Turk. J. Math. Comput. Sci.
15(2)(2023) 433–442

DOI : 10.47000/tjmcs.1092714

Tubular Surfaces According to a Focal Curve in E^3

Abdullah Yildirim

Harran University, Science and Arts Faculty, Mathematics Department, 63300, Şanlıurfa, Türkiye.

Received: 24-03-2022 • Accepted: 11-08-2023

Abstract. A spine curve moves through the middle of a canal or a tubular surface. It might be asked whether it is possible to carry a spine curve over a tubular surface. For a tubular surface, we have seen that it can be done. In this study, we have given the general equations of a canal surface and a tubular surface according to a focal curve. In this case, we found the fundamental curvatures of a tubular surface. We gave theorems and proofs about the focal curve being a special curve.

2020 AMS Classification: 53A04, 53A05, 53C22

Keywords: Focal curve, tube surface, parameric curve, geodesic.

1. Introduction

A canal surface appears as the envelope of a collection of 1-parameter spheres. The center of these spheres is on a curve, which is named the center curve or the spine curve. If the radii of these spheres are constant, they are named a tubular surfaces. Canal surfaces occupy an important place in the surface theory of differential geometry. These surfaces provide a great convenience in engineering applications. Since canal surfaces were first described by the French mathematician Gaspard Monge in 1850, much work has been done algebraic and geometrical in Euclidean and non-Euclidean spaces, [1–3, 5, 8, 11, 12]. For these purposes, we carry out this study.

In studies done so far, the spine curve moves through the middle of a canal or a tubular surface. In this study, we have seen that if we take the focal curve instead of the spine curve of a tubular surface , it can be expressed according to a curve traveling on a tubular surface. We reinforced this with results, theorems and examples.

2. Preliminaries

In this section, some basic notions of regular curves, which have all velocity vectors different from zero, are given.

Definition 2.1. Let us give a function $f : \mathrm{I} \longmapsto \mathbf{R}$. The derivative of the function $f(\delta)$ of a variable δ is

$$f'(\delta) = \frac{\mathrm{d}f}{\mathrm{d}\delta} = \lim_{\triangle\delta\to 0} \frac{f(\delta+\triangle\delta) - f(\delta)}{\triangle\delta},$$

provided that the limit exists and the function $f(\delta)$ is said to differentiable at δ_0. If it is differenetiable at every point in domain I, then it is said to be differntiable on I.

Email address: abdullahyildirim@harran.edu.tr (A. Yıldırım)

For $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbf{R}^3$, the scalar product of X, and Y is defined by

$$g(X, Y) = x_1y_1 + x_2y_2 + x_3y_3.$$

The cross product of X, and Y is defined as

$$X \times Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Let a curve $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength δ. In other words, let the curve α be a unit speed curve. Its velocity vector is $T(\delta) = \alpha'(\delta) = \dfrac{\mathrm{d}\alpha(\delta)}{\mathrm{d}\delta}$. The principal normal N is defined as $\dfrac{T'}{\|T'\|}$ and the binormal vectors B is defined as $N \times T$. The family $\{T, N, B\}$ is an orthonormal triad and is named the Frenet frame.

For a curve α, the rate of change of the Frenet-Serret vector equations may be expressed as

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N, \end{aligned}$$

where the coefficients κ, and τ are the first, and the second curvatures of the curve α, respectively [8].

In $\mathbf{R}^3$ curvatures of an arbitrary curve X is derived as

$$\kappa = \frac{\|X' \times X''\|}{\|X'\|^3}, \qquad \tau = \frac{g(X' \times X'', X''')}{\|X' \times X''\|^2}, \tag{2.1}$$

where $\times$ is the cross product in $\mathbf{R}^3$ [5, 9, 10]. According to (2.1), the first, and the second curvatures of the curve α

$$\kappa_\alpha = \left\|\alpha''\right\| \text{ and } \tau_\alpha = \frac{g\left(\alpha' \times \alpha'', \alpha'''\right)}{\|\alpha''\|^2},$$

respectively

If α', and α'' are linearly independent in I, then the curve α is named well defined [11].

From now on, we will assume that the given curves are good.

3. Focal Curves in $\mathbf{E}^3$

Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength δ. The points of C_α are the centers of the osculating spheres of α, and it is named the focal curve of the curve α. The sphere $\|C_\alpha - \alpha\|^2 = r^2$ with the center C_α is maned the osculating sphere [12].

Let us assume

$$f(\delta) = \frac{1}{2}\left(\|C_\alpha(\delta) - \alpha(\delta)\|^2 - r^2\right).$$

If there are infinitely close joint 4-points between the curve α with its osculating sphere at $\delta = \delta_0$, then we have

$$f(\delta_0) = f^{'}(\delta_0) = f^{''}(\delta_0) = f^{'''}(\delta_0) = 0.$$

The plane spanned by both the tangent vector, and the principal normal vector of a curve is named the osculating plane. A point of a smooth curve in $\mathbf{E}^3$ for which the derivative of the curve of order 3 belongs to the osculating plane is named a flattening.

If there are infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $\delta = \delta_0$ of the curve α, it is named a vertex of the curve. Conversely, unless there are infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $\delta = \delta_0$ of the curve α, it is named non-vertex of the curve.

From now on, we assume that all points of the given curves are non-vertex.

Lemma 3.1. [1] *Let α be given by arclength in $\mathbf{E}^3$, and its Frenet frame be $\{T, N, B\}$. Then, the focal curve C_α of α is*

$$C_\alpha = \alpha + c_1N + c_2B, \tag{3.1}$$

and the focal coefficients of C_α are given by

$$c_1 = \frac{1}{\kappa_\alpha}, \qquad c_2 = c_1^{'}\frac{1}{\tau_\alpha}, \tag{3.2}$$

where $\kappa_\alpha \neq 0$, and $\tau_\alpha \neq 0$ are the curvature, and the torsion of the curve α.

Lemma 3.2 ([1]). *Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength. If a non-flattening point of α is a vertex, then*

$$c_2' + c_1\tau_\alpha = 0.$$

The opposite is also possible.

The forthcoming theorem, lemmas, and corollaries state the relations between a curve α, and its focal curve C_α.

Theorem 3.3. *Let $\alpha : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength. Let $\{T, N, B\}$ (resp. $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$) be the Frenet frame of the curve α (resp. C_α). Let κ_α, and τ_α be the curvature, and the torsion of the curve α, respectively. Then, we have the connections as follows;*

$$\mathbf{t} = \varepsilon_{\mathbf{t}} B,$$
$$\mathbf{n} = \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N,$$
$$\mathbf{b} = -\varepsilon_{\mathbf{n}} T,$$

where $\varepsilon_{\mathbf{t}} = \dfrac{c_2' + c_1\tau_\alpha}{\left|c_2' + c_1\tau_\alpha\right|}$, and $\varepsilon_{\mathbf{n}} = \dfrac{\tau_\alpha}{|\tau_\alpha|}$.

Proof. Let σ be the arc length parameter of the focal curve C_α. If we take the derivative of both sides of (3.1) concerning the arclength parameter δ, we reach

$$\frac{\mathrm{d}C_\alpha}{\mathrm{d}\delta} = \frac{\mathrm{d}C_\alpha}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\delta} = \left[c_2' + c_1\tau_\alpha\right] B, \tag{3.3}$$

and if we take the norm of both sides of (3.3), we reach

$$\frac{\mathrm{d}\delta}{\mathrm{d}\sigma} = \frac{1}{\left|c_2' + c_1\tau_\alpha\right|},$$

and

$$\mathbf{t} = \varepsilon_{\mathbf{t}} B = \frac{\left(c_2' + c_1\tau_\alpha\right)}{\left|c_2' + c_1\tau_\alpha\right|} B = \frac{dC_\alpha}{d\sigma}. \tag{3.4}$$

Now, differentiating both sides of (3.4) concerning the arclength parameter δ we obtain

$$\mathbf{n} = \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N,$$

and

$$\kappa_{C_\alpha} = \frac{|\tau_\alpha|}{\left|c_2' + c_1\tau_\alpha\right|}. \tag{3.5}$$

On the other hand, we may write

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = (\varepsilon_{\mathbf{t}} B) \times (\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N) = -\varepsilon_{\mathbf{n}} T.$$

□

Let κ_α, and τ_α (resp. κ_{C_α}, and τ_{C_α}) be the curvature, and the torsion of the curve α (resp. the first and the second curvatures of the focal curve C_α). Accordingly to this, we can express the following corollaries.

Corollary 3.4. *Taking the derivative of (3) concerning the arclength parameter δ, we obtain*

$$\kappa_\alpha = \left|\tau_{C_\alpha}\right| \left|c_2' + c_1\tau_\alpha\right|. \tag{3.6}$$

Corollary 3.5. *Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength. If the curve α is spherical, then*

$$\begin{aligned} r^2 &= \|C_\alpha - \alpha\|^2, \\ &= \|c_1 N + c_2 B\|^2, \\ &= c_1^2 + c_2^2, \end{aligned} \tag{3.7}$$

where r is the radius of spherical, and differentiating (3.7) with respect to the arc length parameter δ, we obtain

$$\left(r^2\right)' = 2c_2\left(c_2' + c_1\tau_\alpha\right). \tag{3.8}$$

The opposite is also true. According to (3.8), if r is a constant, then

$$c_2 = 0.$$

Since the curve α is a non-vertex curve, $c_2' + c_1\tau_\alpha \neq 0$.

Corollary 3.6. *If we consider (3.2) and (3.8), the focal coefficients of c_1, c_2 of the curve α satisfy the following matrix-vector equation*

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' - \dfrac{\left(r^2\right)'}{2c_2} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix}.$$

Corollary 3.7. *If we consider (3.5) and (3.6), then we obtain that*

$$\frac{\kappa_{C_\alpha}}{|\tau_\alpha|} = \frac{\left|\tau_{C_\alpha}\right|}{\kappa_\alpha} = \frac{1}{\left|c_2' + c_1\tau_\alpha\right|} = \frac{2\,|c_2|}{\left|(r^2)'\right|}.$$

Lemma 3.8. *Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength. If its osculating sphere radius r is constant, then its curvature κ is constant, and*

$$r = |c_1| = \frac{1}{\kappa_\alpha},$$

where c_1 is the first focal coefficient of the focal curve C_α, so

$$C_\alpha' = c_1\tau_\alpha B.$$

Proof. Since the curve α is a non-vertex curve, $c_2' + c_1\tau_\alpha \neq 0$. Using (3.2), and (3.8) we find that $c_1 = \dfrac{1}{\kappa_\alpha}$ is a constant. □

Lemma 3.9. *According to the parameter δ, derivative changes of Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \nu\kappa_{C_\alpha} & 0 \\ -\nu\kappa_{C_\alpha} & 0 & \nu\tau_{C_\alpha} \\ 0 & -\nu\tau_{C_\alpha} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where $\nu = \dfrac{\mathrm{d}\sigma}{\mathrm{d}\delta} = \left|c_2' + c_1\tau_\alpha\right|$. If the radius of the osculating sphere r is a constant, then

$$\nu = \frac{\mathrm{d}\sigma}{\mathrm{d}\delta} = r\,|\tau_\alpha|\,,$$

where δ and σ are the arc length parameters of the curve α , and the focal curve C_α, respectively.

4. Canal Surfaces in $\mathbf{E}^3$

Let us state equations for canal or tubular surface around any good curve in $\mathbf{E}^3$. Let a regular and unite speed curve $\alpha : \mathrm{I} \longrightarrow \mathbf{E}^3$ be parametrized by s and the Frenet frame of the curve α be $\{T, N, B\}$. The derivative changes of these vector fields are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where κ_α and τ_α are the curvature, and the torsion of the curve α, respectively.

If we refer to the concepts in [2, 12], a canal surface appears as the envelope of a collection of 1-parameter spheres. Let these spheres have varying radii $r(t)$ and the centers on a curve $\alpha(t)$ which is named the spine curve. The circle where the canal surface is tangent to the spheres is named the characteristic circle denoted by $K(t)$. In this framework, with the help of Figure 1, a canal surface is parameterized as

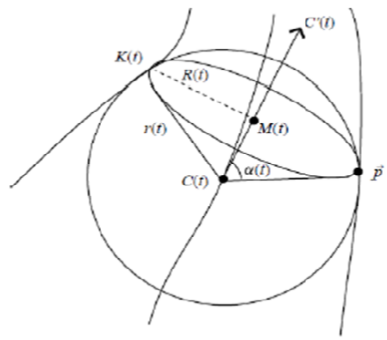

Figure 1. The part of a canal surface, [2]

$$\begin{aligned} K(t,\phi) &= \alpha(t) - r(t)r^{'}(t)\frac{\alpha^{'}(t)}{\|\alpha^{'}(t)\|} \\ &\pm \cos\phi r(t)\frac{\sqrt{\|\alpha^{'}(t)\|^{2} - r^{'}(t)^{2}}}{\|\alpha^{'}(t)\|}N(t) \\ &\pm \sin\phi r(t)\frac{\sqrt{\|\alpha^{'}(t)\|^{2} - r^{'}(t)^{2}}}{\|\alpha^{'}(t)\|}B(t). \end{aligned}$$

This surface is named a canal surface around the curve α. The principal vector field N and the binormal vector field B of the curve α are in the plane with the characteristic circle. If the curve α is a unite speed curve by an arc length δ, then the canal surface is reparameterized as

$$\begin{aligned} K(\delta,\phi) &= \alpha(\delta) - r(\delta)r^{'}(\delta)T(\delta) \\ &\pm \cos\phi r(\delta)\sqrt{1 - r^{'}(\delta)^{2}}N(\delta) \\ &\pm \sin\phi r(\delta)\sqrt{1 - r^{'}(\delta)^{2}}B(\delta). \end{aligned}$$

A canal surface is a tubular surface provided that the radius $r(\delta) = r$ is a constand, and in this case the equation will be as follows;

$$L(\delta,\phi) = \alpha(\delta) + r\left(\cos\phi N(\delta) + \sin\phi B(\delta)\right), \tag{4.1}$$

where $0 \leq \phi \leq 2\pi$.

If we take the focal curve C_α instead of the curve α in $\mathbf{E}^3$, (4.1) will be as follows;

$$L(\delta,\phi) = C_\alpha(\delta) + r\left(\mp\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\cos\phi N(\delta) \pm \varepsilon_{\mathbf{n}}\sin\phi T(\delta)\right). \tag{4.2}$$

Without loss of generality in (4.2) we can take $\varepsilon_{\mathbf{t}} = \varepsilon_{\mathbf{n}} = 1$. By such preference, (4.2) reads as

$$L(\delta,\phi) = C_\alpha(\delta) + r\left(\sin\phi T(\delta) - \cos\phi N(\delta)\right). \tag{4.3}$$

Example 4.1. Let β be a circular helix defined by

$$\beta(\delta) = \left(\frac{1}{\sqrt{2}}\cos(\delta), \frac{1}{\sqrt{2}}\sin(\delta), \frac{\delta}{\sqrt{2}}\right).$$

Since $\|\beta'(\delta)\| = 1$, this curve is regular curve where $0 \leq \delta \leq 2\pi$.

The Frenet vectors $\left\{T_\beta, N_\beta, B_\beta\right\}$ of the curve β are as follows;

$$\begin{aligned} T_\beta(\delta) &= \left(-\frac{1}{\sqrt{2}}\sin(\delta), \frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\right), \\ N_\beta(\delta) &= \left(-\cos(\delta), -\sin(\delta), 0\right), \\ B_\beta(\delta) &= \left(\frac{1}{\sqrt{2}}\sin(\delta), -\frac{1}{\sqrt{2}}\cos(\delta), \frac{1}{\sqrt{2}}\right). \end{aligned}$$

The curvatures of the curve β are found to be $\kappa_\beta = \frac{1}{\sqrt{2}}$, and $\tau_\beta = \frac{1}{\sqrt{2}}$ by using in (2.1). The focal coefficients of the curve β can be computed from (3.2) as $c_1 = \sqrt{2}$, and $c_2 = 0$.

For this specific example, using (3.1) the focal curve C_β of the curve β may be computed as

$$C_\beta = \beta + \sqrt{2}N. \tag{4.4}$$

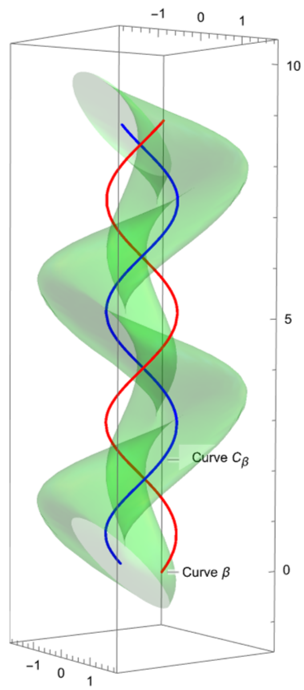

Figure 2. For $0 \le \delta \le 4\pi$, $0 \le \phi \le 2\pi$ tubular surface around the focal curve C_β of the curve β.

Using (4.3) and (4.4) with $r = \sqrt{2}$ lead to the components

$$\begin{aligned}
x_\beta(\delta,\phi) &= \frac{1}{\sqrt{2}}\cos(\delta) - \sqrt{2}\cos(\delta) - \sin(\phi)\sin(\delta) - \sqrt{2}\cos(\phi)\cos(\delta),\\
y_\beta(\delta,\phi) &= \frac{1}{\sqrt{2}}\sin(\delta) - \sqrt{2}\sin(\delta) + \sin(\phi)\cos(\delta) - \sqrt{2}\cos(\phi)\sin(\delta),\\
z_\beta(\delta,\phi) &= \frac{\delta}{\sqrt{2}} + \sin(\phi)
\end{aligned}$$

of the tubular surface $L_\beta(\delta,\phi) = \left(x_\beta(\delta,\phi), y_\beta(\delta,\phi), z_\beta(\delta,\phi)\right)$. According to the focal curve C_β the tubular surface $L_\beta(\delta,\phi)$ is given in Figure 2.

Example 4.2. Let γ be the Salkowski curve γ [6] defined by

$$\gamma(\delta) = \begin{pmatrix}
\frac{8}{\sqrt{65}}\left(-\frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)}\sin\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)}\sin\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1}{2}\sin\delta\right),\\
\frac{8}{\sqrt{65}}\left(\frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)}\cos\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right) + \frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)}\cos\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1}{2}\cos\delta\right),\\
\frac{16}{\sqrt{65}}\cos\left(\frac{2}{\sqrt{65}}\delta\right)
\end{pmatrix}.$$

Since $\|\gamma'(\delta)\| = \dfrac{8}{\sqrt{65}}\cos\left(\frac{\delta}{\sqrt{65}}\right)$, this curve is regular curve in $\left(-\dfrac{\pi\sqrt{65}}{2}, \dfrac{\pi\sqrt{65}}{2}\right)$, where $0 \leq \delta \leq \dfrac{\pi\sqrt{65}}{2}$ is shown in Figure 3.

The Frenet vectors $\left\{T_\gamma, N_\gamma, B_\gamma\right\}$ of the curve γ are

$$
\begin{aligned}
T_\gamma(\delta) &= -\begin{pmatrix} \cos(\delta)\cos(\frac{\delta}{\sqrt{65}}) + \frac{1}{\sqrt{65}}\sin(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ \cos(\frac{\delta}{\sqrt{65}})\sin(\delta) - \frac{1}{\sqrt{65}}\cos(t)\sin(\frac{\delta}{\sqrt{65}}), \\ \frac{1}{\sqrt{65}}\sin(\frac{\delta}{\sqrt{65}}) \end{pmatrix}, \\
N_\gamma(\delta) &= \left(\frac{1}{\sqrt{65}}8\sin(\delta), -\frac{1}{\sqrt{65}}8\cos(\delta), -\frac{1}{\sqrt{65}}\right), \\
B_\gamma(\delta) &= \begin{pmatrix} \frac{1}{\sqrt{65}}\cos(\frac{\delta}{\sqrt{65}})\sin(\delta) - \cos(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ -\frac{1}{\sqrt{65}}\cos(\delta)\cos(\frac{\delta}{\sqrt{65}}) - \sin(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ \frac{1}{\sqrt{65}}\cos(\frac{\delta}{\sqrt{65}}) \end{pmatrix}.
\end{aligned}
$$

The curvatures of γ are found to be $\kappa_\gamma = 1$, and $\tau_\gamma = \tan\left(\frac{\delta}{\sqrt{65}}\right)$ using in (2.1). The focal coefficients of the curve γ can be computed from (3.2) as $c_1 = 1$, and $c_2 = 0$. For this specific example, using (3.1) the focal curve C_γ of the curve γ may be computed as

$$C_\gamma = \gamma + N_\gamma. \tag{4.5}$$

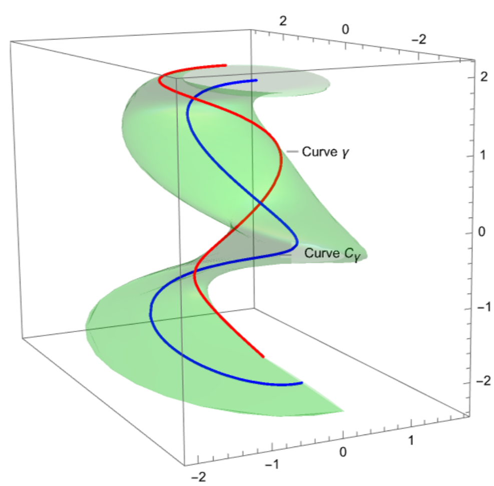

FIGURE 3. For $0 \leq \delta \leq \dfrac{\pi\sqrt{65}}{2}$, $0 \leq \phi \leq 2\pi$ tubular surface around focal curve C_γ of the curve γ .

Using (4.3) and (4.5) with $r = 1$ lead to the components

$$
\begin{aligned}
x_\gamma(\delta,\phi) &= -\frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)}\frac{8}{\sqrt{65}}\sin\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right)\\
&\quad -\frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)}\frac{8}{\sqrt{65}}\sin\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right)\\
&\quad -\frac{1}{2}\frac{8}{\sqrt{65}}\sin(\delta)+\frac{1}{\sqrt{65}}\frac{8}{\sqrt{65}}8\sin(\delta)\\
&\quad +\sin(\phi)\cos(\delta)\cos(\frac{\delta}{\sqrt{65}})+\frac{1}{\sqrt{65}}\sin(\phi)\sin(\delta)\sin(\frac{\delta}{\sqrt{65}})\\
&\quad -\frac{1}{\sqrt{65}}8\cos(\phi)\sin(\delta),
\end{aligned}
$$

$$
\begin{aligned}
y_\gamma(\delta,\phi) &= \frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)}\frac{8}{\sqrt{65}}\cos\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right)\\
&\quad +\frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)}\frac{8}{\sqrt{65}}\cos\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right)\\
&\quad -\frac{1}{2}\frac{8}{\sqrt{65}}\cos(\delta)-\frac{1}{\sqrt{65}}\frac{8}{\sqrt{65}}8\cos(\delta)\\
&\quad +\sin(\phi)\cos(\frac{\delta}{\sqrt{65}})\sin(\delta)-\frac{1}{\sqrt{65}}\sin(\phi)\cos(\delta)\sin(\frac{\delta}{\sqrt{65}})\\
&\quad +\frac{1}{\sqrt{65}}8\cos(\phi)\cos(\delta),
\end{aligned}
$$

$$
\begin{aligned}
z_\gamma(\delta,\phi) &= \frac{16}{\sqrt{65}}\cos\left(\frac{2}{\sqrt{65}}\delta\right)-\frac{1}{\sqrt{65}}\\
&\quad +\frac{1}{\sqrt{65}}\sin(\phi)\sin(\frac{\delta}{\sqrt{65}})\frac{1}{\sqrt{65}}\sin(\phi)\sin(\frac{\delta}{\sqrt{65}})\\
&\quad +\frac{1}{\sqrt{65}}\cos(\phi)
\end{aligned}
$$

of the tubular surface $L_\gamma(\delta,\phi)=\left(x_\gamma(\delta,\phi),y_\gamma(\delta,\phi),z_\gamma(\delta,\phi)\right)$. According to the focal curve C_γ the tubular surface $L_\gamma(\delta,\phi)$ is given in Figure 3.

5. Fundamental Forms of the Tubular Surfaces

Let $\alpha=\alpha(\delta):\mathrm{I}\longrightarrow\mathbf{E}^3$ be any unit speed curve. A parametrization of the tubular surface $L(s,\phi)$ around focal curve $C_\alpha(\delta)$ is presented (4.3). The partial derivatives of L concerning the surface parameters δ , and ϕ can be written as follows

$$
\begin{aligned}
L_\delta &= \cos\phi T+\sin\phi N+r\tau(1-\cos\phi)B,\\
L_\phi &= r\cos\phi T+r\sin\phi N,\\
L_{\delta\delta} &= -\kappa\sin\phi T+\left[\kappa\cos\phi-r\tau^2(1-\cos\phi)\right]N+\left[\tau\sin\phi+r\tau'(1-\cos\phi)\right]B,\\
L_{\delta\phi} &= -\sin\phi T+\cos\phi N+r\tau\sin\phi B,\\
L_{\phi\phi} &= -r\sin\phi T+r\cos\phi N.
\end{aligned}
$$

and

$$\mathbf{N} = \frac{L_\delta \times L_\phi}{\left\| L_\delta \times L_\phi \right\|} = -\sin\phi T + \cos\phi N,$$

where $\mathbf{N}$ is a unit normal vector field of the tubular surface $L(\delta, \phi)$. We can easly obtain

$$\left\| L_\delta \times L_\phi \right\|^2 = EG - F^2 = r^4\tau^2 (1 - \cos\phi)^2 . \tag{5.1}$$

The first fundamental form I of the tubular surface L is defined as

$$\mathbf{I} = \mathbf{E}\mathrm{d}\delta^2 + 2\mathbf{F}\mathrm{d}\delta\mathrm{d}\phi + \mathbf{G}\mathrm{d}\phi^2,$$

where

$$\begin{aligned}
\mathbf{E} &= g(L_\delta, L_\delta) = 1 + r^2\tau^2 (1 - \cos\phi)^2 ,\\
\mathbf{F} &= g(L_\delta, L_\phi) = r,\\
\mathbf{G} &= g(L_\phi, L_\phi) = r^2.
\end{aligned}$$

The second fundamental form II of the tubular surface L is defined as

$$\mathbf{II} = \mathbf{e}\mathrm{d}\delta^2 + 2\mathbf{f}\mathrm{d}\delta\mathrm{d}\phi + \mathbf{g}\mathrm{d}\phi^2,$$

where

$$\begin{aligned}
\mathbf{e} &= g(\mathbf{N}, L_{\delta\delta}) = \kappa - r\tau^2\cos\phi(1 - \cos\phi),\\
\mathbf{f} &= g(\mathbf{N}, L_{s\phi}) = 1,\\
\mathbf{g} &= g(\mathbf{N}, L_{\phi\phi}) = r.
\end{aligned}$$

Definition 5.1. [2] Let M be any surface, and the set $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ be the coefficients of its first fundamental form. M is named a regular surface if $\mathbf{EG} - \mathbf{F}^2 \neq 0$.

Lemma 5.2. *$L(\delta, \phi)$ is a regular tubular surface, when $\cos\phi \neq 1$, $\kappa \neq 0$, and $\tau \neq 0$.*

Proof. If we use (5.1) and consider definition 5.1, the proof ends. □

Theorem 5.3. *For the tubular surface $L(\delta, \phi)$, Gaussian, and the mean curvatures are*

$$K = \frac{\boldsymbol{eg} - \boldsymbol{f}^2}{\boldsymbol{EG} - \boldsymbol{F}^2} = \frac{(1 - 2\cos\phi)}{2r^2 (1 - \cos\phi)},$$

and

$$H = \frac{\boldsymbol{eG} - 2\boldsymbol{fF} + \boldsymbol{gE}}{2\left(\boldsymbol{EG} - \boldsymbol{F}^2\right)} = -\frac{\cos\phi}{r^2 (1 - \cos\phi)^2}$$

respectively.

Let us now give the following theorems about parameter curves on a tubular surface.

Theorem 5.4. [7] *Let a curve γ be on a regular surface. The curve γ is an asymptotic curve provided that its acceleration vector γ'' is orthogonal to the normal vector $\mathbf{N}$ of the aforementioned surface. Namely, $g(\mathbf{N}, \gamma'') = e = 0$.*

Theorem 5.5. *Let $L(\delta, \phi)$ be a tubular surface and the curve $\gamma(\delta)$ be its focal curve.*
(1) If $\delta-$parameter curves L_δ are asymptotic, then $\kappa = r\tau^2\cos\phi(1 - \cos\phi)$, and $\cos\phi \neq 0$
(2) $\phi-$parameter curves L_ϕ are not asymptotic.

Proof. The first coefficient $\mathbf{e}$ of the second fundamental form $\mathbf{II}$ as follows

$$\mathbf{e} = g(\mathbf{N}, L_{\delta\delta}) = \kappa - r\tau^2\cos\phi(1 - \cos\phi).$$

If $\delta-$parameter curves L_δ are asymptotic, then $\kappa = r\tau^2\cos\phi(1 - \cos\phi)$. Due to $\kappa \neq 0$,then $\cos\phi \neq 0$. Similarly, the third coefficient g of the second fundamental form $\mathbf{II}$ as follows

$$\mathbf{g} = g(\mathbf{N}, L_{\phi\phi}) = r \neq 0.$$

If so, the $\phi-$parameter curves L_ϕ are not asymptotic curves. □

Theorem 5.6 ([4]). *Let a curve γ be on a regular surface. The curve γ is a geodesic curve provided its acceleration vector γ'' and the normal vector* $\mathbf{N}$ *of the surface is linearly dependent. Namely,* $\mathbf{N} \times \gamma'' = 0$.

Theorem 5.7. *Let $L(\delta, \phi)$ be a tubular surface and the curve $\gamma(\delta)$ be its focal curve.*
(1) The $\delta-$parameter curves L_δ can not be geodesic,
(2) The $\phi-$parameter curves L_ϕ curves are geodesic.

Proof. For any δ and ϕ, $r^2\tau(1-\cos\phi) \neq 0$ since $L(\delta,\phi)$ is a regular tubular surface (see (5.1)). Therefore,

$$\begin{aligned} \mathbf{N} \times L_{\delta\delta} &= \cos\phi\left[\tau\sin\phi + r\tau'(1-\cos\phi)\right]T \\ &\quad + \sin\phi\left[\tau\sin\phi + r\tau'(1-\cos\phi)\right]N \\ &\quad + r\tau^2\sin\phi(1-\cos\phi)B \neq 0. \end{aligned} \tag{5.2}$$

On the other hand, since

$$\mathbf{N} \times L_{\phi\phi} = \mathbf{N} \times r\mathbf{N} = 0$$

the $\phi-$parameter curves L_ϕ are geodesics. The opposite is also true. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

References

[1] Alegre, P., Arslan, K., Carriazo, A., Murathan, C., Öztürk, G., *Some special types of developable rued surface*, Hacettepe Journal of Mathematics and Statistics, **39**(3)(2010), 319–325.

[2] Doğan, F., Yaylı, Y., *On the curvatures of the tubular surface with Bishop* , Commun. Fac. Sci. Univ. Ank. Series A1, **60**(1)(2011), 59–69.

[3] Öztürk, G., Arslan, K., *On focal curves in Euclidean n-space* $\mathbf{R}^n$, Novi Sad J. Math, **46**(1)(2016), 35-44.

[4] Hacısalihoğlu, H.H., Differensiyel Geometri Cilt II, Hacısalihoğlu Yayınları, Ankara, 2000.

[5] Karacan, M.K., Bukcu, B., *An alternative moving frame for tubular surfaces around spacelike curves with a spacelike Binormal in the Minkowski 3-space*, Mathematica Moravica, **12**(2)(2007), 47–54.

[6] Monterde, J., *Salkowski curves revised: A family of curves with constant curvature and non-constant torsion*, Computer Aided Geometric Design, **26**(3)(2009), 271–278.

[7] Oprea, J., Differential Geometry and Its Applications, Prentice-Hall Inc., New Jersey, 1997.

[8] Özdemir, M., Ergin, A.A., *Spacelike Darboux curves in Minkowski 3-space*, Differential Geometry-Dynamical Systems, **9**(2007), 131–137.

[9] Saffak, A.G., Ayvacı, K.H., *Surface family with a common Mannheim-B isogeodesic curve*, Balkan Journal of Geometry and Its Applications, **26**(2)2021, 1–12.

[10] Şenyurt, S., Ayvacı, K.H., Canlı, D., *Family of Surfaces with a Common Special Involute and Evolute Curves*, Internatıonal Electronıc Journal of Geometry,**15**(1)2022, 160–174.

[11] Uribe-Vargas, R., *On vertices focal curvatures and differential geometry of space curves*, Bull. Brazilian Math. Soc., **36**(3)(2005), 285–307.

[12] Yıldırım, A., *Tubular surface around a Legendre curve in BCV spaces, New Trends in Mathematical Sciences*, **4**(2)(2016), 61–71.