Abstract. The concept of strong $w[\rho,f,q]$-summability of order $(\alpha,\beta)$ for sequences of complex (or real) numbers is introduced in this work. We also give some inclusion relations between the sets of $\rho$-statistical convergence of order $(\alpha,\beta)$, strong $w^\beta[\rho,f,q]$-summability and strong $w^\beta[\rho,q]$-summability.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [28] and Fast [13] and later in 1959, Schoenberg [27] reintroduced independently. Afterwards there has appeared much research with some arguments related of this concept (see Caserta et al. [3], Connor [4], Çakallı [5],[6], Çolak [7], Et et al. [8],[9],[10], Fridy [14], Gadjiev and Orhan [15], Kolk [17], Salat [26], Şengüll et al.(2, 29, 30, 31, 32, 33, 34) and many others).

The statistical convergence order $\alpha$ was introduced by Çolak [7] as follows:

The sequence $x = (x_k)$ is said to be statistically convergent of order $\alpha$ to $L$ if there is a complex number $L$ such that

$$\lim_{n \to \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : |x_k - L| \geq \varepsilon \right\} \right| = 0.$$  

Let $0 < \alpha \leq \beta \leq 1$. Then the $(\alpha,\beta)$-density of the subset $E$ of $\mathbb{N}$ is defined by

$$\delta^\beta_\alpha (E) = \lim_{n \to \infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : k \in E \right\} \right|^\beta$$

if the limit exists (finite or infinite), where $\left| \left\{ k \leq n : k \in E \right\} \right|^\beta$ denotes the $\beta$th power of number of elements of $E$ not exceeding $n$.

If $x = (x_k)$ is a sequence such that satisfies property $P(k)$ for all $k$ except a set of $(\alpha,\beta)$-density zero, then we say that $x_k$ satisfies $P(k)$ for “almost all $k$ according to $\beta$” and we denote this by “a.a.k $(\alpha,\beta)$”.

Throughout this study, we shall denote the space of sequences of real number by $w$.

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Let $0 < \beta \leq 1, 0 < \alpha \leq 1$, $\alpha \leq \beta$ and $x = (x_k) \in w$. Then we say the sequence $x = (x_k)$ is statistically convergent of order $(\alpha, \beta)$ if there is a complex number $L$ such that
\[
\lim_{n \to \infty} \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}|^\beta = 0
\]
i.e. for a.a.k $(\alpha, \beta)$ $|x_k - L| < \varepsilon$ for every $\varepsilon > 0$, in that case a sequence $x$ is said to be statistically convergent of order $(\alpha, \beta)$ to $L$. This limit is denoted by $S_\alpha^\beta - \lim x_k = L$ (29).

Let $0 < \alpha \leq 1$. A sequence $(x_k)$ of points in $\mathbb{R}$, the set of real numbers, is called $\rho$–statistically convergent of order $\alpha$ to an element $L$ of $\mathbb{R}$ if
\[
\lim_{n \to \infty} \frac{1}{\rho_n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0
\]
for each $\varepsilon > 0$, where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to $\infty$ such that $\limsup \rho_n \frac{\alpha}{\pi} < \infty$, $\Delta \rho_n = O(1)$ and $\Delta \rho_n = \rho_{n+1} - x_n$ for each positive integer $n$. In this case we write $st_\rho^\alpha - \lim x_k = L$. If $\rho = (\rho_n) = n$ and $\alpha = 1$, then $\rho$–statistically convergent of order $\alpha$ is coincide statistical convergence (5).

Here and in what follows we suppose that the sequence $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to $\infty$ such that $\limsup \rho_n \frac{\alpha}{\pi} < \infty$, $\Delta \rho_n = O(1)$ where $0 < \alpha \leq 1$ and $\Delta \rho_n = \rho_{n+1} - \rho_n$ for each positive integer $n$.

The notion of a modulus function was given by Nakano [21]. Following Maddox [19] and Ruckle [25], we recall that a modulus $\rho$ such that $\rho_n$ is increasing, such that $\limsup \rho_n \frac{\alpha}{\pi} < \infty$, $\Delta \rho_n = O(1)$ where $0 < \alpha \leq 1$ and $\Delta \rho_n = \rho_{n+1} - \rho_n$ for each positive integer $n$.

Let $(x_k)$ be a sequence of points in $\mathbb{R}$, the set of real numbers, and assume that $f$ is a function from $[0, \infty)$ to $[0, \infty)$ such that
\begin{itemize}
  \item[i)] $f(x) = 0$ if and only if $x = 0$,
  \item[ii)] $f(x + y) \leq f(x) + f(y)$ for $x, y \geq 0$,
  \item[iii)] $f$ is increasing,
  \item[iv)] $f$ is continuous from the right at 0.
\end{itemize}

It follows that $f$ must be continuous everywhere on $[0, \infty)$.

Altın [1], Et ([11], [12]), Gaur and Mursaleen [20], Isik [16], Nuray and Savaş [22], Pehlivan and Fisher [24] and some others have been studied with some sequence $f$ and Ruckle [25], we recall that a modulus $\rho$ such that $\rho_n$ is continuous from the right at 0.

The following inequality will be used frequently throught the paper:
\[
|a_k + b_k|^p = A (|a_k|^p + |b_k|^p)
\]
where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup p_k = B$, $A = \max (1, 2B^{-1})$ (15).

2. Main Results

In this section we first give the sets of strongly $w_\alpha^\beta (\rho, q)$–summable sequences and strongly $w_\alpha^\beta [\rho, f, q]$–summable sequences with respect to the modulus function $f$. Secondly we state and prove the theorems on some inclusion relations between the $S_\alpha^\beta (\rho)$–statistical convergence, strong $w_\alpha^\beta [\rho, f, q]$–summability and strong $w_\alpha^\beta (\rho, q)$–summability.

**Definition 2.1.** Let $0 < \alpha \leq \beta \leq 1$ be given. A sequence $x = (x_k)$ is said to be $S_\alpha^\beta (\rho)$–statistically convergent (or $\rho$–statistically convergent sequences of order $(\alpha, \beta)$) if there is a real number $L$ such that
\[
\lim_{n \to \infty} \frac{1}{\rho_n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}|^\beta = 0,
\]
\( \rho_n^\alpha \) denotes the \( \alpha \)th power \((\rho_n^\alpha)\) of \( \rho_n \), that is \( \rho_n^\alpha = (\rho_n^\alpha) = (\rho_1^\alpha, \rho_2^\alpha, \ldots, \rho_n^\alpha, \ldots) \) and \( \{\{k \leq n : k \in E\}\}^\beta \) denotes the \( \beta \)th power of number of elements of \( E \) not exceeding \( n \). In the present case this convergence is indicated by \( S_\alpha^\beta (\rho) - \lim x_k = L \). \( S_\alpha^\beta (\rho) \) will indicate the set of all \( S_\alpha^\beta (\rho) - \)statistically convergent sequences.

**Definition 2.2.** Let \( 0 < \alpha \leq \beta \leq 1 \) and \( q \) be a positive real number. A sequence \( x = (x_k) \) is said to be strongly \( N_\alpha^\beta (\rho, q) - \)summable (or strongly \( N (\rho, q) - \)summable of order \((\alpha, \beta)\)) if there is a real number \( L \) such that

\[
\lim_{n \to \infty} \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} |x_k - L|^q \right)^\beta = 0.
\]

We denote it by \( N_\alpha^\beta (\rho, q) - \lim x_k = L \). \( N_\alpha^\beta (\rho, q) \) will denote the set of all strongly \( N (\rho, q) - \)summable sequences of order \((\alpha, \beta)\). If \( \alpha = \beta = 1 \), then we will write \( N_1 (\rho, q) \) in the place of \( N_\alpha^\beta (\rho, q) \). If \( L = 0 \), then we will write \( w_\alpha^\beta (\rho, q) \) in the place of \( w_\alpha^\beta (\rho, q) \). \( N_\alpha^\beta (\rho, q) \) will denote the set of all strongly \( N_\alpha (\rho, q) - \)summable sequences of order \((\alpha, \beta)\) to \( 0 \).

**Definition 2.3.** Let \( f \) be a modulus function, \( q = (q_k) \) be a sequence of strictly positive real numbers and \( 0 < \alpha \leq \beta \leq 1 \) be real numbers. A sequence \( x = (x_k) \) is said to be strongly \( w_\alpha^\beta [\rho, f, q] - \)summable of order \((\alpha, \beta)\) if there is a real number \( \overline{L} \) such that

\[
\lim_{n \to \infty} \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} |f \left( |x_k - \overline{L}| \right)|^q \right)^\beta = 0.
\]

In this case, we write \( w_\alpha^\beta [\rho, f, q] - \lim x_k = \overline{L} \). We denote the set of all strongly \( w_\alpha^\beta [\rho, f, q] - \)summable sequences of order \((\alpha, \beta)\) by \( w_\alpha^\beta [\rho, f, q] \). In the special case \( q_k = q \), for all \( k \in \mathbb{N} \) and \( f(x) = x \) we will denote \( N_\alpha^\beta (\rho, q) \) in the place of \( w_\alpha^\beta [\rho, f, q] \). \( w_\alpha^\beta [\rho, f, q] \) will denote the set of all strongly \( w [\rho, f, q] - \)summable sequences of order \((\alpha, \beta)\) to \( 0 \).

In the following theorems, we shall assume that the sequence \( q = (q_k) \) is bounded and \( 0 < d = \inf_k q_k \leq q_k \leq \sup_k q_k = D < \infty \).

**Theorem 2.1.** The class of sequences \( w_\alpha^\beta [\rho, f, q] \) is linear space.

**Proof.** Omitted. \( \square \)

**Theorem 2.2.** The space \( w_\alpha^\beta [\rho, f, q] \) is paranormed by

\[
g(x) = \sup_n \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} |f \left( |x_k| \right)|^q \right)^\beta \]

where \( 0 < \alpha \leq \beta \leq 1 \) and \( \overline{H} = \max(1, D) \).

**Proof.** Clearly \( g(0) = 0 \) and \( g(x) = g(-x) \). Let \( x, y \in w_\alpha^\beta [\rho, f, q] \) be two sequences. Since \( \frac{\alpha}{\beta} \leq 1 \) and \( \frac{\alpha}{\beta} \geq 1 \), using the Minkowski’s inequality and definition of \( f \), we have
\[
\left\{ \frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|x_k + y_k|)]^{q_k} \right)^{\beta} \right\}^{\frac{1}{\mu}} \leq \left\{ \frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|x_k|) + f (|y_k|)]^{q_k} \right)^{\beta} \right\}^{\frac{1}{\mu}}
\]

\[
= \frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|x_k|) + f (|y_k|)]^{q_k} \right)^{\beta} \frac{1}{\mu}
\]

\[
\leq \frac{1}{\rho_n^\mu} \left\{ \left( \sum_{k=1}^{n} [f (|x_k|)]^{q_k} \right)^{\beta} \right\}^{\frac{1}{\mu}}
\]

\[
+ \frac{1}{\rho_n^\mu} \left\{ \left( \sum_{k=1}^{n} [f (|y_k|)]^{q_k} \right)^{\beta} \right\}^{\frac{1}{\mu}}.
\]

Hence, we have \( g(x + y) \leq g(x) + g(y) \) for \( x, y \in w_{\alpha,0}^\mu [\rho, f, q] \). Let \( \mu \) be complex number. By definition of \( f \) we have

\[
g(\mu x) = \sup_n \left\{ \frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|\mu x_k|)]^{q_k} \right)^{\beta} \right\}^{\frac{1}{\mu}} \leq K \frac{\rho}{\mu} g(x)
\]

where \([\mu]\) denotes the integer part of \( \mu \), and \( K = 1 + |[\mu]| \). Now, let \( \mu \to 0 \) for any fixed \( x \) with \( g(x) \neq 0 \). By definition of \( f \), for \(|\mu| < 1\) and \( 0 < \alpha \leq \beta \leq 1 \), we have

\[
\frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|\mu x_k|)]^{q_k} \right)^{\beta} < \varepsilon \quad \text{for} \quad n > N(\varepsilon).
\]

Also, for \( 1 \leq n \leq N \), taking \( \mu \) small enough, since \( f \) is continuous we have

\[
\frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|\mu x_k|)]^{q_k} \right)^{\beta} < \varepsilon.
\]

Therefore, \( \text{[2.1]} \) and \( \text{[2.2]} \) imply that \( g(\mu x) \to 0 \) as \( \mu \to 0 \). \( \square \)

**Proposition 2.3.** \( \text{[2.3]} \) Let \( f \) be a modulus and \( 0 < \delta < 1 \). Then for each \( ||u|| \geq \delta \), we have \( f(||u||) \leq 2f(1) \delta^{-1} ||u|| \).

**Theorem 2.4.** If \( 0 < \alpha = \beta \leq 1 \), \( q > 1 \) and \( \liminf_{u \to \infty} \frac{f(u)}{u} > 0 \), then \( w_{\alpha}^\mu [\rho, f, q] = w_{\alpha}^\beta (\rho, q) \).

**Proof.** If \( \liminf_{u \to \infty} \frac{f(u)}{u} > 0 \) then there exists a number \( c > 0 \) such that \( f(u) > cu \) for \( u > 0 \). Let \( x \in w_{\beta}^\alpha [\rho, f, q] \), then

\[
\frac{1}{\rho_n^\mu} \left( \sum_{k=1}^{n} [f (|x_k - L|)]^{q_k} \right)^{\beta} \geq \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [c|x_k - L|]^{q_k} \right)^{\beta} = \frac{c^{\alpha\beta}}{\rho_n^\alpha} \left( \sum_{k=1}^{n} |x_k - L|^{q_k} \right)^{\beta}.
\]

This means that \( w_{\alpha}^\beta [\rho, f, q] \subseteq w_{\alpha}^\beta (\rho, q) \).

Let \( x \in w_{\alpha}^\beta (\rho, q) \). Thus we have
\[
\frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^n |x_k - L|^q \right)^\beta \to 0 \quad \text{as} \quad n \to \infty.
\]

Let \( \varepsilon > 0, \beta = \alpha \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( cu < f(u) < \varepsilon \) for every \( u \) with \( 0 \leq u \leq \delta \). Therefore, by Proposition 1, we have

\[
\frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^n [f(|x_k - L|)]^q \right)^\beta \leq \frac{1}{\rho_n^\alpha} \varepsilon^{q\beta} n^\beta + \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^n [2f(1) \delta^{-1} |x_k - L|]^q \right)^\beta \leq \frac{1}{\rho_n^\alpha} \varepsilon^{q\beta} n^\beta + \frac{2^{q\beta} f(1)^{q\beta}}{\rho_n^\alpha \delta^{q\beta}} \left( \sum_{k=1}^n |x_k - L|^q \right)^\beta.
\]

This gives \( x \in w_{\alpha}^\beta [\rho, f, q) \).

**Example 2.1.** We now give an example to show that \( w_{\alpha}^\beta [\rho, f, q] \neq w_{\alpha}^\beta (\rho, q) \) in this case \( \lim_{n \to \infty} f(n) = 0 \). Consider the sequence \( f(x) = \frac{x}{1+x} \) of modulus function. Define \( x = (x_k) \) by

\[
x_k = \begin{cases} 
  k, & \text{if } k = m^3 \\
  0, & \text{if } k \neq m^3.
\end{cases}
\]

Then we have, for \( L = 0, q = 1, (\rho_n) = (n) \) and \( \alpha = \beta \)

\[
\frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^n [f(|x_k|)]^q \right)^\beta \leq \frac{1}{n^\frac{1}{2} \beta} \to 0 \quad \text{as} \quad n \to \infty.
\]

and so \( x \in w_{\alpha}^\beta [\theta, f, q] \). But

\[
\frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^n |x_k|^q \right)^\beta = \frac{1}{n^\alpha} \left( 1 + 2^3 + 3^3 + \cdots + \left[ \sqrt[n]{n} \right]^3 \right)^\beta \geq \frac{1}{n^\alpha} \left( \frac{(\sqrt[n]{n} - 1)(\sqrt[n]{n})}{2} \right)^{2\beta} \to \infty \quad \text{as} \quad n \to \infty
\]

and so \( x \notin w_{\alpha}^\beta (p) \).

**Theorem 2.5.** Let \( 0 < \alpha \leq \beta \leq 1 \) and \( \lim \inf q_k > 0 \). If a sequence is convergent to \( L \), then it is strongly \( w_{\alpha}^\beta [\rho, f, q] \)-summable of order \( (\alpha, \beta) \) to \( L \).

**Proof.** We assume that \( x_k \to L \). Since \( f \) be a modulus function, we have \( f(|x_k - L|) \to 0 \). Since \( \lim \inf q_k > 0 \), we have \( [f(|x_k - L|)]^{q_k} \to 0 \). Hence \( w_{\alpha}^\beta [\rho, f, q] - \lim x_k = L \). \( \square \)
Theorem 2.6. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, 1]$ be real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, $f$ be a modulus function, then $w_{\alpha_1}^{\beta_1} [\rho, f, q] \subset S_{\alpha_2}^{\beta_2} (\rho)$.

Proof. Let $x \in w_{\alpha_1}^{\beta_1} [\rho, f, q]$ and let $\varepsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k \leq n$ with $|x_k - L| \geq \varepsilon$ and $k \leq n$ with $|x_k - L| < \varepsilon$ respectively. Since $\rho_{\alpha_1}^n \leq \rho_{\alpha_2}^n$ for each $n$ we have

$$
\frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \leq \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \\
= \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} + \sum_2 [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \\
\leq \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n \max (R^d, R^D) + \sum_2 [f(\varepsilon)]^{\rho_k} \right)^{\beta_1} \\
\leq \left( \max (R^d, R^D) \right)^{\beta_2} \frac{1}{\rho_{\alpha_1}^n} \left( \max (f(\varepsilon)^d, f(\varepsilon)^D) \right)^{\beta_2} \\
+ \frac{\rho_{\alpha_2}^n}{\rho_{\alpha_1}^n} \left( \max (f(\varepsilon)^d, f(\varepsilon)^D) \right)^{\beta_2}.
$$

We get $x \in S_{\alpha_2}^{\beta_2} (\rho)$.

Theorem 2.7. If $f$ is a bounded modulus function and $\lim_{n \to \infty} \frac{\rho_{\alpha_2}^n}{\rho_{\alpha_1}^n} = 1$ then $S_{\alpha_2}^{\beta_2} (\rho) \subset w_{\alpha_1}^{\beta_1} [\rho, f, q]$.

Proof. Let $x \in S_{\alpha_2}^{\beta_2} (\rho)$. Suppose that $f$ be bounded. Therefore $f(x) \leq R$, for a positive integer $R$ and all $x \geq 0$. Then for each $\varepsilon > 0$ we can write

$$
\frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \leq \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \\
= \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} + \sum_2 [f(|x_k - L|)]^{\rho_k} \right)^{\beta_1} \\
\leq \frac{1}{\rho_{\alpha_1}^n} \left( \sum_{k=1}^n \max (R^d, R^D) + \sum_2 [f(\varepsilon)]^{\rho_k} \right)^{\beta_1} \\
\leq \left( \max (R^d, R^D) \right)^{\beta_2} \frac{1}{\rho_{\alpha_1}^n} \left( \max (f(\varepsilon)^d, f(\varepsilon)^D) \right)^{\beta_2} \\
+ \frac{\rho_{\alpha_2}^n}{\rho_{\alpha_1}^n} \left( \max (f(\varepsilon)^d, f(\varepsilon)^D) \right)^{\beta_2}.
$$

Hence $x \in w_{\alpha_1}^{\beta_1} [\rho, f, q]$.

Theorem 2.8. Let $f$ be a modulus function. If $\lim q_k > 0$, then $w_{\alpha}^{\beta} [\rho, f, q] - \lim x_k = L$ uniquely.

Proof. Let $\lim q_k = t > 0$. Suppose that $w_{\alpha}^{\beta} [\rho, f, q] - \lim x_k = L_1$ and $w_{\alpha}^{\beta} [\rho, f, q] - \lim x_k = L_2$. Then

$$
\lim_{n} \frac{1}{\rho_{\alpha}^n} \left( \sum_{k=1}^n [f(|x_k - L|)]^{\rho_k} \right)^{\beta} = 0,
$$

and
\[ \lim \frac{1}{n} \left( \sum_{k=1}^{n} [f(|x_k - L_2|)]^{q_k} \right)^{\beta} = 0. \]

By definition of \( f \) and using (1.1), we may write
\[ \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [f(|L_1 - L_2|)]^{q_k} \right)^{\beta} \leq \frac{A}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} + \sum_{k=1}^{n} [f(|x_k - L_2|)]^{q_k} \right)^{\beta} \]
\[ \leq \frac{A}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta} + \frac{A}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [f(|x_k - L_2|)]^{q_k} \right)^{\beta} \]

where \( \sup_k q_k = D, 0 < \beta \leq 1 \) and \( A = \max \{1, 2^{D-1}\} \). Hence
\[ \lim \frac{1}{n} \frac{1}{\rho_n^\alpha} \left( \sum_{k=1}^{n} [f(|L_1 - L_2|)]^{q_k} \right)^{\beta} = 0. \]

Since \( \lim_{k \to \infty} q_k = t \) we have \( L_1 - L_2 = 0 \). Hence the limit is unique. \( \square \)

**Theorem 2.9.** Let \( \rho = (\rho_n) \) and \( \tau = (\tau_n) \) be two sequences such that \( \rho_n \leq \tau_n \) for all \( n \in \mathbb{N} \) and let \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) be such that \( 0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1 \).

(i) If
\[ \lim \inf_{n \to \infty} \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} > 0 \quad (2.3) \]
then \( w_{\alpha_1}^{\beta_2} [\tau, f, q] \subset w_{\alpha_1}^{\beta_1} [\rho, f, q] \).

(ii) If
\[ \lim \sup_{n \to \infty} \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} < \infty \quad (2.4) \]
then \( w_{\alpha_1}^{\beta_2} [\rho, f, q] \subset w_{\alpha_1}^{\beta_1} [\tau, f, q] \).

**Proof.**

(i) Let \( x \in w_{\alpha_2}^{\beta_2} [\tau, f, q] \). We have
\[ \frac{1}{\tau_n^{\alpha_1}} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta_2} \leq \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} \frac{1}{\rho_n^{\alpha_1}} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta_1} . \]
Thus if \( x \in w_{\alpha_2}^{\beta_2} [\tau, f, q] \), then \( x \in w_{\alpha_1}^{\beta_1} [\rho, f, q] \).

(ii) Let \( x = (x_k) \in w_{\alpha_2}^{\beta_2} [\rho, f, q] \) and (2.4) holds. Now, since \( \rho_n \leq \tau_n \) for all \( n \in \mathbb{N} \), we have
\[ \frac{1}{\tau_n^{\alpha_2}} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta_1} \leq \frac{1}{\tau_n^{\alpha_2}} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta_2} \]
\[ = \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} \frac{1}{\rho_n^{\alpha_1}} \left( \sum_{k=1}^{n} [f(|x_k - L_1|)]^{q_k} \right)^{\beta_2} \]
for every \( n \in \mathbb{N} \). Therefore \( w_{\alpha_1}^{\beta_2} [\rho, f, q] \subset w_{\alpha_1}^{\beta_1} [\tau, f, q] \). \( \square \)
References


**Nazlım Deniz Aral**

Department of Mathematics; Bitlis Eren University 13000; Bitlis; TURKEY. ORCID ID: 0000-0002-8984-2620

*Email address: ndaral@beu.edu.tr*