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AN APPROACH TO THE FAITH-MENAL CONJECTURE

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Abstract. The Faith-Menal conjecture is one of the three main open conjectures on *QF* rings. It says that every right noetherian and left *FP*-injective ring is *QF*. In this paper, it is proved that the conjecture is true if every nonzero complement left ideal of the ring R is not small (or not singular). Several known results are then obtained as corollaries.

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1. Introduction

Throughout this paper rings are associative with identity. For a subset X of a ring R, the left annihilator of X in R is $\mathbf{l}(X) = \{r \in R\colon\ rx = 0 \text{ for all } x \in X\}$. For any $a \in R$, we write $\mathbf{l}(a)$ for $\mathbf{l}(\{a\})$. Right annihilators are defined analogously. We write J, Z_l, Z_r, S_l and S_r for the Jacobson radical, the left singular ideal, the right singular ideal, the left socle and the right socle of R, respectively. $I \subseteq^{ess} R_R$ means that I is an essential right ideal of R. $f = c\cdot$ (resp., $f = \cdot c$) means that f is a left (resp., right) multiplication map by the element $c \in R$.

Recall that a ring R is *quasi-Frobenius (QF)* if R is one-sided noetherian and one-sided self-injective. There are three famous conjectures on *QF* rings (see [11]). One of them is the Faith-Menal conjecture, which was raised by Faith and Menal in [4]. A ring R is called *right Johns* if R is right noetherian and every right ideal of R is a right annihilator. R is called *strongly right Johns* if the matrix ring $\mathrm{M}_n(R)$ is right Johns for all $n \geq 1$. In [7], Johns used a false result of Kurshan [8, Theorem 3.3] to show that right Johns rings are right artinian. In [3], Faith and Menal gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left *FP*-injective rings (see [4, Theorem 1.1]). Recall that a ring R is called *left FP-injective* if every R-homomorphism from a submodule N of a free left R-module F to ${}_RR$ can be

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extended to one from F to ${}_RR$.

In this paper, we apply McCoy's Lemma (see Lemma 2.1) to show that Faith-Menal conjecture is true if every nonzero complement left ideal of R is not small (or not singular). A left ideal I of R is said to be a *complement* if it is maximal with respect to the property that $I \cap K$=0 for some left ideal K of R. A left ideal I of R is *small* if, for any proper left ideal K of R, $I + K \neq R$.

2. Results

Let R be a ring. An element $a \in R$ is called *regular* if there exists an element $b \in R$ such that $a = aba$. R is called *regular* if every element of R is regular.

Lemma 2.1. (McCoy's Lemma) *Let R be a ring and a, $c \in R$. If b=a-aca is a regular element of R, then a is also a regular element of R.*

Proof. This follows easily from the definition. □

A ring R is called *left P-injective* if every homomorphism from a principal left ideal Rt to R can be extended to one from ${}_RR$ to ${}_RR$.

Proposition 2.2. *Suppose R is a right noetherian and left P-injective ring. Then J =Z_l is a nilpotent right annihilator, and $\mathbf{l}(J)$ is essential both as a left and as a right ideal of R.*

Proof. Since R is left P-injective, J=Z_l (see [11, Theorem 5.14]). By [5, Theorem 2.7], J is nilpotent and $\mathbf{l}(J)$ is essential both as a left and as a right ideal of R. Since $\mathbf{l}(J)$ is an essential left ideal of R, $\mathbf{rl}(J) \subseteq Z_l = J$. Thus $J = \mathbf{rl}(J)$ is a right annihilator. □

Lemma 2.3. [6, Lemma 5.8] *If R is a semiprime right Goldie ring, then R has DCC on right annihilators.*

Lemma 2.4. *Suppose R is a right noetherian and left P-injective ring such that every nonzero complement left ideal of R is not small (or not singular). Then R is right artinian.*

Proof. First, we prove that $\overline{R} = R/J$ is a regular ring. Assume $a \notin J$. Then, since $J = \mathbf{rl}(J) = Z_l$ by Proposition 2.2, there exists a nonzero complement left ideal I of R such that $\mathbf{l}(a) \cap I = 0$. We claim that there exists some $b \in I$ such that $ba \notin J$. If not, then $Ia \subseteq J$. This implies that $\mathbf{l}(J)Ia = 0$. Since $\mathbf{l}(a) \cap I = 0$, $\mathbf{l}(J)I \subseteq \mathbf{l}(a) \cap I = 0$. Thus $I \subseteq \mathbf{rl}(J)$=$J = Z_l$. So I is a small and singular left ideal of R. This is a contradiction. Since R is left P-injective, every homomorphism from

Rba to ${}_RR$ is a right multiplication map by some $c \in R$. Define $f: Rba \to R$ by $f(rba) = rb$ for all $r \in R$. Then f is well-defined and there exists $0 \neq c \in R$ such that $f = \cdot c$. So $b = bac$. This implies that $\bar{b} \in \mathbf{l}_{\overline{R}}(\bar{a} - \overline{aca})$, where $\bar{r}$ denotes $r + J$ in R/J for any $r \in R$. Since $\overline{ba} \neq \bar{0}$, $\mathbf{l}_{\overline{R}}(\bar{a})$ is properly contained in $\mathbf{l}_{\overline{R}}(\bar{a} - \overline{aca})$. If $a - aca \in J$, then $\bar{a}$ is a regular element of $\overline{R}$. If not, let $a_1 = a - aca$. Then $\mathbf{l}(a_1)$ is not essential in ${}_RR$. In the same way, we get $a_2 = a_1 - a_1c_1a_1$ for some $c_1 \in R$ and $\mathbf{l}_{\overline{R}}(\overline{a_1})$ is properly contained in $\mathbf{l}_{\overline{R}}(\overline{a_2})$. If $a_2 \in J$, then $\bar{a}_1$ is a regular element of $\overline{R}$. Thus, by Lemma 2.1, $\bar{a}$ is a regular element of $\overline{R}$. If $a_2 \notin J$, we have $a_3 = a_2 - a_2c_2a_2$ for some $c_2 \in R$ and $\mathbf{l}_{\overline{R}}(\overline{a_2})$ is properly contained in $\mathbf{l}_{\overline{R}}(\overline{a_3})$. Therefore we have such $a_k \in R$ step by step, $k = 1, 2, \cdots$. Since R is right noetherian and $J(\overline{R}) = \bar{0}$, $\overline{R}$ is a semiprime and right Goldie ring. By Lemma 2.3, $\overline{R}$ satisfies ACC on left annihilators. So there exists some positive integer m such that $a_m \in J$ and $a_k = a_{k-1} - a_{k-1}c_{k-1}a_{k-1}$ for some $c_{k-1} \in R, k = 2, 3, \cdots, m$. By Lemma 2.1, $\bar{a}$ is a regular element of $\overline{R}$. Since $\bar{a}$ is an arbitrary nonzero element of $\overline{R}$, $\overline{R}$ is a regular ring. Then $\overline{R}$ is semisimple because $\overline{R}$ is right noetherian. Moreover, by Proposition 2.2, J is nilpotent and so R is semiprimary. Thus R is right artinian. □

A ring R is called *left 2-injective* if every left R-homomorphism from a 2-generated left ideal I of R to ${}_RR$ can be extended to one from ${}_RR$ to ${}_RR$. It is clear that every left 2-injective ring is left P-injective, but the converse is not true (see [11, Example 2.5.]).

Lemma 2.5. [12, Corollary 3] *If R is a left 2-injective ring satisfying ACC on left annihilators, then R is QF.*

Theorem 2.6. *R is QF if and only if R is right noetherian, left 2-injective and every nonzero complement left ideal of R is not small (or not singular).*

Proof. “ $\Rightarrow$ ”. This is obvious.
“ $\Leftarrow$ ”. By Lemma 2.4, R is right artinian. So R has ACC on left annihilators. Thus Lemma 2.5 implies that R is QF. □

Since every left *FP-injective* ring is left 2-injective, we have

Corollary 2.7. *The Faith-Menal conjecture is true if every nonzero complement left ideal of R is not small (or not singular).*

A ring R is called *left CS* (resp. *left min-CS*) if every left ideal (resp. minimal left ideal) is essential in a direct summand of ${}_RR$. The left CS condition is also equivalent to saying that every complement left ideal of R is a direct summand of ${}_RR$. So if R is left CS, then every nonzero complement left ideal is neither small nor singular. But the converse is not true (see Example 2.11). R is called left *C2* if every left ideal that is isomorphic to a direct summand of ${}_RR$ is also a direct summand of ${}_RR$. Every left P-injective ring is left *C2* (see [11, Proposition 5.10]). A left *CS* and left *C2* ring is called a *left continuous* ring.

Theorem 2.8. *Suppose R is a right noetherian, left P-injective, and left min-CS ring such that every nonzero complement left ideal is not small (or not singular). Then R is QF.*

Proof. By Lemma 2.4, R is right artinian. So R is a left *GPF* ring (i.e., R is left P-injective, semiperfect, and $S_l \subseteq^{ess} {}_RR$). Then R is left Kasch and $S_r = S_l$ by [11, Theorem 5.31]. Thus $Soc(Re)$ is simple for each local idempotent e of R (see [11, Lemma 4.5]). By [11, Lemma 3.1] the dual, $(eR/eJ)^* \cong l(J)e = S_re = S_le = Soc(Re)$ is simple for every local idempotent e of R. Since R is semiperfect, each simple right R-module is isomorphic to eR/eJ for some local idempotent e of R (see [1, Theorem 27.10]). Thus R is right mininjective by [11, Theorem 2.29]. Since a left P-injective ring is left mininjective, R is a two-sided mininjective and right artinian ring. So R is QF by [13, Theorem 2.5]. □

Corollary 2.9. [2, Theorem 2.21] *If R is right noetherian, left CS and left P-injective, then R is QF.*

Corollary 2.10. [10, Theorem 3.2] *The following are equivalent for a ring R.*
(1) *R is QF.*
(2) *R is a right Johns and left CS ring.*

Example 2.11. *There is a left min-CS ring S satisfying every nonzero complement left ideal of S is neither small nor singular. But it is not left CS.*

Proof. Let k be a division ring and V_k be a right k-vector space of infinite dimension. Take R=End(V_k), then R is regular but not left self-injective (see [9, Example 3.74B]). Let S=$M_{2\times 2}(R)$, then S is also regular. This implies S is left P-injective and $J(S) = Z_l(S) = 0$. Thus S is left C2 and every nonzero complement left ideal of S is neither small nor singular. And every minimal left ideal of S is a direct summand of ${}_SS$. So S is left *min-CS*. But S is not left *CS*. For if S is left *CS*, then S is left continuous. Thus R is left self-injective by [11, Theorem 1.35]. This is a contradiction. □

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References

[1] F.W. Anderson and K.R. Fuller, Rings and Categories of Modules, Springer-Verlag, Berlin-Heidelberg-New York, 1992.

[2] J.L. Chen and W.X. Li, *On artiness of right CF rings*, Comm. Algebra, 32(11) (2004), 4485-4494.

[3] C. Faith and P. Menal, *A counter example to a conjecture of Johns*, Proc. Amer. Math. Soc., 116 (1992), 21-26.

[4] C. Faith and P. Menal, *The structure of Johns rings*, Proc. Amer. Math. Soc., 120 (1994), 1071-1081.

[5] J. L. Gómez Pardo and P. A. Guil Asensio, *Torsionless modules and rings with finite essential socle*, Lecture Notes in Pure and Appl. Math., 201, New York, Marcel Dekker, 1998, 261-278.

[6] K.R.Goodearl and R.B. Warfield, An Introduction to Noncommutative Noetherian Rings, London Mathematical Society Student Texts 16, Cambridge University Press, 1989.

[7] B. Johns, *Annihilator conditions in noetherian rings*, J. Algebra, 49 (1977), 222-224.

[8] R.P. Kurshan, *Rings whose cyclic modules have finitely genrated socle*, J. Algebra, 15 (1970), 376-386.

[9] T. Y. Lam, Lectures on Modules and Rings, Springer/Verlag, New York, 1998.

[10] W.K. Nicholson and M.F.Yousif, *Annihilators and the CS-condition*, Glasgow Math. J., 40 (1998), 548-578.

[11] W.K. Nicholson and M.F. Yousif, Quasi-Frobenius Rings, Cambridge University Press, Cambridge, 2003.

[12] E. A. Rutter, *Rings with the principal extension property*, Comm. Algebra, 3(3) (1975), 203-212.

[13] L. Shen and J.L. Chen, *New characterizations of quasi-Frobenius rings*, Comm. Algebra, 34 (2006), 2157-2165.

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