}]], [{A_1}^{S,\leq}]])-$*homomorphism*

$$\psi^S : [[{M_{32}}^{S,\leq}]] \bigotimes_{[[{A_2}^{S,\leq}]]} [[{M_{21}}^{S,\leq}]] \longrightarrow [[{M_{31}}^{S,\leq}]]$$

such that with the usual matrix operations , T^S *is a ring.*

Proof. Since M_{32}, M_{21} is $(A_3, A_2)-, (A_3, A_1)-$bimodule, respectively, according to [9], it is easy to see that $[[{M_{32}}^{S,\leq}]]$ is a $([[{A_3}^{S,\leq}]], [[{A_2}^{S,\leq}]])-$ bimodule, and $[[{M_{21}}^{S,\leq}]]$ is a $([[{A_2}^{S,\leq}]], [[{A_1}^{S,\leq}]])-$bimodule. Consider following diagram:



Let $n \in [[M_{32}^{S,\leq}]]$ and $m \in [[M_{21}^{S,\leq}]]$. Define a map

$$\alpha_{[n,m]} : S \longrightarrow M_{31}, \quad \alpha_{[n,m]}(s) = \sum_{(u,\, v) \in X_s(n,m)} \psi(n(u) \bigotimes m(v))$$

for any $s \in S$. It is clearly that $supp(\alpha_{[n,m]}) \subseteq supp(n) + supp(m)$, thus $\alpha_{[n,m]} \in [[M_{31}^{S,\leq}]]$.

Define a map $f : [[M_{32}^{S,\leq}]] \times [[M_{21}^{S,\leq}]] \longrightarrow [[M_{31}^{S,\leq}]]$, where $f((n,m)) = \alpha_{[n,m]}$ for any $(n,m) \in [[M_{32}^{S,\leq}]] \times [[M_{21}^{S,\leq}]]$. Let $n_1, n_2 \in [[M_{32}^{S,\leq}]], m \in [[M_{21}^{S,\leq}]]$. By the preceding discussions, there exist $\alpha_{[n_1,m]}, \alpha_{[n_2,m]}, \alpha_{[n_1+n_2,m]} \in [[M_{31}^{S,\leq}]]$. For all $s \in S$,

$$\begin{aligned}
\alpha_{[n_1+n_2,m]}(s) &= \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi((n_1+n_2)(u) \bigotimes m(v)) \\
&= \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi(n_1(u) \bigotimes m(v)) \\
&\quad + \sum_{(u,v)\in X_s(n_1+n_2,m)} \psi(n_2(u) \bigotimes m(v)).
\end{aligned}$$

If $(u',\ v') \in X_s(n_1,\ m)$, but $(u',\ v') \bar{\in} X_s(n_1+n_2,\ m)$, then we have $(n_1+n_2)(u') = 0$. So $n_2(u') \neq 0$, thus $(u',\ v') \in X_s(n_2, m)$ and $\psi(n_1(u') \bigotimes m(v')) + \psi(n_2(u') \bigotimes m(v')) = \psi((n_1(u') + n_2(u')) \bigotimes m(v')) = 0$. Likewise, if $(u',\ v') \in X_s(n_2,\ m)$, but $(u',\ v') \bar{\in} X_s(n_1+n_2,\ m)$, we also have $(u',\ v') \in X_s(n_1,\ m)$ and $\psi(n_1(u') \bigotimes m(v')) + \psi(n_2(u') \bigotimes m(v')) = \psi((n_1(u') + n_2(u')) \bigotimes m(v')) = 0$. So

$$\begin{aligned}
\alpha_{[n_1+n_2,\, m]}(s) &= \sum_{(u,\, v)\in X_s(n_1+n_2,\, m)} \psi(n_1(u) \bigotimes m(v)) \\
&\quad + \sum_{(u,\, v)\in X_s(n_1+n_2,m)} \psi(n_2(u) \bigotimes m(v)) \\
&= \sum_{(u,v)\in X_s(n_1,\ m)} \psi(n_1(u) \bigotimes m(v)) \\
&\quad + \sum_{(u,\, v)\in X_s(n_2,m)} \psi(n_2(u) \bigotimes m(v)) \\
&= \alpha_{[n_1,m]}(s) + \alpha_{[n_2,m]}(s) \\
&= (\alpha_{[n_1,m]} + \alpha_{[n_2,m]})(s).
\end{aligned}$$

Thus $\alpha_{[n_1+n_2,m]} = \alpha_{[n_1,m]} + \alpha_{[n_2,m]}$, hence $f((n_1+n_2,m)) = f((n_1,m)) + f((n_2,m))$. Analogously, we see that $f((n, m_1+m_2)) = f((n,m_1)) + f((n,m_2))$ for all $n \in [[M_{32}^{S,\leq}]], m_1, m_2 \in [[M_{21}^{S,\leq}]]$.

For any $n \in [[M_{32}^{S,\leq}]], \tau \in [[A_2^{S,\leq}]], m \in [[M_{21}^{S,\leq}]]$ and any $s \in S$, we have

$$\begin{aligned}
f((n\tau,\ m))(s) &= \alpha_{[n\tau,m]}(s) \\
&= \sum_{(u',u)\in X_s(n\tau,m)} \psi((n\tau)(u') \bigotimes m(u)) \\
&= \sum_{(u',u)\in X_s(n\tau,m)} \psi(\sum_{(v,w)\in X_{u'}(n,\tau)} (n(v)\tau(w) \bigotimes m(u)) \\
&= \sum_{(u',u)\in X_s(n\tau,m)} \sum_{(v,w)\in X_{u'(n,\tau)}} \psi(n(v)\tau(w) \bigotimes m(u)) \\
&= \sum_{(u',u)\in X_s(n\tau,m)} \sum_{(v,w)\in X_{u'}(n,\tau)} \psi(n(v)\tau(w) \bigotimes m(u)) \\
&\quad + \sum_{(v,w,u)\in X} \psi(n(v)\tau(w) \bigotimes m(u)) \\
&= \sum_{(v,w,u)\in X_s(n,\tau,m)} \psi(n(v)\tau(w) \bigotimes m(u)) \\
&= \sum_{(v,w,u)\in X_s(n,\tau,m)} \psi(n(v) \bigotimes \tau(w)m(u)) \\
&= f((n,\ \tau m))(s).
\end{aligned}$$

Where $X = \{(v,w,u) \in X_s(n,\tau,m) | n\tau(v+w) = 0\}$. Thus we have $f(n\tau,\ m) = f(n,\ \tau m)$ and hence f is a bilinear balanced morphism. Then there exists a homomorphism $\psi^S : [[M_{32}^{S,\leq}]] \bigotimes_{[[A_2^{S,\leq}]]} [[M_{21}^{S,\leq}]] \longrightarrow [[M_{31}^{S,\leq}]]$ such that the preceding diagram commutes.

Next, we check that ψ^S is a bimodule homomorphism. For any $a \in [[A_3^{S,\leq}]], n \in [[M_{32}^{S,\leq}]], m \in [[M_{21}^{S,\leq}]]$ and any $s \in S$.

$$\begin{aligned}
\psi^S(an,\, m)(s) &= \alpha_{[an,\, m]}(s) \\
&= \sum_{(u',u)\in X_s(an,m)} \psi((an)(u') \bigotimes m(u)) \\
&= \sum_{(u',u)\in X_s(an,m)} \psi(\sum_{v,w)\in X_{u'}(a,\, n)} (a(v)n(w) \bigotimes m(u)) \\
&= \sum_{(v,w,u)\in X_s(a,n,m)} \psi(a(v)n(w) \bigotimes m(u)) \\
&= \sum_{(v,w,u)\in X_s(a,n,m)} a(v)\psi(n(w) \bigotimes m(u)) \\
&= a\psi^S(n,m)(s).
\end{aligned}$$

Thus $\psi^S(an, m) = a\psi^S(n, m)$. This implies that ψ^S is a left $[[A_3^{S,\leq}]]-$ module homomorphism. Analogously, it is easy to verify that ψ^S is a right $[[A_1^{S,\leq}]]-$ module homomorphism.Thus ψ^S is a bimodule homomorphism. With the usual matrix operations, T^S is a ring, see [4]. □

Theorem 2.8. *Let A_1, A_2, A_3 be reduced rings, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. Then the following conditions are equivalent:*

(1) $A_1, A_2,$ and A_3 are $GM-$rings.

(2) The formal triangular matrix ring over generalized power series

$$T^S = \begin{pmatrix} [[{A_1}^{S,\leq}]] & 0 & 0 \\ [[{M_{21}}^{S,\leq}]] & [[{A_2}^{S,\leq}]] & 0 \\ [[{M_{31}}^{S,\leq}]] & [[{M_{32}}^{S,\leq}]] & [[{A_3}^{S,\leq}]] \end{pmatrix}$$

is a $GM-$ring.

Proof. (1)⇒ (2) Since $A_1, A_2,$ and A_3 are $GM-$rings, so are rings $[[A_1^{S,\leq}]], [A_2^{S,\leq}]]$ and $[A_3^{S,\leq}]]$ by virtue of Lemma 2.4. According to [2, Theorem 6], the result follows.

(2)⇒ (1) Applying [2, Theorem 6], we have $[[A_1^{S,\leq}]], [A_2^{S,\leq}]]$ and $[A_3^{S,\leq}]]$ are $GM-$rings. Then according to Lemma 2.5, we get the result. □

Example 4 Let A_1, A_2, A_3 be reduced rings and N the semigroup of natural numbers. Let $S = N \cup \{0\}$, with the usual order. then

$$\begin{aligned} T^S &= \begin{pmatrix} [[{A_1}^{S,\leq}]] & 0 & 0 \\ [[{M_{21}}^{S,\leq}]] & [[{A_2}^{S,\leq}]] & 0 \\ [[{M_{31}}^{S,\leq}]] & [[{M_{32}}^{S,\leq}]] & [[{A_3}^{S,\leq}]] \end{pmatrix} \\ &\cong \begin{pmatrix} A_1[[X]] & 0 & 0 \\ M_{21}[[X]] & A_2[[X]] & 0 \\ M_{31}[[X]] & M_{32}[[X]] & A_3[[X]] \end{pmatrix} \end{aligned}$$

where $A_i[[X]](i = 1, 2, 3)$ is the ring of formal power series, and $M_{ij}[[X]](i = 2, 3, j = 1, 2)$ is a bimodule of power series rings. If A_1, A_2, A_3 are $GM-$ rings, then T^S is also a $GM-$ring. Actually, let

$$F = \begin{pmatrix} f_1 & 0 & 0 \\ m_{21} & f_2 & 0 \\ m_{31} & m_{32} & f_3 \end{pmatrix} \in T^S, \quad G = \begin{pmatrix} g_1 & 0 & 0 \\ n_{21} & g_2 & 0 \\ n_{31} & n_{32} & g_3 \end{pmatrix} \in T^S.$$

Since $A_i(i = 1, 2, 3)$ is a $GM-$ ring, by Lemma 2.4, we have $A_i[[X]]$ is also a $GM-$ ring. Thus there exist $e_i^2 = e_i, p_i^2 = p_i \in A_i[[X]], u_i \in U(A_i[[X]])$ and $v_i \in U(A_i[[X]], v_i' \in U(A_i[[X]])$ such that $f_i = e_iu_i + v_i$, and $g_i = p_iu_i^{-1} + v_i'(i = 1, 2, 3)$.

Set

$$F_1 = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}, W = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}, K_1 = \begin{pmatrix} v_1 & 0 & 0 \\ m_{21} & v_2 & 0 \\ m_{31} & m_{32} & v_3 \end{pmatrix}.$$

It is easy to verify that $F_1^2 = F_1 \in T^S$, and

$$\begin{aligned} & K_1 \begin{pmatrix} v_1^{-1} & 0 & 0 \\ -v_2^{-1} m_{21} v_1^{-1} & v_2^{-1} & 0 \\ v_3^{-1} m_{32} v_2^{-1} \otimes m_{21} v_1^{-1} - v_3^{-1} m_{31} v_1^{-1} & -v_3^{-1} m_{32} v_2^{-1} & v_3^{-1} \end{pmatrix} \\ &= \begin{pmatrix} v_1^{-1} & 0 & 0 \\ -v_2^{-1} m_{21} v_1^{-1} & v_2^{-1} & 0 \\ v_3^{-1} m_{32} v_2^{-1} \otimes m_{21} v_1^{-1} - v_3^{-1} m_{31} v_1^{-1} & -v_3^{-1} m_{32} v_2^{-1} & v_3^{-1} \end{pmatrix} K_1 \\ &= diag(1, 1, \cdots, 1), \end{aligned}$$

This means that F_1 is a idempotent and K_1 is a unit. Moreover, $F = F_1 W + K_1$ and W is a unit. Analogously, we have a idempotent $F_2 = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$, and a unit $K_2 = \begin{pmatrix} v_1' & 0 & 0 \\ n_{21} & v_2' & 0 \\ n_{31} & n_{32} & v_3' \end{pmatrix}$ such that $G = F_2 W^{-1} + K_2$. Therefore we conclude that T^S is a $GM-$ ring. Conversely, if T^S is a $GM-$ring, similar to the proof of Theorem 6 in [2], we obtain that $A_i[[X]]$ is a $GM-$ring. Then by Lemma 2.5, we have $A_i (i = 1, 2, 3)$ is a $GM-$ ring.

Corollary 2.9. *Let R be a reduced ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. A ring R is a $GM-$ring if and only if the ring of all $n \times n$ lower triangular matrices over $[[R^{S,\leq}]]$ is a $GM-$ring.*

Proof. According to Theorem 2.8, the result follows. □

Analogously, let R be a reduced ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. we deduce that a ring R is a $GM-$ring if and only if the ring of all $n \times n$ upper triangular matrices over $[[R^{S,\leq}]]$ is a $GM-$ring.

Let M be a $(R, R)-$bimodule, then the module extension of R by M is the ring $R \bowtie M$ with the usual addition and multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Now we investigate $GM-$rings for module extension of $[[R^{S,\leq}]]$ by $[[M^{S,\leq}]]$ and introduce a large class of such rings.

Lemma 2.10. *Let ring $R \bowtie M$ be the module extension of R by M. Let $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]]$ be the module extension of $[[R^{S,\leq}]]$ by $[[M^{S,\leq}]]$. Then $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]] \cong [[(R \bowtie M)^{S,\leq}]]$.*

Proof. Let

$$T(R,M) = \left\{ \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} | r \in R, m \in M \right\},$$
$$T^*(R,M) = \left\{ \begin{pmatrix} f & m \\ 0 & f \end{pmatrix} | f \in [[R^{S,\leq}]], m \in [[M^{S,\leq}]] \right\}.$$

With the usual matrix operations, $T(R,M)$ and $T^*(R,M)$ are rings. As in the proof of [7, Proposition 4.3], it is easy to show that $T^*(R,M) \cong [[T(R,M)^{S,\leq}]]$. Moreover, $R \bowtie M \cong T(R,M)$ and $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]] \cong T^*(R,M)$. So $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]] \cong [[(R \bowtie M)^{S,\leq}]]$, as asserted. □

Theorem 2.11. *Let R be a ring, M a $(R,R)-$bimodule. If R is a $GM-$ring, then $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]]$ is a $GM-$ring.*

Proof. Since R is a $GM-$ring, so is ring $R \bowtie M$ by [2, Theorem 11]. Use the fact that $[[R^{S,\leq}]] \bowtie [[M^{S,\leq}]] \cong [[(R \bowtie M)^{S,\leq}]]$, then the result follows by Lemma 2.4. □

Corollary 2.12. *Let R be a ring. If R is a $GM-$ring, then $[[R^{S,\leq}]] \bowtie [[R^{S,\leq}]]$ is a $GM-$ring.*

Proof. It is a immediate consequence of Theorem 2.11. □

Corollary 2.13. *Let R be an exchange ring with artinian primitive factors. Then $[[R^{S,\leq}]] \bowtie [[R^{S,\leq}]]$ is a $GM-$ring.*

Proof. Since R is an exchange ring with artinian primitive factors, it is a $GM-$ring. Thus we get the result by Corollary 2.12. □

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