Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume IV Issue 1 (2022), Pages 15-23. Doi: https://doi.org/ 10.47087/mjm.1092599

ρ-STATISTICAL CONVERGENCE DEFINED BY A MODULUS FUNCTION OF ORDER (α, β)

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Abstract. The concept of strong $w[\rho, f, q]$ −summability of order (α, β) for sequences of complex (or real) numbers is introduced in this work. We also give some inclusion relations between the sets of ρ-statistical convergence of order (α, β), strong $w_{\alpha}^{\beta}[\rho, f, q]$ −summability and strong $w_{\alpha}^{\beta}(\rho, q)$ −summability.

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [28] and Fast [13] and later in 1959, Schoenberg [27] reintroduced independently. Afterwards there has appeared much research with some arguments related of this concept (see Caserta et al. [3], Connor [4], Çakallı ([5],[6]), Çolak [7], Et et al. ([8],[9],[10]), Fridy [14], Gadjiev and Orhan [15], Kolk [17], Salat [26], Şengül et al.([2],[29],[30],[31],[32],[33],[34]) and many others).

The statistical convergence order α was introduced by Çolak [7] as follows:

The sequence $x = (x_k)$ is said to be statistically convergent of order α to L if there is a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

Let $0 < \alpha \leq \beta \leq 1$. Then the $(\alpha, \beta)-$density of the subset E of $\mathbb{N}$ is defined by

$$\delta_{\alpha}^{\beta}(E) = \lim_{n} \frac{1}{n^{\alpha}} |\{k \leq n : k \in E\}|^{\beta}$$

if the limit exists (finite or infinite), where $|\{k \leq n : k \in E\}|^{\beta}$ denotes the βth power of number of elements of E not exceeding n.

If $x = (x_k)$ is a sequence such that satisfies property $P(k)$ for all k except a set of $(\alpha, \beta)-$density zero, then we say that x_k satisfies $P(k)$ for "almost all k according to β" and we denote this by "$a.a.k \ \ (\alpha, \beta)$".

Throughout this study, we shall denote the space of sequences of real number by w.

2020 *Mathematics Subject Classification.* Primary 40A05, 40C05; Secondary 46A45.
Key words and phrases. Cesàro summability; modulus function; statistical convergence.

Submitted on March 24th, 2022. Published on May 6th, 2022
Communicated by Huseyin CAKALLI, Ibrahim CANAK and Sefa Anıl SEZER.

Let $0<\beta\leq 1, 0<\alpha\leq 1$, $\alpha\leq\beta$ and $x=(x_k)\in w$. Then we say the sequence $x=(x_k)$ is statistically convergent of order (α,β) if there is a complex number L such that

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\left|\{k\leq n:|x_k-L|\geq\varepsilon\}\right|^{\beta}=0$$

i.e. for $a.a.k$ (α,β) $|x_k-L|<\varepsilon$ for every $\varepsilon>0$, in that case a sequence x is said to be statistically convergent of order (α,β) to L. This limit is denoted by $S_{\alpha}^{\beta}-\lim x_k=L$ ([29]).

Let $0<\alpha\leqslant 1$. A sequence (x_k) of points in $\mathbb{R}$, the set of real numbers, is called $\rho-$statistically convergent of order α to an element L of $\mathbb{R}$ if

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}|\{k\leq n:|x_k-L|\geq\varepsilon\}|=0$$

for each $\varepsilon>0$, where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-x_n$ for each positive integer n. In this case we write $st_{\rho}^{\alpha}-\lim x_k=L$. If $\rho=(\rho_n)=n$ and $\alpha=1$, then $\rho-$statistically convergent of order α is coincide statistical convergence ([5]).

Here and in what follows we suppose that the sequence $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ where $0<\alpha\leqslant 1$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n.

The notion of a modulus function was given by Nakano [21]. Following Maddox [19] and Ruckle [25], we recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

i) $f(x)=0$ if and only if $x=0$,
ii) $f(x+y)\leq f(x)+f(y)$ for $x,y\geq 0$,
iii) f is increasing,
iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$.

Altın [1], Et ([11], [12]), Gaur and Mursaleen [20], Işık [16], Nuray and Savaş [22], Pehlivan and Fisher [23] and some others have been studied with some sequence spaces defined by modulus function.

The following inequality will be used frequently throught the paper:

$$|a_k+b_k|^{p_k}\leq A\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{1.1}$$

where $a_k,b_k\in\mathbb{C}$, $0<p_k\leq\sup p_k=B$, $A=\max\left(1,2^{B-1}\right)$ ([18]).

2. Main Results

In this section we first give the sets of strongly $w_{\alpha}^{\beta}(\rho,q)-$summable sequences and strongly $w_{\alpha}^{\beta}[\rho,f,q]-$summable sequences with respect to the modulus function f. Secondly we state and prove the theorems on some inclusion relations between the $S_{\alpha}^{\beta}(\rho)-$ statistical convergence, strong $w_{\alpha}^{\beta}[\rho,f,q]-$summability and strong $w_{\alpha}^{\beta}(\rho,q)-$summability.

Definition 2.1. *Let $0<\alpha\leq\beta\leq 1$ be given. A sequence $x=(x_k)$ is said to be $S_{\alpha}^{\beta}(\rho)-$statistically convergent (or ρ- statistically convergent sequences of order (α,β)) if there is a real number L such that*

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}\left|\{k\leqslant n:|x_k-L|\geq\varepsilon\}\right|^{\beta}=0,$$

where ρ_n^α *denotes the* α*th power* $(\rho_n)^\alpha$ *of* ρ_n, *that is* $\rho^\alpha=(\rho_n^\alpha)=(\rho_1^\alpha,\rho_2^\alpha,...,\rho_n^\alpha,...)$ *and* $|\{k\leq n:k\in E\}|^\beta$ *denotes the* β*th power of number of elements of* E *not exceeding* n. *In the present case this convergence is indicated by* $S_\alpha^\beta(\rho)-\lim x_k=L$. $S_\alpha^\beta(\rho)$ *will indicate the set of all* $S_\alpha^\beta(\rho)-$*statistically convergent sequences.*

Definition 2.2. *Let* $0<\alpha\leq\beta\leq1$ *and* q *be a positive real number. A sequence* $x=(x_k)$ *is said to be strongly* $N_\alpha^\beta(\rho,q)-$*summable (or strongly* $N(\rho,q)-$*summable of order* (α,β)*) if there is a real number* L *such that*

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^{n}|x_k-L|^q\right)^\beta=0.$$

We denote it by $N_\alpha^\beta(\rho,q)-\lim x_k=L$. $N_\alpha^\beta(\rho,q)$ *will denote the set of all strongly* $N(\rho,q)-$*summable sequences of order* (α,β). *If* $\alpha=\beta=1$, *then we will write* $N(\rho,q)$ *in the place of* $N_\alpha^\beta(\rho,q)$. *If* $L=0$, *then we will write* $w_{\alpha,0}^\beta(\rho,q)$ *in the place of* $w_\alpha^\beta(\rho,q)$. $N_{\alpha,0}^\beta(\rho,q)$ *will denote the set of all strongly* $N_\rho(q)-$*summable sequences of order* (α,β) *to* 0.

Definition 2.3. *Let* f *be a modulus function,* $q=(q_k)$ *be a sequence of strictly positive real numbers and* $0<\alpha\leq\beta\leq1$ *be real numbers. A sequence* $x=(x_k)$ *is said to be strongly* $w_\alpha^\beta[\rho,f,q]-$*summable of order* (α,β) *if there is a real number* L *such that*

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^{q_k}\right)^\beta=0.$$

In this case, we write $w_\alpha^\beta[\rho,f,q]-\lim x_k=L$. *We denote the set of all strongly* $w_\alpha^\beta[\rho,f,q]-$*summable sequences of order* (α,β) *by* $w_\alpha^\beta[\rho,f,q]$. *In the special case* $q_k=q$, *for all* $k\in\mathbb{N}$ *and* $f(x)=x$ *we will denote* $N_\alpha^\beta(\rho,q)$ *in the place of* $w_\alpha^\beta[\rho,f,q]$. $w_{\alpha,0}^\beta[\rho,f,q]$ *will denote the set of all strongly* $w[\rho,f,q]-$*summable sequences of order* (α,β) *to* 0.

In the following theorems, we shall assume that the sequence $q=(q_k)$ is bounded and $0<d=\inf_k q_k\leq q_k\leq\sup_k q_k=D<\infty$.

Theorem 2.1. *The class of sequences* $w_{\alpha,0}^\beta[\rho,f,q]$ *is linear space.*

Proof. Omitted. □

Theorem 2.2. *The space* $w_{\alpha,0}^\beta[\rho,f,q]$ *is paranormed by*

$$g(x)=\sup_n\left\{\frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^{n}[f(|x_k|)]^{q_k}\right)^\beta\right\}^{\frac{1}{H}}$$

where $0<\alpha\leq\beta\leq1$ *and* H=$\max(1,D)$.

Proof. Clearly $g(0)=0$ and $g(x)=g(-x)$. Let $x,y\in w_{\alpha,0}^\beta[\rho,f,q]$ be two sequences. Since $\frac{q_k}{\frac{H}{\beta}}\leq1$ and $\frac{H}{\beta}\geq1$, using the Minkowski's inequality and definition of f, we have

$$\begin{aligned}\left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k+y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} &\leq \left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)+f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \\ &= \frac{1}{\rho_n^{\frac{\alpha}{H}}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)+f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\frac{\frac{1}{H}}{\beta}} \\ &\leq \frac{1}{\rho_n^{\frac{\alpha}{H}}}\left\{\left(\sum_{k=1}^{n}\left[f\left(\left|x_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \\ &\quad + \frac{1}{\rho_n^{\frac{\alpha}{H}}}\left\{\left(\sum_{k=1}^{n}\left[f\left(\left|y_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}}.\end{aligned}$$

Hence, we have $g(x+y) \leq g(x)+g(y)$ for $x, y \in w_{\alpha,0}^{\beta}[\rho, f, q]$. Let μ be complex number. By defnition of f we have

$$g(\mu x)=\sup_n\left\{\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}\right\}^{\frac{1}{H}} \leq K^{\frac{\frac{D}{H}}{\beta}} g(x)$$

where $[\mu]$ denotes the integer part of μ, and $K=1+[|\mu|]$. Now, let $\mu \to 0$ for any fixed x with $g(x) \neq 0$. By definition of f, for $|\mu|<1$ and $0<\alpha \leq \beta \leq 1$, we have

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}<\varepsilon \quad \text{for} \quad n>N(\varepsilon). \tag{2.1}$$

Also, for $1 \leq n \leq N$, taking μ small enough, since f is continuous we have

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|\mu x_k\right|\right)\right]^{q_k}\right)^{\beta}<\varepsilon. \tag{2.2}$$

Therefore, (2.1) and (2.2) imply that $g(\mu x) \to 0$ as $\mu \to 0$. □

Proposition 2.3. *([24]) Let f be a modulus and $0<\delta<1$. Then for each $\|u\| \geq \delta$, we have $f(\|u\|) \leq 2f(1)\,\delta^{-1}\|u\|$.*

Theorem 2.4. *If $0<\alpha=\beta \leq 1$, $q>1$ and $\liminf_{u\to\infty}\frac{f(u)}{u}>0$, then $w_{\alpha}^{\beta}[\rho, f, q]=w_{\alpha}^{\beta}(\rho, q)$.*

Proof. If $\liminf_{u\to\infty}\frac{f(u)}{u}>0$ then there exists a number $c>0$ such that $f(u)>cu$ for $u>0$. Let $x \in w_{\alpha}^{\beta}[\rho, f, q]$, then

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(\left|x_k-L\right|\right)\right]^{q}\right)^{\beta} \geq \frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[c\left|x_k-L\right|\right]^{q}\right)^{\beta}=\frac{c^{q\alpha\beta}}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left|x_k-L\right|^{q}\right)^{\beta}.$$

This means that $w_{\alpha}^{\beta}[\rho, f, q] \subseteq w_{\alpha}^{\beta}(\rho, q)$.

Let $x \in w_{\alpha}^{\beta}(\rho, q)$. Thus we have

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}|x_k-L|^q\right)^{\beta}\to 0 \text{ as } n\to\infty.$$

Let $\varepsilon>0$, $\beta=\alpha$ and choose δ with $0<\delta<1$ such that $cu<f(u)<\varepsilon$ for every u with $0\le u\le\delta$. Therefore, by Proposition 1, we have

$$\begin{aligned}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^q\right)^{\beta} &= \frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|\le\delta}}^{n}[f(|x_k-L|)]^q\right)^{\beta}+\frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|>\delta}}^{n}[f(|x_k-L|)]^q\right)^{\beta}\\ &\le \frac{1}{\rho_n^{\alpha}}\varepsilon^{q\beta}n^{\beta}+\frac{1}{\rho_n^{\alpha}}\left(\sum_{\substack{k=1\\|x_k-L|>\delta}}^{n}\left[2f(1)\,\delta^{-1}|x_k-L|\right]^q\right)^{\beta}\\ &\le \frac{1}{\rho_n^{\alpha}}\varepsilon^{q\alpha}n^{\beta}+\frac{2^{q\beta}f(1)^{q\beta}}{\rho_n^{\alpha}\delta^{q\beta}}\left(\sum_{k=1}^{n}|x_k-L|^q\right)^{\beta}.\end{aligned}$$

This gives $x\in w_{\alpha}^{\beta}[\rho,f,q]$. □

Example 2.1. *We now give an example to show that $w_{\alpha}^{\beta}[\rho,f,q]\neq w_{\alpha}^{\beta}(\rho,q)$ in this case $\liminf_{u\to\infty}\frac{f(u)}{u}=0$. Consider the sequence $f(x)=\frac{x}{1+x}$ of modulus function. Define $x=(x_k)$ by*

$$x_k=\begin{cases}k, & \text{if } k=m^3\\ 0, & \text{if } k\neq m^3.\end{cases}$$

Then we have, for $L=0$, $q=1$, $(\rho_n)=(n)$ and $\alpha=\beta$

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[f(|x_k|)]^q\right)^{\beta}\leqslant\frac{1}{n^{\alpha}}n^{\frac{1}{3}\beta}\to 0 \text{ as } n\to\infty$$

and so $x\in w_{\alpha}^{\beta}[\theta,f,q]$. But

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}|x_k|^q\right)^{\beta}=\frac{1}{n^{\alpha}}\left(1+2^3+3^3+\cdots+\left[\sqrt[3]{n}\right]\right)^{\beta}$$

$$\geqslant\frac{1}{n^{\alpha}}\left[\frac{(\sqrt[3]{n}-1)(\sqrt[3]{n})}{2}\right]^{2\beta}=\frac{1}{n^{\alpha}}\frac{\left(n^{4/3}-2n+n^{2/3}\right)^{\beta}}{4^{\beta}}\to\infty \text{ as } n\to\infty$$

and so $x\notin w_{\alpha}^{\beta}(p)$.

Theorem 2.5. *Let $0<\alpha\le\beta\le1$ and $\liminf q_k>0$. If a sequence is convergent to L, then it is strongly $w_{\alpha}^{\beta}[\rho,f,q]$ −summable of order (α,β) to L.*

Proof. We assume that $x_k\to L$. Since f be a modulus function, we have $f(|x_k-L|)\to 0$. Since $\liminf q_k>0$, we have $[f(|x_k-L|)]^{q_k}\to 0$. Hence $w_{\alpha}^{\beta}[\rho,f,q]-\lim x_k=L$. □

Theorem 2.6. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0,1]$ be real numbers such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$, f be a modulus function , then $w_{\alpha_1}^{\beta_2}[\rho, f, q] \subset S_{\alpha_2}^{\beta_1}(\rho)$.*

Proof. Let $x \in w_{\alpha_1}^{\beta_2}[\rho, f, q]$ and let $\varepsilon > 0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k \leqslant n$ with $|x_k - L| \geq \varepsilon$ and $k \leqslant n$ with $|x_k - L| < \varepsilon$ respectively. Since $\rho_n^{\alpha_1} \leq \rho_n^{\alpha_2}$ for each n we have

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^{q_k}\right)^{\beta_2} &= \frac{1}{\rho_n^{\alpha_1}}\left[\sum\nolimits_1 [f(|x_k-L|)]^{q_k} + \sum\nolimits_2 [f(|x_k-L|)]^{q_k}\right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}}\left[\sum\nolimits_1 [f(|x_k-L|)]^{q_k} + \sum\nolimits_2 [f(|x_k-L|)]^{q_k}\right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}}\left[\sum\nolimits_1 [f(\varepsilon)]^{q_k}\right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}}\left[\sum\nolimits_1 \min([f(\varepsilon)]^d, [f(\varepsilon)]^D)\right]^{\beta_2} \\
&\geq \frac{1}{\rho_n^{\alpha_2}}\left|\{k \leqslant n : |x_k - L| \geq \varepsilon\}\right|^{\beta_1} \left[\min([f(\varepsilon)]^d, [f(\varepsilon)]^D)\right]^{\beta_1}.
\end{aligned}$$

We get $x \in S_{\alpha_2}^{\beta_1}(\rho)$. □

Theorem 2.7. *If f is a bounded modulus function and $\lim_{n\to\infty} \frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}} = 1$ then $S_{\alpha_1}^{\beta_2}(\rho) \subset w_{\alpha_2}^{\beta_1}[\rho, f, q]$.*

Proof. Let $x \in S_{\alpha_1}^{\beta_2}(\rho)$. Suppose that f be bounded. Therefore $f(x) \leq R$, for a positive integer R and all $x \geq 0$. Then for each $\varepsilon > 0$ we can write

$$\begin{aligned}
\frac{1}{\rho_n^{\alpha_2}}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^{q_k}\right)^{\beta_1} &\leq \frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}[f(|x_k-L|)]^{q_k}\right)^{\beta_1} \\
&= \frac{1}{\rho_n^{\alpha_1}}\left(\sum\nolimits_1 [f(|x_k-L|)]^{q_k} + \sum\nolimits_2 [f(|x_k-L|)]^{q_k}\right)^{\beta_1} \\
&\leq \frac{1}{\rho_n^{\alpha_1}}\left(\sum\nolimits_1 \max\left(R^d, R^D\right) + \sum\nolimits_2 [f(\varepsilon)]^{q_k}\right)^{\beta_1} \\
&\leq \left(\max\left(R^d, R^D\right)\right)^{\beta_2} \frac{1}{\rho_n^{\alpha_1}}\left|\{k \leqslant n : f(|x_k - L|) \geq \varepsilon\}\right|^{\beta_2} \\
&\quad + \frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}}\left(\max\left(f(\varepsilon)^d,\ f(\varepsilon)^D\right)\right)^{\beta_2}.
\end{aligned}$$

Hence $x \in w_{\alpha_2}^{\beta_1}[\rho, f, q]$. □

Theorem 2.8. *Let f be a modulus function. If $\lim q_k > 0$, then $w_\alpha^\beta[\rho, f, q] - \lim x_k = L$ uniquely.*

Proof. Let $\lim q_k = t > 0$. Suppose that $w_\alpha^\beta[\rho, f, q] - \lim x_k = L_1$ and $w_\alpha^\beta[\rho, f, q] - \lim x_k = L_2$. Then

$$\lim_n \frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^{n}[f(|x_k-L_1|)]^{q_k}\right)^{\beta} = 0,$$

and

$$\lim_n \frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|x_k-L_2|)]^{q_k}\right)^\beta = 0.$$

By definition of f and using (1.1), we may write

$$\begin{aligned}\frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|L_1-L_2|)]^{q_k}\right)^\beta &\leq \frac{A}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|x_k-L_1|)]^{q_k}+\sum_{k=1}^n [f(|x_k-L_2|)]^{q_k}\right)^\beta \\ &\leq \frac{A}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|x_k-L_1|)]^{q_k}\right)^\beta+\frac{A}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|x_k-L_2|)]^{q_k}\right)^\beta\end{aligned}$$

where $\sup_k q_k = D$, $0<\beta\leq\alpha\leq 1$ and $A=\max\left(1,2^{D-1}\right)$. Hence

$$\lim_n \frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^n [f(|L_1-L_2|)]^{q_k}\right)^\beta = 0.$$

Since $\lim_{k\to\infty} q_k = t$ we have $L_1-L_2=0$. Hence the limit is unique. □

Theorem 2.9. *Let $\rho=(\rho_n)$ and $\tau=(\tau_n)$ be two sequences such that $\rho_n \leqslant \tau_n$ for all $n\in\mathbb{N}$ and let $\alpha_1,\alpha_2,\beta_1$ and β_2 be such that $0<\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2\leq 1$,*

(i) If

$$\lim\inf_{n\to\infty} \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} > 0 \tag{2.3}$$

then $w_{\alpha_2}^{\beta_2}[\tau,f,q]\subset w_{\alpha_1}^{\beta_1}[\rho,f,q]$,

(ii) If

$$\lim\sup_{n\to\infty} \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}} < \infty \tag{2.4}$$

then $w_{\alpha_1}^{\beta_2}[\rho,f,q]\subset w_{\alpha_2}^{\beta_1}[\tau,f,q]$.

Proof. (i) Let $x\in w_{\alpha_2}^{\beta_2}[\tau,f,q]$. We have

$$\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^n [f(|x_k-L|)]^{q_k}\right)^{\beta_2} \geq \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^n [f(|x_k-L|)]^{q_k}\right)^{\beta_1}.$$

Thus if $x\in w_{\alpha_2}^{\beta_2}[\tau,f,q]$, then $x\in w_{\alpha_1}^{\beta_1}[\rho,f,q]$.

(ii) Let $x=(x_k)\in w_{\alpha_1}^{\beta_2}[\rho,f,q]$ and (2.4) holds. Now, since $\rho_n\leq\tau_n$ for all $n\in\mathbb{N}$, we have

$$\begin{aligned}\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^n [f(|x_k-L|)]^{q_k}\right)^{\beta_1} &\leq \frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^n [f(|x_k-L|)]^{q_k}\right)^{\beta_2} \\ &= \frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^n [f(|x_k-L|)]^{q_k}\right)^{\beta_2}\end{aligned}$$

for every $n\in\mathbb{N}$. Therefore $w_{\alpha_1}^{\beta_2}[\rho,f,q]\subset w_{\alpha_2}^{\beta_1}[\tau,f,q]$. □

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