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HOM-ALTERNATIVE ALGEBRAS AND HOM-JORDAN ALGEBRAS

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Abstract. The main feature of Hom-algebras is that the identities defining the structures are twisted by homomorphisms. The purpose of this paper is to introduce Hom-alternative algebras and Hom-Jordan algebras. We discuss some of their properties and provide construction procedures using ordinary alternative algebras or Jordan algebras. Also, we show that a plus algebra of Hom-associative algebra leads to Hom-Jordan algebra.

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1. Introduction

Hom-algebraic structures are algebras where the identities defining the structure are twisted by a homomorphism. They have been intensively investigated in the literature recently. The Hom-Lie algebras were introduced and discussed in [12,14,15,16], motivated by quasi-deformations of Lie algebras of vector fields, in particular q-deformations of Witt and Virasoro algebras. Hom-associative algebras were introduced in [17], where it is shown that the commutator bracket of a Hom-associative algebra gives rise to a Hom-Lie algebra and where a classification of Hom-Lie admissible algebras is established. Given a Hom-Lie algebra, there is a universal enveloping Hom-associative algebra (see [28]). Dualizing Hom-associative algebras, one can define Hom-coassociative coalgebras, Hom-bialgebras and Hom-Hopf algebras which were introduced in [20,18], see also [31,32,33,34,35]. It is shown in [30] that the universal enveloping Hom-associative algebra carries a structure of Hom-bialgebra. A study from the monoidal category point of view is given in [5]. See also [2,3,8,9,10,19,29] for other works on twisted algebraic structures. We refer for classical theory of Alternative algebras and related structures to [6,7,13,21,22,23,24,25,26,27].

The purpose of this paper is to introduce Hom-alternative algebras and Hom-Jordan algebras which are twisted versions of the ordinary alternative algebras and Jordan algebras. We discuss some of their properties and provide construction procedures using ordinary alternative algebras or Jordan algebras. Also, we show that a plus algebra of Hom-associative algebra leads to Hom-Jordan algebra.

In the first section of this paper we introduce Hom-alternative algebras, respectively left and right Hom-alternative algebras, and study their properties. In particular, we define a twisted version of the associator and show that for alternative algebras, the Hom-associator is an alternating function of its arguments. The second section is devoted to construction of Hom-alternative algebras. We show that an ordinary alternative algebra and one of its algebra endomorphisms lead to a Hom-alternative algebra where the twisting map is actually the algebra endomorphism. This process was introduced in [29] for Lie and associative algebras and more generally for G-associative algebras (see [17] for this class of algebras). It was also generalized to coalgebras in [20], [18] and to n-ary algebras of Lie and associative types in [3]. We derive examples of Hom-alternative algebras from 4-dimensional alternative algebras which are not associative and from algebra of octonions. The last section is dedicated to Jordan algebras. We introduce a notion of Hom-Jordan algebras and show that it fits with the Hom-associative structure, that is a Hom-associative algebra leads to Hom-Jordan algebra by considering plus algebra. Also, we provide a way to construct Hom-Jordan algebras starting from ordinary Jordan algebras and algebra endomorphisms.

2. Definitions and properties

Throughout this paper $\mathbb{K}$ denote a field of characteristic 0 and V a $\mathbb{K}$-linear space. The definitions and properties provided in this paper are still valid for any field of characteristic different from 2. First we recall the notion of Hom-associative algebra introduced in [17] and provide an example.

Definition 2.1. *A* **Hom-associative algebra** *over V is a triple (V, μ, α) where $\mu : V \times V \to V$ is a bilinear map and $\alpha : V \to V$ is a linear map, satisfying*

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)). \tag{1}$$

Example 2.2. *Let $\{e_1, e_2, e_3\}$ be a basis of a 3-dimensional linear space V over $\mathbb{K}$. The following multiplication μ and linear map α on V define Hom-associative algebras over $\mathbb{K}^3$:*

$$\begin{array}{lllllll}
\mu(e_1, e_1) & = & a\ e_1, & \qquad & \mu(e_2, e_2) & = & a\ e_2, \\
\mu(e_1, e_2) & = & \mu(e_2, e_1) = a\ e_2, & \qquad & \mu(e_2, e_3) & = & b\ e_3, \\
\mu(e_1, e_3) & = & \mu(e_3, e_1) = b\ e_3, & \qquad & \mu(e_3, e_2) & = & \mu(e_3, e_3) = 0,
\end{array}$$

$$\alpha(e_1) = a\ e_1, \quad \alpha(e_2) = a\ e_2, \quad \alpha(e_3) = b\ e_3,$$

where a, b are parameters in $\mathbb{K}$.

The algebras are not associative when $a \neq b$ and $b \neq 0$, since

$$\mu(\mu(e_1, e_1), e_3)) - \mu(e_1, \mu(e_1, e_3)) = (a - b)be_3.$$

Now, we introduce the notions of left Hom-alternative algebra, right Hom-alternative algebra and Hom-alternative algebra.

Definition 2.3. *A* **left Hom-alternative** *algebra (resp.* ***right Hom-alternative algebra****) is a triple (V, μ, α) consisting of a $\mathbb{K}$-linear space V, a multiplication $\mu : V \otimes V \to V$ and a linear map $\alpha : V \to V$ satisfying, for any x, y in V, the left Hom-alternative identity, that is*

$$\mu(\alpha(x), \mu(x, y)) = \mu(\mu(x, x), \alpha(y)), \tag{2}$$

respectively, right Hom-alternative identity, that is

$$\mu(\alpha(x), \mu(y, y)) = \mu(\mu(x, y), \alpha(y)). \tag{3}$$

A Hom-alternative algebra is one which is both left and right Hom-alternative algebra.

Remark 2.4. *Any Hom associative algebra is a Hom-alternative algebra.*

Definition 2.5. *Let (V, μ, α) and (V', μ', α') be two Hom-alternative algebras. A linear map $f : V \to V'$ is said to be a* **morphism of Hom-alternative algebras** *if*

$$\mu' \circ (f \otimes f) = f \circ \mu, \qquad f \circ \alpha = \alpha' \circ f.$$

Let (V, μ, α) be a Hom-algebra. We call ***Hom-associator*** associated to the Hom-algebra, which we denote by $\mathfrak{as}_\alpha$, the trilinear map defined for any $x, y, z \in V$ by

$$\mathfrak{as}_\alpha(x, y, z) = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)). \tag{4}$$

The condition (2) (resp. (3)) may be written using Hom-associator respectively

$$\mathfrak{as}_\alpha(x, x, y) = 0, \quad \mathfrak{as}_\alpha(y, x, x) = 0.$$

By linearization, we have the following equivalent definition of left and right Hom-alternative algebras.

Proposition 2.6. *A triple (V, μ, α) is a left Hom-alternative algebra (resp. right alternative algebra) if and only if the identity*

$$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(y), \mu(x, z)) - \mu(\mu(y, x), \alpha(z)) = 0 \tag{5}$$

respectively,

$$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(x), \mu(z, y)) - \mu(\mu(x, z), \alpha(y)) = 0 \tag{6}$$

holds.

Proof. We assume that, for any $x, y, z \in V$, $\mathfrak{as}_\alpha(x, x, z) = 0$ (left alternativity). Then we expand $\mathfrak{as}_\alpha(x + y, x + y, z) = 0$. The proof for right Hom-alternativity is obtained by expanding $\mathfrak{as}_\alpha(x, y + z, y + z) = 0$.

Conversely, we set $x = y$ in (5), respectively $y = z$ in (6). $\square$

Remark 2.7. *The multiplication could be considered as a linear map $\mu : V \otimes V \to V$, then the condition* (5) *and* (6) *may be written*

$$\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (id^{\otimes 3} + \sigma_1) = 0, \tag{7}$$

respectively

$$\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (id^{\otimes 3} + \sigma_2) = 0 \tag{8}$$

where id stands for the identity map and σ_1 and σ_2 stands for trilinear maps defined for any $x, y, z \in V$ by

$$\sigma_1(x \otimes y \otimes z) = y \otimes x \otimes z,$$
$$\sigma_2(x \otimes y \otimes z) = x \otimes z \otimes y.$$

In terms of associators, the identities (5) *and* (6) *are equivalent respectively to*

$$\mathfrak{as}_\alpha + \mathfrak{as}_\alpha \circ \sigma_1 = 0 \quad \textit{and} \quad \mathfrak{as}_\alpha + \mathfrak{as}_\alpha \circ \sigma_2 = 0. \tag{9}$$

Hence, for any $x, y, z \in V$, we have

$$\mathfrak{as}_\alpha(x, y, z) = -\mathfrak{as}_\alpha(y, x, z) \quad \text{and} \quad \mathfrak{as}_\alpha(x, y, z) = -\mathfrak{as}_\alpha(x, z, y). \tag{10}$$

We have also the following property.

Lemma 2.8. *Let (V, μ, α) be a Hom-alternative algebra. Then*

$$\mathfrak{as}_\alpha(x, y, z) = -\mathfrak{as}_\alpha(z, y, x). \tag{11}$$

Proof. Using (10), we have

$$\begin{aligned} \mathfrak{as}_\alpha(x, y, z) + \mathfrak{as}_\alpha(z, y, x) &= -\mathfrak{as}_\alpha(y, x, z) - \mathfrak{as}_\alpha(y, z, x) \\ &= 0. \end{aligned}$$

$\square$

Remark 2.9. *The identities* (10) *and* (11) *lead to the fact that an algebra is Hom-alternative if and only if the Hom-associator $\mathfrak{as}_\alpha(x, y, z)$ is an alternating function of its arguments, that is*

$$\mathfrak{as}_\alpha(x, y, z) = -\mathfrak{as}_\alpha(y, x, z) = -\mathfrak{as}_\alpha(x, z, y) = -\mathfrak{as}_\alpha(z, y, x).$$

Proposition 2.10. *A Hom-alternative algebra is Hom-flexible, that is $\mathfrak{as}_\alpha(x, y, x) = 0$.*

Proof. Using Lemma 2.8, we have $\mathfrak{as}_\alpha(x,y,x) = -\mathfrak{as}_\alpha(x,y,x)$. Therefore, $\mathfrak{as}_\alpha(x,y,x) = 0$. $\square$

Proposition 2.11. *Let (V,μ,α) be a Hom-alternative algebra and $x,y,z \in V$. If x and y anticommute, that is $\mu(x,y) = -\mu(y,x)$, then we have*

$$\mu(\alpha(x),\mu(y,z)) = -\mu(\alpha(y),\mu(x,z), \tag{12}$$

and

$$\mu(\mu(z,x),\alpha(y)) = -\mu(\mu(z,y),\alpha(x)). \tag{13}$$

Proof. The left alternativity leads to

$$\mu(\alpha(x),\mu(y,z)) - \mu(\mu(x,y),\alpha(z)) + \mu(\alpha(y),\mu(x,z)) - \mu(\mu(y,x),\alpha(z)) = 0. \tag{14}$$

Since $\mu(x,y) = -\mu(y,x)$, then the previous identity becomes

$$\mu(\alpha(x),\mu(y,z)) + \mu(\alpha(y),\mu(x,z)) = 0. \tag{15}$$

Similarly, using the right alternativity and the assumption of anticommutativity, we get the second identity. $\square$

Remark 2.12. *A subalgebra of a Hom-alternative algebra (V,μ,α) is given by a subspace W of V such that for any $x,y \in W$, we have $\mu(x,y) \in W$ and $\alpha(x) \in W$. The multiplication and the twisting map being the same. The notions of ideal, quotient algebra are defined as usual and similarly.*

3. Construction theorem and examples

In this section, we provide a way to construct Hom-alternative algebras starting from an alternative algebra and an algebra endomorphism. This procedure was applied to associative algebras, G-associative algebras and Lie algebra in [29]. It was extended to coalgebras in [20] and to n-ary algebras of Lie type respectively associative type in [3].

Theorem 3.1. *Let (V,μ) be a left alternative algebra (resp. a right alternative algebra) and $\alpha : V \to V$ be an algebra endomorphism. Then (V,μ_α,α), where $\mu_\alpha = \alpha \circ \mu$, is a left Hom-alternative algebra (resp. right Hom-alternative algebra).*

Moreover, suppose that (V',μ') is another left alternative algebra (resp. a right alternative algebra) and $\alpha' : V' \to V'$ is an algebra endomorphism. If $f : V \to V'$ is an algebras morphism that satisfies $f \circ \alpha = \alpha' \circ f$ then

$$f : (V,\mu_\alpha,\alpha) \longrightarrow (V',\mu'_{\alpha'},\alpha')$$

is a morphism of left Hom-alternative algebras (resp. right Hom-alternative algebras).

Proof. We show that (V, μ_α, α) satisfies the left Hom-alternative identity (2). Indeed

$$\begin{aligned}\mu_\alpha(\alpha(x)\otimes\mu_\alpha(x\otimes y)) &= \alpha(\mu(\alpha(x)\otimes\alpha(\mu(x)\otimes y)))\\ &= \alpha(\mu(\alpha(x)\otimes\mu(\alpha(x)\otimes\alpha(y)))\\ &= \alpha(\mu(\mu(\alpha(x)\otimes\alpha(x))\otimes\alpha(y)))\\ &= \alpha(\mu(\alpha(\mu(x\otimes x))\otimes\alpha(y)))\\ &= \mu_\alpha(\mu_\alpha(x\otimes x)\otimes\alpha(y)).\end{aligned}$$

The proof for right alternativity is obtained similarly. The second assertion follows from

$$f\circ\mu_\alpha = f\circ\alpha\circ\mu = \alpha'\circ f\circ\mu = \alpha'\circ\mu'\circ(f\times f) = \mu'_{\alpha'}\circ(f\times f). \qquad \square$$

The Theorem 3.1 gives a procedure to construct Hom-alternative algebras using ordinary alternative algebras and their algebras endomorphisms.

Remark 3.2. *Let (V, μ, α) be a Hom-alternative algebra, one may ask whether this Hom-alternative algebra is induced by an ordinary alternative algebra $(V, \widetilde{\mu})$, that is α is an algebra endomorphism with respect to $\widetilde{\mu}$ and $\mu = \alpha\circ\widetilde{\mu}$. This question was addressed and discussed for Hom-associative algebras in* [9,10].

First observation, if α is an algebra endomorphism with respect to $\widetilde{\mu}$ then α is also an algebra endomorphism with respect to μ. Indeed,

$$\mu(\alpha(x),\alpha(y)) = \alpha\circ\widetilde{\mu}(\alpha(x),\alpha(y)) = \alpha\circ\alpha\circ\widetilde{\mu}(x,y) = \alpha\circ\mu(x,y).$$

Second observation, if α is bijective then α^{-1} is also an algebra automorphism. Therefore one may use an untwist operation on the Hom-alternative algebra in order to recover the alternative algebra ($\widetilde{\mu} = \alpha^{-1}\circ\mu$).

3.1. Examples of Hom-Alternative algebras. We construct examples of Hom-alternative algebras. We use to this end the classification of 4-dimensional alternative algebras which are not associative (see [11]) and the algebra of octonions (see [4]). For each algebra, algebra endomorphisms are provided. Therefore, Hom-alternative algebras are attached according to Theorem 3.1.

Example 3.3 (Hom-alternative algebras of dimension 4)**.** *According to* [11]*, p 144, there are exactly two alternative but not associative algebras of dimension 4 over any field. With respect to a basis $\{e_0, e_1, e_2, e_3\}$, one algebra is given by the following multiplication (the unspecified products are zeros)*

$$\mu_1(e_0,e_0) = e_0,\ \mu_1(e_0,e_1) = e_1,\ \mu_1(e_2,e_0) = e_2,$$
$$\mu_1(e_2,e_3) = e_1,\ \mu_1(e_3,e_0) = e_3,\ \mu_1(e_3,e_2) = -e_1.$$

The other algebra is given by

$$\mu_2(e_0,e_0)=e_0,\ \mu_2(e_0,e_2)=e_2,\ \mu_2(e_0,e_3)=e_3,$$
$$\mu_2(e_1,e_0)=e_1,\ \mu_2(e_2,e_3)=e_1,\ \mu_2(e_3,e_2)=-e_1.$$

These two alternative algebras are anti-isomorphic, that is the first one is isomorphic to the opposite of the second one. The algebra endomorphisms of μ_1 and μ_2 are exactly the same. We provide two examples of algebra endomorphisms for these algebras.

(1) *The algebra endomorphism α_1 with respect to the same basis is defined by*

$$\alpha_1(e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\quad \alpha_1(e_1)=0,$$
$$\alpha_1(e_2)=a_4\ e_2+\frac{a_4a_3}{a_2}\ e_3,\quad \alpha_1(e_3)=a_5\ e_2+\frac{a_5a_3}{a_2}\ e_3,$$

with $a_1,\cdots,a_5\in\mathbb{K}$ and $a_2\neq 0$.

(2) *The algebra endomorphism α_2 with respect to the same basis is defined by*

$$\alpha_2(e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\quad \alpha_2(e_1)=a_4\ e_1,$$
$$\alpha_2(e_2)=-\frac{a_4a_2}{a_5}\ e_2-\frac{a_4a_3}{a_5}\ e_3,\quad \alpha_2(e_3)=a_5\ e_1+a_6\ e_2+\frac{a_6a_3-a_5}{a_2}\ e_3,$$

with $a_1,\cdots,a_6\in\mathbb{K}$ and $a_2,a_5\neq 0$.

According to Theorem 3.1, the linear map α_1 an the following multiplications

- $\mu_1^1(e_0,e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\ \mu_1^1(e_0,e_1)=0,\ \mu_1^1(e_2,e_0)=a_4\ e_2+\dfrac{a_4a_3}{a_2}\ e_3,$
  $\mu_1^1(e_2,e_3)=0,\ \mu_1^1(e_3,e_0)=a_5\ e_2+\dfrac{a_5a_3}{a_2}\ e_3,\ \mu_1(e_3,e_2)=0.$

- $\mu_2^1(e_0,e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\ \mu_2^1(e_0,e_2)=a_4\ e_2+\dfrac{a_4a_3}{a_2}\ e_3,$
  $\mu_2^1(e_0,e_3)=a_5\ e_2+\dfrac{a_5a_3}{a_2}\ e_3,\ \mu_2^1(e_1,e_0)=0,\ \mu_2^1(e_2,e_3)=0,\ \mu_2^1(e_3,e_2)=0.$

determine 4-dimensional Hom-alternative algebras.

The linear map α_2 leads to the following multiplications

- $\mu_1^2(e_0,e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\ \mu_1^2(e_0,e_1)=a_4\ e_1,\ \mu_1^2(e_2,e_0)=-\dfrac{a_4a_2}{a_5}\ e_2-\dfrac{a_4a_3}{a_5}\ e_3,$
  $\mu_1^2(e_2,e_3)=a_4\ e_1,\ \mu_1^2(e_3,e_0)=e_3,\ \mu_1^2(e_3,e_2)=-a_4\ e_1.$

- $\mu_2^2(e_0,e_0)=e_0+a_1\ e_1+a_2\ e_2+a_3\ e_3,\ \mu_2^2(e_0,e_2)=-\dfrac{a_4a_2}{a_5}\ e_2-\dfrac{a_4a_3}{a_5}\ e_3,$
  $\mu_2^2(e_0,e_3)=a_5\ e_1+a_6\ e_2+\dfrac{a_6a_3-a_5}{a_2}\ e_3,$
  $\mu_2^2(e_1,e_0)=a_4\ e_1,\ \mu_2(e_2,e_3)=a_4\ e_1,\ \mu_2^2(e_3,e_2)=-a_4\ e_1.$

Example 3.4 (Octonions)**.** Octonions are typical example of alternative algebra. They were discovered in 1843 by John T. Graves who called them Octaves and independently by Arthur Cayley in 1845. See [4] for the role of the octonions in algebra, geometry and topology. See also [1] where octonions are viewed as a quasialgebra. The octonions algebra which is also called Cayley Octaves or Cayley algebra is 8-dimensional and defined with respect to a basis $\{u, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$, where u is the identity for the multiplication, by the following multiplication table. The table describes multiplying the ith row elements by the jth column elements.

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-u$	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	e_2	$-e_4$	$-u$	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	e_3	$-e_7$	$-e_5$	$-u$	e_6	e_2	$-e_4$	e_1
e_4	e_4	e_2	$-e_1$	$-e_6$	$-u$	e_7	e_3	$-e_5$
e_5	e_5	$-e_6$	e_3	$-e_2$	$-e_7$	$-u$	e_1	e_4
e_6	e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	$-u$	e_2
e_7	e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	$-u$

The diagonal algebra endomorphism of octonions are give by maps α defined with respect to the basis $\{u, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ by

$$\alpha(u) = u, \quad \alpha(e_1) = a\ e_1, \quad \alpha(e_2) = b\ e_2, \quad \alpha(e_3) = c\ e_3,$$
$$\alpha(e_4) = ab\ e_4, \quad \alpha(e_5) = bc\ e_5, \quad \alpha(e_6) = abc\ e_6, \quad \alpha(e_7) = ac\ e_7,$$

where a, b, c are parameter in $\mathbb{K}$ satisfying $a^2 = 1$, $b^2 = 1$ and $c^2 = 1$. The associated Hom-alternative algebra to the octonions algebra, for $a = b = c = -1$, according to Theorem 3.1 is described by the map α and the multiplication defined by the following table. The table describes multiplying the ith row elements by the jth column elements.

	u	e_1	e_2	e_3	e_4	e_5	e_6	e_7
u	u	$-e_1$	$-e_2$	$-e_3$	e_4	e_5	$-e_6$	e_7
e_1	$-e_1$	$-u$	e_4	e_7	e_2	$-e_6$	$-e_5$	e_3
e_2	$-e_2$	$-e_4$	$-u$	e_5	$-e_1$	e_3	e_7	e_6
e_3	$-e_3$	$-e_7$	$-e_5$	$-u$	$-e_6$	$-e_2$	$-e_4$	$-e_1$
e_4	e_4	$-e_2$	e_1	e_6	$-u$	e_7	$-e_3$	$-e_5$
e_5	e_5	e_6	$-e_3$	e_2	$-e_7$	$-u$	$-e_1$	e_4
e_6	$-e_6$	e_5	$-e_7$	e_4	e_3	e_1	$-u$	$-e_2$
e_7	e_7	$-e_3$	$-e_6$	e_1	e_5	$-e_4$	e_2	$-u$

Notice that the new algebra is no longer unital neither an alternative algebra since

$$\mu(u, \mu(u, e_1)) - \mu(\mu(u, u), e_1) = 2e_1.$$

4. Hom-Jordan algebras

In this section, we introduce a generalization of Jordan algebra by twisting the usual Jordan identity

$$(x \cdot y) \cdot x^2 = x \cdot (y \cdot x^2). \tag{16}$$

We show that this generalization fits with Hom-associative algebras. Also, we provide a procedure to construct examples starting from an ordinary Jordan algebra.

Definition 4.1. *A* ***Hom-Jordan algebra*** *is a triple (V, μ, α) consisting of a linear space V, a multiplication $\mu : V \times V \to V$ which is commutative and a homomorphism $\alpha : V \to V$ satisfying for any $x, y, z \in V$*

$$\mu(\alpha^2(x), \mu(y, \mu(x, x))) = \mu(\mu(\alpha(x), y), \alpha(\mu(x, x))) \tag{17}$$

where $\alpha^2 = \alpha \circ \alpha$.

Let (V, μ, α) and (V', μ', α') be two Hom-Jordan algebras. A linear map $\phi : V \to V'$ is a morphism of Hom-Jordan algebras if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad \text{and} \quad \phi \circ \alpha = \alpha' \circ \phi.$$

Remark 4.2. *Since the multiplication is commutative, one may write the identity* (17) *as*

$$\mu(\mu(y, \mu(x, x)), \alpha^2(x)) = \mu(\mu(y, \alpha(x)), \alpha(\mu(x, x))). \tag{18}$$

When the twisting map α is the identity map, we recover the classical notion of Jordan algebra.

The identity (17) is motivated by the following functor which associates to a Hom-associative algebra a Hom-Jordan algebra by considering the plus Hom-algebra.

Theorem 4.3. *Let (V, m, α) be a Hom-associative algebra. Then the Hom-algebra (V, μ, α), where the multiplication μ is defined for $x, y \in V$ by*

$$\mu(x, y) = \frac{1}{2}(m(x, y) + m(y, x)).$$

is a Hom-Jordan algebra.

Proof. The commutativity of μ is obvious. We compute the difference

$$D = \mu(\alpha^2(x), \mu(y, \mu(x, x))) - \mu(\mu(\alpha(x), y), \alpha(\mu(x, x)))$$

A straightforward computation gives

$$\begin{aligned} D &= m(\alpha^2(x), m(y, m(x,x))) + m(m(y, m(x,x)), \alpha^2(x)) + m(\alpha^2(x), m(m(x,x), y)) \\ &\quad + m(m(m(x,x), y), \alpha^2(x)) - m(m(\alpha(x), y), \alpha(m(x,x))) - m(\alpha(m(x,x)), m(\alpha(x), y)) \\ &\quad - m(m(y, \alpha(x)), \alpha(m(x,x))) - m(\alpha(m(x,x)), m(y, \alpha(x))). \end{aligned}$$

We have by Hom-associativity

$$m(\alpha^2(x), m(y, m(x,x))) - m(m(\alpha(x), y), \alpha(m(x,x))) = 0$$
$$m(m(m(x,x), y), \alpha^2(x)) - m(\alpha(m(x,x)), m(y, \alpha(x))) = 0.$$

Therefore

$$\begin{aligned} D &= m(m(y, m(x,x)), \alpha^2(x)) + m(\alpha^2(x), m(m(x,x), y)) \\ &\quad - m(\alpha(m(x,x)), m(\alpha(x), y)) - m(m(y, \alpha(x)), \alpha(m(x,x))). \end{aligned}$$

One may show that for any Hom-associative algebra we have

$$\begin{aligned} m(\alpha(m(x,x)), m(\alpha(x), y)) &= m(m(m(x,x), \alpha(x)), \alpha(y)) \\ &= m(m(\alpha(x), m(x,x)), \alpha(y)) \\ &= m(\alpha^2(x), m(m(x,x), y)), \end{aligned}$$

and similarly

$$m(m(y, \alpha(x)), \alpha(m(x,x))) = m(m(y, m(x,x)), \alpha^2(x)).$$

Thus

$$\begin{aligned} D &= m(m(y, m(x,x)), \alpha^2(x)) + m(\alpha^2(x), m(m(x,x), y)) \\ &\quad - m(\alpha^2(x), m(m(x,x), y)) - m(m(y, m(x,x)), \alpha^2(x)) \\ &= 0. \end{aligned}$$

□

Remark 4.4. *Notice also that in general a Hom-alternative algebra doesn't lead to a Hom-Jordan algebra. In a recent publication* [36]*, following the preprint version of this paper, D. Yau considered a subclass of these Hom-Jordan algebras. The identity* (18) *is replaced by*

$$\mu(\mu(\alpha(y), \mu(x,x)), \alpha^2(x)) = \mu(\mu(\alpha(y), \alpha(x)), \alpha(\mu(x,x))). \tag{19}$$

where instead of any y, he takes $\alpha(y)$.

Then any Hom-alternative algebra gives rise to a Hom-Jordan of this type.

The following theorem gives a procedure to construct Hom-Jordan algebras using ordinary Jordan algebras and their algebra endomorphisms.

Theorem 4.5. *Let (V,μ) be a Jordan algebra and $\alpha : V \to V$ be an algebra endomorphism. Then (V,μ_α,α), where $\mu_\alpha = \alpha\circ\mu$, is a Hom-Jordan algebra.*

Moreover, suppose that (V',μ') is another Jordan algebra and $\alpha' : V' \to V'$ is an algebra endomorphism. If $f : V \to V'$ is an algebras morphism that satisfies $f\circ\alpha = \alpha'\circ f$ then

$$f : (V,\mu_\alpha,\alpha) \longrightarrow (V',\mu'_{\alpha'},\alpha')$$

is a morphism of Hom-Jordan algebras.

Proof. We show that (V,μ_α,α) satisfies the Hom-Jordan identity (17) while (V,μ) satisfies the Jordan identity (16). Indeed

$$\begin{aligned}
&\mu_\alpha(\alpha^2(x),\mu_\alpha(y,\mu_\alpha(x,x))) - \mu_\alpha(\mu_\alpha(\alpha(x),y),\alpha(\mu_\alpha(x,x)))\\
&= \alpha\circ\mu(\alpha^2(x),\alpha\circ\mu(y,\alpha\circ\mu(x,x))) - \alpha\circ\mu(\alpha\circ\mu(\alpha(x),y),\alpha^2\circ\mu(x,x))\\
&= \alpha^2(\mu(\alpha(x),\mu(y,\alpha\circ\mu(x,x))) - \mu(\mu(\alpha(x),y),\alpha\circ\mu(x,x)))\\
&= \alpha^2(\mu(\alpha(x),\mu(y,\mu(\alpha(x),\alpha(x)))) - \mu(\mu(\alpha(x),y),\mu(\alpha(x),\alpha(x))))\\
&= 0.
\end{aligned}$$

the second assertion follows from

$$f\circ\mu_\alpha = f\circ\alpha\circ\mu = \alpha'\circ f\circ\mu = \alpha'\circ\mu'\circ(f\times f) = \mu'_{\alpha'}\circ(f\times f). \qquad \square$$

Remark 4.6. *We may give here similar observations as in the remark 3.2 concerning Hom-Jordan algebra induced by an ordinary Jordan algebra.*

We provide in the sequel examples of Hom-Jordan algebras using Theorem 4.3.

Example 4.7. *We consider Hom-Jordan algebras associated to Hom-associative algebras described in example (2.2). Let $\{e_1,e_2,e_3\}$ be a basis of a 3-dimensional linear space V over $\mathbb{K}$. The following multiplication $\widetilde{\mu}$ and linear map α on V define Hom-Jordan algebras over $\mathbb{K}^3$:*

$$\begin{array}{lllllll}
\widetilde{\mu}(e_1,e_1) &=& a\ e_1, & \qquad & \widetilde{\mu}(e_2,e_2) &=& a\ e_2,\\
\widetilde{\mu}(e_1,e_2) &=& \widetilde{\mu}(e_2,e_1) = a\ e_2, & & \widetilde{\mu}(e_2,e_3) &=& \frac{1}{2}b\ e_3,\\
\widetilde{\mu}(e_1,e_3) &=& \widetilde{\mu}(e_3,e_1) = b\ e_3, & & \widetilde{\mu}(e_3,e_2) &=& \widetilde{\mu}(e_3,e_3) = 0,
\end{array}$$

$$\alpha(e_1) = a\ e_1, \quad \alpha(e_2) = a\ e_2, \quad \alpha(e_3) = b\ e_3$$

where a,b are parameters in $\mathbb{K}$.

It turns out that the multiplications of this Hom-Jordan algebras define Jordan algebras.

Remark 4.8. *We may define noncommutative Jordan algebras as triples (V,μ,α) satisfying the identity (17) and the flexibility condition, which is a generalization of* *the commutativity. Eventually, we may consider the Hom-flexibilty defined by the identity* $\mu(\alpha(x), \mu(y,x)) = \mu(\mu(x,y), \alpha(x))$.

Remark 4.9. *A* $\mathbb{Z}_2$*-graded version of Hom-alternative algebras and Hom-Jordan algebras might be defined in a natural way.*

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