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Research Article

Finite dimensional realization of a parameter choice strategy for fractional Tikhonov regularization method in Hilbert scales

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Abstract

One of the most crucial parts of applying a regularization method to solve an ill-posed problem is choosing a regularization parameter to obtain an optimal order error estimate. In this paper, we consider the finite dimensional realization of the parameter choice strategy proposed in [C. Mekoth, S. George and P. Jidesh, Appl. Math. Comput. **392**, 125701, 2021] for Fractional Tikhonov regularization method for linear ill-posed operator equations in the setting of Hilbert scales.

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1. Introduction

In this paper our aim is to attain a stable approximation for the solution $\hat{x}$ of the ill-posed equation (i.e., the solution $\hat{x}$ is not depending continuously on the data y (see Hadamard [9]))

$$Tx = y, \tag{1.1}$$

where X and Y are Hilbert spaces and $T : X \longrightarrow Y$ is a bounded linear operator between them. Over the years many methods have been developed to find solutions to such equations as it has a wide range of applications in a variety of fields. However in practical applications, y may not be precisely known and the available data is some y^δ so that

$$\|y - y^\delta\| \leq \delta \tag{1.2}$$

and one ends up dealing with the equation

$$Tx = y^\delta \tag{1.3}$$

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instead of (1.1). The notations $\langle ., . \rangle$ and $\|.\|$ are used for inner product and the corresponding norm respectively. As (1.1) is ill-posed the standard Tikhonov regularization method [4, 5, 8, 13, 20] can be employed to solve equation (1.1) wherein the minimizer x_α^δ of the functional

$$J_\alpha(x) = \|Tx - y^\delta\|^2 + \alpha\,\|x\|^2, \tag{1.4}$$

is taken as an approximation for the solution $\hat{x}$ of (1.1)($\hat{x}$ is assumed to exist). In (1.4), $\alpha > 0$ is the regularization parameter which needs to be chosen appropriately in order to get an optimum regularized solution [5]. The methods for choosing a suitable regularization parameter can be divided into two main types, such as the methods based on knowledge of the error and methods that do not require any knowledge of the error. The discrepancy principle is an example of the first type and the Cross-Validation and L-curve are examples from the second type (see [23]). Since it has been observed that x_α^δ oversmoothens the solution $\hat{x}$ [12], the Fractional Tikhonov regularization (FTR) method [1, 2, 7, 10–12, 19] was considered, wherein the minimizer $x_{\alpha,\beta}^\delta$ of the functional

$$J_\alpha^\beta(x) = |||Tx - y^\delta|||_\beta^2 + \alpha\,\|x\|^2, \tag{1.5}$$

is taken as an approximate for $\hat{x}$, with $|||x|||_\beta = \|(TT^*)^{(\beta-1)/4}x\|$, for some parameter $0 \leq \beta \leq 1$ (see [7, 11]). One drawback of the FTR method is that it reduces the order of convergence of the solutions. However, to overcome this problem too we study FTR method in the setting of Hilbert scales [16]. Our approach reduces the oversmoothing and gives a more preferable convergence rate as well.

Recall, a family of Hilbert spaces, say $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ is called a Hilbert scale [14] if :

- For $t < s$, $\mathcal{X}_s \subseteq \mathcal{X}_t$ and $\mathcal{X}_s$ is a dense subset of $\mathcal{X}_t$.
- As Hilbert spaces, the above inclusion is a continuous embedding, i.e. there exists $c_{t,s} > 0$ such that

$$\|x\|_t \leq c_{t,s}\|x\|_s \text{ for all } x \in \mathcal{X}_s. \tag{1.6}$$

It is known that the fractional power has lower damping at small singular values and that fractional power has a greater effect on the filtering function[26] (also see [16]). While performing numerical calculations in practical applications, $x_{\alpha,\beta}^\delta$ being the minimizer of J_α^β over an infinite dimensional space is not easy to compute. However a computable approximation can be obtained using the finite dimensional realization of the infinite dimensional method [18]. The aim of this paper is to study the finite dimensional realization of the parameter choice rule introduced in [16] (motivated reader can refer [19] for finite dimentional fractional Tikhonov regularization).

The rest of the paper is arranged as follows. Section 2 contains Definitions and Preliminary results. The parameter choice strategy is given in Section 3, numerical examples are given in Section 4 and a conclusion in Section 5.

2. Definitions and preliminary results

In our work, it is assumed that the Hilbert scale is induced by a strictly positive definite operator L [6,22,27]. Let $L : D(L) \subset X \longrightarrow X$ be an unbounded, strictly positive definite, densely defined, self-adjoint operator i.e.,

$$\langle Lx, x\rangle > 0,$$

$D(L)$ is dense in X and

$$\|Lx\| \geq \|x\|,\ x \in D(L).$$

Let X_t be defined as the completion of $D := \cap_{k=0}^\infty D(L^k)$ with respect to the norm $\|x\|_t = \|L^t x\|$, induced by the inner product

$$\langle u, v\rangle_t = \langle L^t u, L^t v\rangle,\ \ u, v \in D.$$

Then, $\{X_s\}_{s\in\mathbb{R}}$ satisfies the definition of Hilbert scales (cf. [3,6,15]) given in the beginning of the paper. Throughout, we assume that, the operator T satisfies:

$$b_1\|x\|_{-a} \leq \|Tx\| \leq b_2\|x\|_{-a}, \; x \in \mathcal{X} \tag{2.1}$$

for some $a > 0, b_1 > 0$ and $b_2 > 0$.

Using the above notation, we recall the following results from [16].

Proposition 2.1 ([16], Proposition 1)**.** *Let T satisfy (2.1). Let $f(t) := \min\{b_1^t, b_2^t\}$, $g(t) := \max\{b_1^t, b_2^t\}$, $t \in \mathbb{R}$ and $|t| \leq 1$. Then, for $|\nu| \leq 1$,*

$$f(\nu)\|x\|_{-\nu a} \leq \|(T^*T)^{\nu/2}x\| \leq g(\nu)\|x\|_{-\nu a}, \; x \in D((T^*T)^{\nu/2}).$$

Further,

$$F(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)} \leq \|(L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}x\| \leq G(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)},$$

$x \in D((L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}), s > 0, 0 \leq \beta \leq 1, |\nu| \leq 1$ where

$$F(t) := \min\{f(\frac{1+\beta}{2})^t, g(\frac{1+\beta}{2})^t\}, \; G(t) := \max\{f(\frac{1+\beta}{2})^t, g(\frac{1+\beta}{2})^t\}.$$

Remark 2.2. Since $b_1 \leq b_2$, one can write $f(t) := \begin{cases} b_1^t, & \text{if } 0 \leq t \\ b_2^t, & \text{if } t < 0 \end{cases}$,

$g(t) := \begin{cases} b_2^t, & \text{if } 0 \leq t \\ b_1^t, & \text{if } t < 0 \end{cases}$,

$$F(t) := \begin{cases} b_1^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ b_2^{(\frac{1+\beta}{2})t}, & \text{if } t < 0 \end{cases}$$

and

$$G(t) := \begin{cases} b_2^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ b_1^{(\frac{1+\beta}{2})t}, & \text{if } t < 0. \end{cases}$$

In [16], we considered $x_{\alpha,\beta}^s$ the minimizer of the functional

$$J_{\alpha,\beta}^s(x) = \||Tx - y^\delta\||_\beta^2 + \alpha\|x\|_s^2 \quad \alpha > 0, \tag{2.2}$$

where $0 \leq \beta \leq 1$ as an approximation for $\hat{x}$. Note that $x_{\alpha,\beta}^s$ satisfies the equation

$$((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})x_{\alpha,\beta}^s = (T^*T)^{\frac{\beta}{2}}y. \tag{2.3}$$

Observe that, for $\beta = 1, s = 0$, (2.2) reduces to (1.4). Following assumption is used to get an error estimate for $\|\hat{x} - x_{\alpha,\beta}^s\|$.

Assumption 2.1. There exists some $E > 0$, $0 < t \leq \frac{1+\beta}{2}a + 2s$ such that $\hat{x} \in M_{t,E} = \{x \in \mathcal{X} : \|x\|_t \leq E\}$.

For the error analysis, we need the following Lemma.

Lemma 2.3 ([16], Lemma 3.3)**.** *Let $x_{\alpha,\beta}^s$ be as in (2.3), $0 \leq \beta \leq 1$ and $T\hat{x} = y$. Suppose Assumption 2.1 and Proposition 2.1 hold. Then*

$$\|\hat{x} - x_{\alpha,\beta}^s\| \leq \psi(s,a,\beta,t)\alpha^{\frac{t}{(1+\beta)a+2s}},$$

where $\psi(s,a,\beta,t) := \dfrac{G(\frac{2(s-t)}{(1+\beta)a+2s})}{F(\frac{2s}{(1+\beta)a+2s})}E := E\begin{cases} (\frac{b_2^{s-t}}{b_1^s})^{\frac{1+\beta}{(1+\beta)a+2s}}, & \text{if } t \leq s \\ (\frac{b_1^{s-t}}{b_2^s})^{\frac{1+\beta}{(1+\beta)a+2s}}, & \text{if } s < t. \end{cases}$

2.1. Finite dimensional realization of FTR method in Hilbert scales

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections of X onto $R(P_h)$, range of P_h. We also impose the condition

$$\varepsilon_h = \|T(I - P_h)\|. \tag{2.4}$$

Assume $\lim_{h\to 0} \varepsilon_h = 0$ which is satisfied if T is a compact operator and $P_h \to I$ pointwise. Let $T_h = TP_h$ and let $h_0 > 0$ be such that

$$\varepsilon_h \leq \frac{b_1\|x\|_{-a}}{2\|x\|} \quad \forall x \neq 0, h \leq h_0. \tag{2.5}$$

Lemma 2.4 ([17], Lemma 3.1). *Let $\bar{b}_1 = \frac{b_1}{2}$, $\bar{b}_2 = b_2 + \frac{b_1}{2}$ and $h \leq h_0$. Then*

$$\bar{b}_1\|x\|_{-a} \leq \|T_h x\| \leq \bar{b}_2\|x\|_{-a}. \tag{2.6}$$

Clearly $\bar{b}_1 \leq \bar{b}_2$. Let $\bar{f}(t) := \begin{cases} \bar{b}_1^t, & \text{if } 0 \leq t \\ \bar{b}_2^t, & \text{if } t < 0 \end{cases}$ and $\bar{g}(t) := \begin{cases} \bar{b}_2^t, & \text{if } 0 \leq t \\ \bar{b}_1^t, & \text{if } t < 0. \end{cases}$

Proposition 2.5 ([17], Proposition 3.2). *Suppose Lemma 2.4 holds and $|\nu| \leq 1$, then*

$$\bar{f}(\nu)\|x\|_{-\nu a} \leq \|(T_h^*T_h)^{\nu/2}x\| \leq \bar{g}(\nu)\|x\|_{-\nu a}, \ x \in D((T_h^*T_h)^{\nu/2}). \tag{2.7}$$

Proposition 2.6 ([17], Proposition 3.3). *Let T_h be a bounded linear operator that satisfies (2.6). Then, from Proposition 2.5 and for $0 \leq \beta \leq 1$, the following holds:*

$$\bar{F}(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)} \leq \|(L^{-s}(T_h^*T_h)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}x\| \leq \bar{G}(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)},$$

$x \in D((L^{-s}(T_h^*T_h)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}), s > 0, |\nu| \leq 1,$ *where* $\bar{F}(t) := \begin{cases} \bar{b}_1^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ \bar{b}_2^{(\frac{1+\beta}{2})t}, & \text{if } t < 0 \end{cases}$ *and*

$\bar{G}(t) := \begin{cases} \bar{b}_2^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ \bar{b}_1^{(\frac{1+\beta}{2})t}, & \text{if } t < 0. \end{cases}$

We consider the unique solution $x_{\alpha,\beta,h}^{s,\delta}$ of the equation

$$((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})x_{\alpha,\beta,h}^{s,\delta} = (T_h^*T_h)^{\frac{\beta}{2}}y^\delta, \tag{2.8}$$

as an approximation for $\hat{x}$.

Let

$$A_{s,\beta} := L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s},$$

and

$$A_{s,\beta,h} := L^{-s}(T_h^*T_h)^{\frac{1+\beta}{2}}L^{-s}.$$

Then

$$x_{\alpha,\beta}^s = L^{-s}(A_{s,\beta} + \alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y, \tag{2.9}$$

$$x_{\alpha,\beta,h}^s = L^{-s}(A_{s,\beta,h} + \alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y \tag{2.10}$$

and

$$x_{\alpha,\beta,h}^{s,\delta} = L^{-s}(A_{s,\beta,h} + \alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta. \tag{2.11}$$

Furthermore, by the spectral properties of self-adjoint operators, for $A_{s,\beta}$, $A_{s,\beta,h}$, $s > 0$, $\beta \in [0,1]$,
we have

$$\|(A_{s,\beta} + \alpha I)^{-1}A_{s,\beta}^\mu\| \leq \alpha^{\mu-1}, \alpha > 0, \ 0 \leq \mu \leq 1. \tag{2.12}$$

$$\|(A_{s,\beta,h} + \alpha I)^{-1}A_{s,\beta,h}^\mu\| \leq \alpha^{\mu-1}, \alpha > 0, \ 0 \leq \mu \leq 1. \tag{2.13}$$

Lemma 2.7 ([17], Lemma 3.4). *Let $x^s_{\alpha,\beta,h}, x^{s,\delta}_{\alpha,\beta,h}$ be as in (2.10) and (2.11), respectively. If the assumptions in Proposition 2.6 holds, then for $0 \le \beta \le 1$,*

$$\|x^s_{\alpha,\beta,h} - x^{s,\delta}_{\alpha,\beta,h}\| \le \varphi(s,a,\beta,h)\alpha^{\frac{-a}{(1+\beta)a+2s}}\delta,$$

where $\varphi(s,a,\beta,h) := \dfrac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)\bar{f}(-\beta)} = \dfrac{(\bar{b}_1^{-\beta a-2s})^{\frac{1+\beta}{(1+\beta)a+2s}}}{\bar{b}_2^{-\beta}}.$

Lemma 2.8 ([17], Lemma 3.5). *Let $x^s_{\alpha,\beta}$ and $x^s_{\alpha,\beta,h}$ be as in (2.9) and (2.10), respectively and Assumption 2.1 holds. Further, let the assumptions in Proposition 2.1, Proposition 2.5 and Proposition 2.6 hold. Then for $0 \le t \le \frac{(1+\beta)}{2}a + 2s$*

$$\|x^s_{\alpha,\beta,h} - x^s_{\alpha,\beta}\| \le \varphi_1(s,a,\beta,h,t)\alpha^{\frac{-a}{(1+\beta)a+2s}}\varepsilon_h,$$

where

$$\varphi_1(s,a,\beta,h,t) = \frac{\sin\pi(\frac{1+\beta}{2})}{\pi b_2^{-\beta}} \frac{\bar{b}_1^{\frac{-(1+\beta)(2s+\beta a)}{(1+\beta)a+2s}}}{b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}}} \left(C_h b_1^{\frac{-t}{a}} \frac{2aE}{t} + b_2^{\frac{(1+\beta)s}{(1+\beta)a+2s}} 2\|T\|\|\hat{x}\| \right)$$
$$+ \frac{(\bar{b}_1)^{\frac{-(1+\beta)(\beta a+2s)}{(1+\beta)a+2s}}}{\bar{b}_2^{-\beta}}\|\hat{x}\|,$$

with

$$C_h = \begin{cases} b_2^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ b_1^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t. \end{cases}$$

Combining the Lemma 2.7, Lemma 2.8 and Lemma 2.3 we have the following Theorem.

Theorem 2.9 ([17], Theorem 3.6). *Let $x^s_{\alpha,\beta,h}, x^{s,\delta}_{\alpha,\beta,}$ be as in (2.9) and (2.10), respectively, Assumption 2.1 holds. Further, let Lemma 2.7, Lemma 2.8 and Lemma 2.3 hold. Then*

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| \le 2\varphi_2(s,a,\beta,h)\alpha^{\frac{-a}{(1+\beta)a+2s}}(\delta+\varepsilon_h) + \psi(s,a,\beta,t)\alpha^{\frac{t}{(1+\beta)a+2s}}, \tag{2.14}$$

where $\varphi_2(s,a,\beta,h) = \max\{\varphi(s,a,\beta,h), \varphi_1(s,a,\beta,h)\}$. In particular, if $\alpha := \alpha(s,a,\beta,t) = c_0(\delta+\varepsilon_h)^{\frac{(1+\beta)a+2s}{t+a}}$ for some $c_0 > 0$, then

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| \le \eta(s,a,\beta,t)(\delta+\varepsilon_h)^{\frac{t}{t+a}},$$

where $\eta(s,a,\beta,t) = \max\{\varphi(s,a,\beta)c_0^{\frac{-a}{(1+\beta)a+2s}}, \psi(s,a,\beta,t)c_0^{\frac{t}{(1+\beta)a+2s}}\}.$

□

3. Parameter choice strategy

In [16] the authors considered the following parameter choice rule, i.e. choose α satisfying

$$\|\alpha^2(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s} = c\delta \tag{3.1}$$

where $c > 0$ is a constant for FTR method in Hilbert scales. In this section, we study the finite dimensional realization of the parameter choice strategy (3.1). Let

$$\phi(\alpha, y^\delta, h) = \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s}. \tag{3.2}$$

The proof of the following theorem is analogous to the proof of the theorem in [16], but for sake of completion we give the proof as well.

Theorem 3.1. *For each $y^\delta \neq 0$, the function $\alpha \longrightarrow \phi(\alpha, y^\delta, h)$ for $\alpha > 0$, as defined in (3.2), is increasing and continuous. In addition*

$$\lim_{\alpha\to 0} \phi(\alpha, y^\delta, h) = 0, \lim_{\alpha\to\infty} \phi(\alpha, y^\delta, h) = \|(T_h^*T_h)^{\frac{\beta}{2}} y^\delta\|_{\beta a}. \tag{3.3}$$

Proof. Let $\{E_\lambda : 0 \leq \lambda \leq \|A_{s,\beta,h}\|\}$ be the spectral family of $A_{s,\beta,h}$. Then

$$\phi(\alpha, y^\delta, h)^2 = \int_0^{\|A_{s,\beta,h}\|} \left(\frac{\alpha}{\lambda+\alpha}\right)^4 d\langle E_\lambda L^{-s}(T_h^*T_h)^{\beta/2} y^\delta, L^{-s}(T_h^*T_h)^{\beta/2} y^\delta\rangle_{\beta a+s}. \tag{3.4}$$

Now since $\alpha \longrightarrow \left(\frac{\alpha}{\lambda+\alpha}\right)^4$ for $\lambda > 0$ is strictly increasing, $\lim_{\alpha\to 0}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 0$ and $\lim_{\alpha\to\infty}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 1$, by Dominated convergence theorem, we have

$$\lim_{\alpha\to 0} \phi(\alpha, y^\delta, h) = 0, \lim_{\alpha\to\infty} \phi(\alpha, y^\delta, h) = \|(T_h^*T_h)^{\frac{\beta}{2}} y^\delta\|_{\beta a}. \tag{3.5}$$

□

Theorem 3.2. *Suppose (1.2) holds and*

$$\|(T_h^*T_h)^{\frac{\beta}{2}} y^\delta\|_{\beta a} \geq c\delta + d\epsilon_h > 0 \tag{3.6}$$

for some $c > 0$ and $d > 0$. Then, there exists a unique $\alpha = \alpha(\delta, h)$ satisfying

$$\phi(\alpha, y^\delta, h) = c\delta + d\epsilon_h. \tag{3.7}$$

Proof. Follows from Theorem 3.1, (3.6) and the Intermediate value theorem.

□

Remark 3.3. Note that, by (1.2) and Proposition 2.5, we have

$$\|y^\delta\| = \|(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h)^{\frac{\beta}{2}} y^\delta\| \leq \bar{g}(-\beta)\|(T_h^*T_h)^{\frac{\beta}{2}} y^\delta\|_{\beta a}. \tag{3.8}$$

Now, since $\|y^\delta\| \geq \|y\| - \delta$, if

$$\delta \leq \frac{\|y\|}{\bar{g}(-\beta)(c + d\epsilon_h/\delta) + 1}, \tag{3.9}$$

then equation (3.6) is satisfied.

In further analysis, we make use of the following formula;

$$\begin{aligned} B^z(x) &= \frac{\sin \pi z}{\pi}\int_0^\infty t^z[(B+tI)^{-1}x - \frac{\theta(t)}{t}x + \cdots + (-1)^n\frac{\theta(t)}{t^n}B^{n-1}x]dt \\ &\quad + \frac{\sin \pi z}{\pi}[\frac{x}{z} - \frac{Bz}{z-1} + \cdots + (-1)^{n-1}\frac{B^{n-1}x}{z-n+1}], x \in X, \end{aligned} \tag{3.10}$$

where

$$\theta(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1 \\ 1, & \text{if } 1 < t \leq \infty \end{cases}$$

for any positive self-adjoint operator B and complex number z such that $0 < Rez < n$. Substituting $z = \frac{1+\beta}{2}$, $0 \leq \beta < 1$, in (3.10) one can see that

$$B^{\frac{1+\beta}{2}}x = \frac{\sin \pi(\frac{1+\beta}{2})}{\pi}\left[\frac{2x}{1+\beta} + \int_0^\infty \lambda^{\frac{1+\beta}{2}}(B+\lambda I)^{-1}x d\lambda - \int_1^\infty \frac{x}{\lambda^{1-(\frac{1+\beta}{2})}}d\lambda\right]. \tag{3.11}$$

Using (3.11), for $z \in X$ we have

$$[(T_h^* T_h)^{\frac{1+\beta}{2}} - (T^*T)^{\frac{1+\beta}{2}}]z = \frac{\sin \pi(\frac{1+\beta}{2})}{\pi} \int_0^\infty \lambda^{\frac{1+\beta}{2}} ((T_h^*T_h) + \lambda I)^{-1} [(T^*T) - (T_h^*T_h)]((T^*T) + \lambda I)^{-1} z d\lambda. \tag{3.12}$$

For convenience we use the following notations

$$c_1 = \frac{\bar{b}_1^{\frac{-(a+s)(1+\beta)}{(1+\beta)a+2s}} b_1^{\frac{-t}{a}} 2a}{\bar{b}_2^{-1}((\beta-1)a+t)} \|\hat{x}\|_t \begin{cases} (\frac{b_2}{b_1})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (\frac{b_1}{b_2})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t \end{cases}$$

$$c_2 = \frac{\bar{b}_1^{\frac{-(1+\beta)(a+s)}{(1+\beta)a+2s}} b_2^{\frac{(1+\beta)s}{(1+\beta)a+2s}} 2}{\bar{b}_2^{-1} b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}} (1-\beta)} \|\hat{x}\|$$

$$c_3 = \frac{b_1^{\frac{-t}{a}} 2a}{((\beta-1)a+t)} \|\hat{x}\|_t \begin{cases} (\frac{1}{b_1})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (\frac{1}{b_2})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t \end{cases}$$

$$c_4 = \frac{2}{b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}} (1-\beta)} \|\hat{x}\|$$

$$\tilde{C}_1 = \frac{\sin \pi(\frac{\beta}{2})}{\pi \bar{b}_2^{-\beta}} \left(b_1^{\frac{-t}{a}} \frac{2a}{t} E + 2\|y\| \right)$$

$$_1 = (\frac{\bar{b}_1}{\bar{b}_2})^{\frac{-(1+\beta)(\beta a+s)}{(1+\beta)a+2s}} \tilde{C}_1$$

$$_2 = \frac{\sin(\frac{1+\beta}{2})}{\pi} (c_1 + c_2) c_{(\beta-1)a,0}$$

$$_3 = \frac{\bar{b}_1^{\frac{-(1+\beta)(a+s)}{(1+\beta)a+2s}} \sin \pi(\frac{1+\beta}{2}) b_2^{\frac{(1+\beta)(1-\beta)a}{(1+\beta)a+2s}}}{\bar{b}_2^{-1}\pi} c_{(\beta-1)a,0}(c_3 + c_4) \begin{cases} (b_2)^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (b_1)^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t \end{cases}$$

and

$$_4 = \bar{b}_2^{\frac{(1+\beta)(\beta a+s)}{(1+\beta)a+2s}} (_2 + _3). \tag{3.13}$$

Lemma 3.4. *Suppose Assumption 2.1 holds and* $\alpha := \alpha(\delta, h) > 0$ *is the unique solution of (3.7) with* $c \ge c_0$ *and* $d \ge d_0$ *where* $c_0 = \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}$, $d_0 = _1 + _4$. *Then,*

$$\alpha \ge c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}} \tag{3.14}$$

where $c_{\beta,a,s,h} = \frac{\bar{G}\left(\frac{-(\beta a+s)}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-(\beta a+s)}{(1+\beta)a+2s}\right) \bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} E \min\{c - c_0, d - d_0\}.$

Proof. Note that, from Proposition 2.5, Proposition 2.6, (2.13) and (3.2) we have

$$
\begin{aligned}
c\delta + d\epsilon_h &= \phi(\alpha, y^\delta, h) \\
&\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\| \\
&\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \\
&+ \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\frac{\|y^\delta - y\|}{\bar{f}(-\beta)} \\
&\quad + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})\bar{f}(-\beta)}\delta + G,
\end{aligned}
\tag{3.15}
$$

where $G = \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|$. Note that

$$
\begin{aligned}
G &\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}[(T_h^*T_h)^{\frac{\beta}{2}} - (T^*T)^{\frac{\beta}{2}}]y\| \\
&\quad + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2 \|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}G_1 + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}G_2,
\end{aligned}
\tag{3.16}
$$

where $G_1 = \|[(T_h^*T_h)^{\frac{\beta}{2}} - (T^*T)^{\frac{\beta}{2}}]y\|_{\beta a}$ and $G_2 = \alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$. One can prove (see Appendix A)

$$G_1 \leq \tilde{C}_1\epsilon_h \tag{3.17}$$

and (see Appendix B)

$$G_2 \leq ({}_1 + {}_4)\epsilon_h + \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}E\alpha^{\frac{a+t}{(1+\beta)a+2s}}. \tag{3.18}$$

Hence from (3.16), (3.17) and (3.18), we have

$$(c-c_0)\delta + (d-d_0)\epsilon_h \leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}E\alpha^{\frac{a+t}{(1+\beta)a+2s}}, \tag{3.19}$$

which implies that

$$\alpha \geq c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}. \tag{3.20}$$

□

Theorem 3.5. *Suppose conditions in Lemma 2.8 hold with $c \geq c_0$ and $d \geq d_0$ and $\alpha := \alpha(\delta, h) > 0$ is the unique solution of (3.7). Then*

$$\|\hat{x} - x^s_{\alpha,\beta}\| = O((\delta + \epsilon_h)^{\frac{t}{t+a}}). \tag{3.21}$$

Proof. As in [16, Lemma 3.3] , we have

$$\begin{aligned} \hat{x} - x^s_{\alpha,\beta} &= \hat{x} - ((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T^*T)^{\frac{\beta}{2}} y \\ &= \alpha((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1} L^{2s}\hat{x} \\ &= \alpha L^{-s}(A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}. \end{aligned} \tag{3.22}$$

By Proposition 2.1 we have

$$\|\hat{x} - x^s_{\alpha,\beta}\| \leq \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\|. \tag{3.23}$$

In order to proceed with obtaining an error estimate for $\|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\|$ the following inequality is used

$$\|B^u z\| \leq \|B^v z\|^{u/v} \|z\|^{1-u/v} \quad ,0 \leq u \leq v, \tag{3.24}$$

where B is a positive self-adjoint operator. The inequality in (3.24) is called the moment inequality. Let
$u = \frac{t}{p}, v = \frac{t+a}{p}, B = \alpha A_{s,\beta}^{\frac{p}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1}$ and $z = \alpha^{1-\frac{t}{p}} (A_{s,\beta} + \alpha I)^{-1+\frac{t}{p}} A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}$
where $p = (\frac{1+\beta}{2})a + 2s$.
Then $B^u z = \alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}$. Also, from (3.25) we have

$$\begin{aligned} \|B^u z\| &\leq \|B^v z\|^{\frac{t}{t+a}} \|z\|^{\frac{a}{t+a}} \\ &= \|\alpha^{1+\frac{a}{p}} (A_{s,\beta} + \alpha I)^{-(1+\frac{a}{p})} A_{s,\beta}^{\frac{s+a}{(1+\beta)a+2s}} L^s \hat{x}\|^{\frac{t}{t+a}} \\ &\quad \times \|\alpha^{1-\frac{t}{p}} (A_{s,\beta} + \alpha I)^{-1+\frac{t}{p}} A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\|^{\frac{t}{t+a}}. \end{aligned}$$

From Proposition 2.1 we have

$$\begin{aligned} \|B^v z\| &= \|\alpha^{\frac{a}{p}-1} (A_{s,\beta} + \alpha I)^{-(\frac{a}{p}-1)} \alpha^2 (A_{s,\beta} + \alpha I)^{-2} A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} A_{s,\beta} L^s \hat{x}\| \\ &\leq \|\alpha^2 (A_{s,\beta} + \alpha I)^{-2} A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} A_{s,\beta} L^s \hat{x}\| \\ &\leq \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &\leq \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} [(A_{s,\beta} + \alpha I)^{-2} - (A_{s,\beta,h} + \alpha I)^{-2}] L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &\quad + \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &= G_{21} + Ģ \leq ({}_2 + {}_3)\epsilon_h + Ģ \quad \text{by (B.9)} \end{aligned} \tag{3.25}$$

where $\underset{,}{G} = \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$. Here we make use of $\|\alpha^{\frac{a}{p}-1}(A_{s,\beta}+\alpha I)^{-(\frac{a}{p}-1)}\| \leq 1$ ($\frac{a}{p}\geq 1$ by the condition $a \geq \frac{4s}{1-\beta}$). Further,

$$\begin{aligned}
\underset{,}{G} &\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|_{\beta a+s}\\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)[\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}[(T^*T)^{\frac{\beta}{2}}-(T_h^*T_h)^{\frac{\beta}{2}}]y\|_{\beta a+s}\\
&\quad +\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}]\\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)(G_1+\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s})\\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)(\tilde{C}_1\epsilon_h+\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}).
\end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned}
&\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}\\
\leq\ &\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y-y^\delta)\|_{\beta a+s}\\
&+\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s}\\
\leq\ &\frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta+\phi(\alpha,y^\delta,h)\ \ (\text{see}\quad (3.15)).
\end{aligned} \tag{3.27}$$

Therefore from (3.25), (3.26) and (3.27) we have

$$\begin{aligned}
\|B^v z\| &\leq ({}_2+{}_3)\epsilon_h+\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1\epsilon_h+\frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta+\phi(\alpha,y^\delta,h)\\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta+c\delta+\left({}_2+{}_3+\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1+d\right)\epsilon_h\\
&\leq \underset{,}{L}(\delta+\epsilon_h)
\end{aligned} \tag{3.28}$$

where

$$\underset{,}{L} = \max\left\{\frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}+c,\ {}_2+{}_3+\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1+d\right\}.$$

We further have

$$\begin{aligned}
\|z\| &\leq \|A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\|\\
&\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)\|L^s\hat{x}\|_{t-s}\\
&\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)\|\hat{x}\|_t.
\end{aligned} \tag{3.29}$$

Hence by (3.24)-(3.29), we have

$$\|\hat{x}-x_{\alpha,\beta}^s\| = O((\delta+\epsilon_h)^{\frac{t}{t+a}}). \tag{3.30}$$

□

Combining Lemma 3.4, Lemma 2.7, Lemma 2.8 and Theorem 3.5, we have the following theorem.

Theorem 3.6. *Suppose conditions in Lemma 3.4, Lemma 2.7, Lemma 2.8 and Theorem 3.5 hold and $\alpha := \alpha(\delta, h) > 0$ is the unique solution of (3.7). Then*

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| = O((\delta + \epsilon_h)^{\frac{t}{t+a}}). \tag{3.31}$$

Proof. By Lemma 2.8, we have $\alpha \geq c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}$. So,

$$\frac{1}{\alpha} \leq \frac{1}{c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}}$$

and hence

$$\begin{aligned} \frac{\delta + \epsilon_h}{\alpha^{\frac{a}{(1+\beta)a+2s}}} &\leq \frac{\delta + \epsilon_h}{c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{a}{a+t}}} \\ &= \frac{1}{c_{\beta,a,s,h}}(\delta + \epsilon_h)^{\frac{t}{a+t}}. \end{aligned} \tag{3.32}$$

Therefore by Lemma 2.7, Lemma 2.8, Theorem 2.9, (3.32) and the following inequality

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| \leq \|\hat{x} - x^{s}_{\alpha,\beta}\| + \|x^{s}_{\alpha,\beta} - x^{s}_{\alpha,\beta,h}\| + \|x^{s}_{\alpha,\beta} - x^{s,\delta}_{\alpha,\beta,h}\|,$$

we have

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| = O((\delta + \epsilon_h)^{\frac{t}{t+a}}). \tag{3.33}$$

□

4. Numerical examples

We now consider two academic examples to validate our theoretical results for the numerical discussion.

We consider the Hilbert scales generated by the linear operator L defined by

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$

where $u_j(t) = \sqrt{2}\sin(j\pi t)$, $j \in \mathbb{N}$, with domain of L as

$$D(L) := \left\{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \right\}.$$

In this case the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$\begin{aligned} \mathcal{X}_s &= \{x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4s} |\langle x, u_j \rangle|^2 < \infty\} \\ &= \{x \in H^{2s}(0,1) : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \ldots \lceil \frac{s}{2} - \frac{1}{4} \rceil\}, \end{aligned} \tag{4.1}$$

where $\lceil p \rceil$ denotes the greatest integer less than or equal to p, $s \in \mathbb{R}$, and H^s is the usual Sobolev space. Also, one can see that $H^0 = L^2[0,1]$, and for $s \in \mathbb{N}, H_s \subset H^s$. We take $s = 1/2$ in our computation. The constants a, b_1 and b_2 are given by $a = 1, b_1 = b_2 = \frac{1}{\pi}$.

We consider a sequence of finite dimensional subspaces of (V_n) of $\mathcal{X}$ and P_h $(h = \frac{1}{n})$ denote the orthogonal projection on $\mathcal{X}$ with $R(P_h) = V_n$. We choose V_n as the linear span of $\{v_1, v_2, \ldots, v_n\}$ with $v_i, i = 1, 2, \ldots, n-1$ are linear splines [24] defined by

$$v_i(t) = \begin{cases} nt + 1 - i & (i-1)h \leq t \leq ih \\ -nt + 1 + i & ih \leq t \leq (i+1)h \\ 0 & \textit{otherwise.} \end{cases}$$

In this case the matrix corresponding to $T_h^*T_h$ is given by

$$M_h := (\langle Tv_i, Tv_j\rangle)_{i,j} = (\int_0^1 [Tv_i(s)][Tv_j(s)]ds)_{i,j}, \ i,j = 1,2,\dots,n-1, \tag{4.2}$$

where

$$Tv_i(s) = \int_{(i-1)h}^{ih} k(s,t)(\frac{t}{h} + 1 - i)dt + \int_{ih}^{(i+1)h} k(s,t)(\frac{-t}{h} + 1 + i)dt.$$

The matrix corresponding to L^{2s} for $s = \frac{1}{2}$ is given by

$$B_h = (\langle Lv_i, v_j\rangle)_{i,j} \tag{4.3}$$

$$= (\sum_{m=1}^{\infty} m^2\langle v_i, u_m\rangle\langle v_j, u_m\rangle)_{i,j} \ , i,j = 1,2,\dots,n-1. \tag{4.4}$$

One can see ([24, Page 165])

$$B_h := \frac{1}{h}\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{bmatrix}.$$

We take the singular value decomposition (SVD) of M_h as

$$M_h = U\Sigma V^T,$$

where $U = [u_1, u_2, \dots, u_n] \in \mathbb{R}^{n\times n}$ and $V = [v_1, v_2, \dots, v_n] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and

$$\Sigma = diag[\lambda_1, \lambda_2, \dots \lambda_n] \in \mathbb{R}^{n\times n},$$

are the singular values of M_h ordered as

$$\lambda_1 \geq \lambda_2 \geq \dots \lambda_r > \lambda_{r+1} = \dots \lambda_n = 0.$$

Substituting the SVD in (2.11) yields

$$x^{s,\delta}_{\alpha,\beta,h} = V(\Sigma^{1+\beta} + \alpha V^T B_h V)^{-1}\Sigma^{\beta} V^T \tilde{y}^{\delta}, \tag{4.5}$$

where $\tilde{y} = [y^{\delta}(t_0), y^{\delta}(t_1), \dots, y^{\delta}(t_n)]^T, \ t_i = \frac{i-1}{n}$.

For a particular β we choose the regularization parameter α satisfying (3.7) as follows. Let

$$f(\alpha) = \alpha^2\|B_h^{\frac{3s+\beta a}{2s}} V^2(\Sigma^{1+\beta} + \alpha V^T B_h V)^{-2} V^T B_h^{\frac{1}{2}} \Sigma^{\beta} V^T \tilde{y}^{\delta}\| - (c\delta + d\epsilon_h). \tag{4.6}$$

By Newton's method, we compute α as the limit of the sequence

$$\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}.$$

We terminate the iterate when $|\alpha_{n+1} - \alpha_n| < 10^{-5}$ in our computation. In both the examples, we have computed the matrix M_h as in (4.2) and computed $x^{s,\delta}_{\alpha,\beta,h}$ as in (4.5). The computed values of α and the error $E_{\alpha,\beta} := \|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\|$ and the $E_{ratio} := \frac{E_{\alpha,\beta}}{\|x^{s,\delta}_{\alpha,\beta,h}\|}$ for different values of $\beta(0.1, 0.2, ..., 0.9, 1)$ with random noise levels $\delta = 0.01$ and $\delta = 0.001$ are given in the Tables.

Example 4.1. ([25, Shaw]) Let

$$[Tx](s) := \int_0^1 k(s,t)x(t)dt = y(s),\ 0 \le s \le 1, \tag{4.7}$$

where $k(s,t) = (cos(s) + cos(t))^2(\frac{sin(u)}{u})^2$, $u = \pi(sin(s) + sin(t))$. We take $y = T\hat{x}$, where $\hat{x}$ is given by $\hat{x}(t) = 2exp(-6(t-0.8)^2) + exp(-2(t+0.5)^2)$.

Relative errors E_{ratio}, the error $E_{\alpha,\beta}$ and α values are displayed in Table 1 for different values of β, n and δ. In Fig:2 - Fig:4 and Fig:6 - Fig:8, the exact solution and computed solution (C.S) for different values of β are plotted and Fig:1 and Fig:5, contain the exact data and noise data.

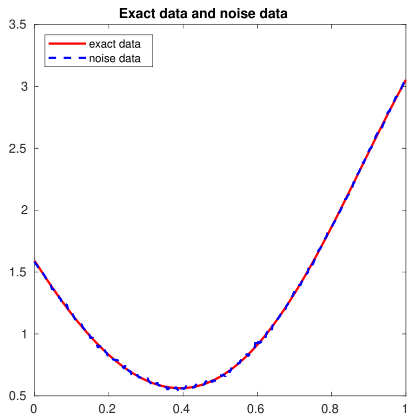

Fig:1 Noise data and exact data with $\delta = 0.01$ and $n = 300$ for the example.

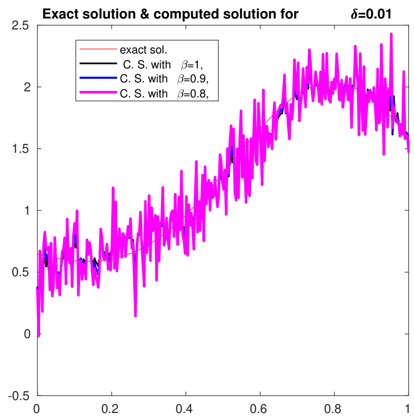

Fig:2 Solutions when $\delta = 0.01$ and $n = 300$ in the example.

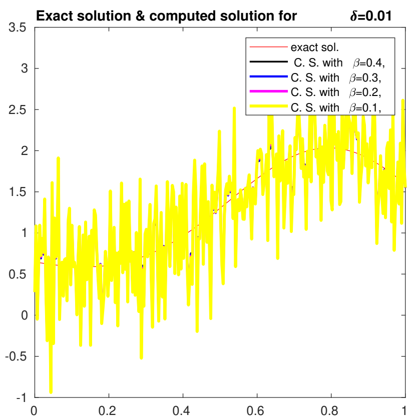

Fig:3 Solutions when $\delta = 0.01$ and $n = 300$ in the Shawn example.

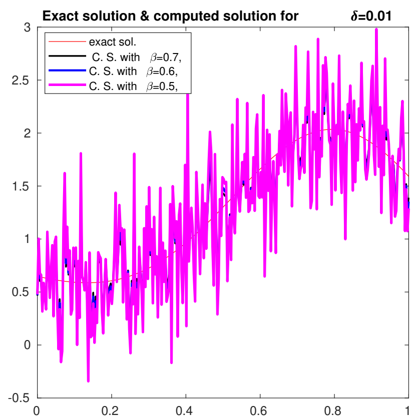

Fig:4 Solutions when $\delta = 0.01$ and $n = 300$ in the Shawn example.

Example 4.2. ([21, Phillips]) Let

$$\int_{-6}^{6} k(s,t)x(t)dt = g(s), \quad -6 \le u \le 6, \tag{4.8}$$

where $k(s,t) = \phi(s-t)$ with

$$\phi(x) = \begin{cases} 1 + \cos(x * \pi/3), & |x| < 3, \\ 0, & |x| \ge 3. \end{cases}$$

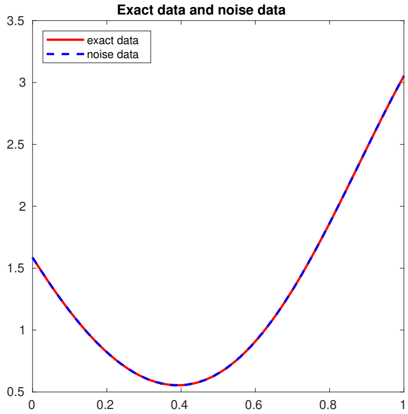

Fig:5 Noise data and exact data with $\delta = 0.001$ and $n = 500$ for the Shawn example.

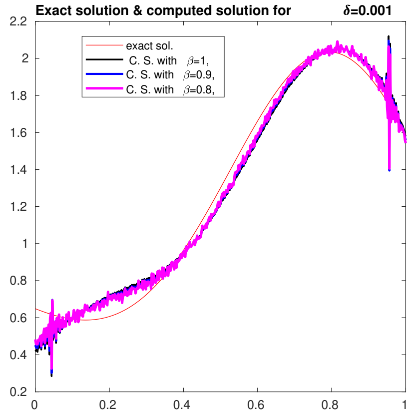

Fig:6 Solutions when $\delta = 0.001$ and $n = 500$ in the Shawn example.

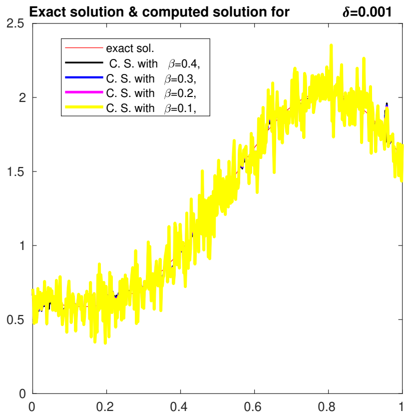

Fig:7 Solutions when $\delta = 0.001$ and $n = 500$ in the Shawn example.

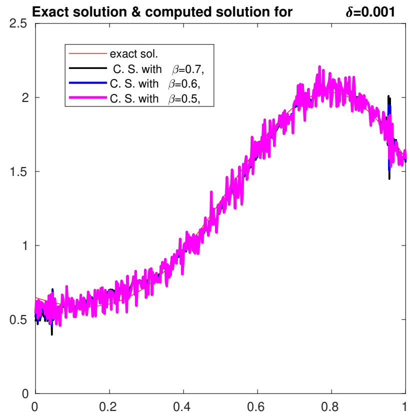

Fig:8 Solutions when $\delta = 0.001$ and $n = 500$ in the Shawn example.

We take $A := T^*T|_{N(T^*T)^\perp}$ and $y = T^*g$, where

$$g(s) = (6 - |s|) * (1 + .5 * \cos(s * \pi/3)) + 9/(2 * \pi) * \sin(|s| * \pi/3)$$

for our computation. The solution $\hat{x}$ is given by $\hat{x}(t) = \phi(t)$. We have taken $s = a = \frac{1}{2}, d_1 = d_2 = \frac{1}{36}$ in our computation. Relative errors E_{ratio}, the error $E_{\alpha,\beta}$ and α values are displayed in Table 1 for different values of β, n and δ.

Fig: 10 - Fig: 12 and Fig: 14 - Fig: 16, contain the exact solution and computed solution (C.S) for different values of β and in Fig: 9 and Fig: 13, the exact data and noise data are plotted.

Remark 4.3. From Table 1 and Table 2, one can observe that the error $E_{\alpha,\beta}$ obtained using the finite dimensional realization of FTR method in Hilbert scales for fractional values of β is smaller than that when $\beta = 1$. However, one can observe that with β the error $E_{\alpha,\beta}$ decreases to a certain limit and then increases after. This can also be observed from the Figures (Fig:1-Fig:16).

Table 1. Relative errors for different values of β.

β		$n = 300$		$n = 500$	
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$
1	α	2.157497e-03	2.111102e-03	2.359467e-03	2.045106e-03
	$E_{\alpha,\beta}$	8.086544e-02	5.580062e-02	6.979050e-02	5.984465e-02
	E_{ratio}	1.966310e+01	4.290699e+01	2.190834e+01	5.940711e+01
0.9	α	2.094812e-03	2.001696e-03	2.335022e-03	1.999903e-03
	$E_{\alpha,\beta}$	1.028586e-01	4.954536e-02	7.844515e-02	5.437206e-02
	E_{ratio}	2.501092e+01	3.809712e+01	2.462517e+01	5.397453e+01
0.8	α	2.002888e-03	1.838909e-03	2.289867e-03	1.925116e-03
	$E_{\alpha,\beta}$	1.423106e-01	4.333680e-02	1.017988e-01	4.881881e-02
	E_{ratio}	3.460401e+01	3.332314e+01	3.195624e+01	4.846188e+01
0.7	α	1.879297e-03	1.606383e-03	2.204036e-03	1.802906e-03
	$E_{\alpha,\beta}$	1.952871e-01	3.860271e-02	1.453034e-01	4.467454e-02
	E_{ratio}	4.748569e+01	2.968294e+01	4.561301e+01	4.434792e+01
0.6	α	2.149673e-03	1.296593e-03	2.073304e-03	1.610127e-03
	$E_{\alpha,\beta}$	2.373810e-01	3.835648e-02	2.223598e-01	4.313786e-02
	E_{ratio}	5.772115e+01	2.949360e+01	6.980223e+01	4.282247e+01
0.5	α	2.062660e-03	1.329601e-03	1.890583e-03	1.327042e-03
	$E_{\alpha,\beta}$	3.099169e-01	4.098500e-02	3.347035e-01	4.923642e-02
	E_{ratio}	7.535886e+01	3.151476e+01	1.050687e+02	4.887644e+01
0.4	α	2.526780e-03	1.333518e-03	2.112389e-03	1.327837e-03
	$E_{\alpha,\beta}$	3.677623e-01	4.264501e-02	4.479871e-01	5.680138e-02
	E_{ratio}	8.942446e+01	3.279120e+01	1.406302e+02	5.638609e+01
0.3	α	3.600039e-03	1.351594e-03	2.365584e-03	1.262176e-03
	$E_{\alpha,\beta}$	3.818477e-01	4.650115e-02	5.419249e-01	6.773099e-02
	E_{ratio}	9.284943e+01	3.575632e+01	1.701188e+02	6.723579e+01
0.2	α	4.540330e-03	1.472437e-03	2.741712e-03	1.168581e-03
	$E_{\alpha,\beta}$	4.142634e-01	4.931446e-02	6.257286e-01	7.788526e-02
	E_{ratio}	1.007316e+02	3.791956e+01	1.964261e+02	7.731582e+01
0.1	α	5.973373e-03	2.219342e-03	3.314624e-03	1.173023e-03
	$E_{\alpha,\beta}$	3.525842e-01	4.961016e-02	6.943178e-01	8.458691e-02
	E_{ratio}	8.573378e+01	3.814694e+01	2.179573e+02	8.396847e+01

5. Conclusion

In this work, the finite dimensional version of the parameter choice strategy introduced in [16] has been studied. We also obtained the optimal order error estimate for the finite dimensional FTR method in Hilbert scales .

As mentioned in the introduction, the FTR method reduces the over-smoothing in the standard Tikhonov regularization method in Hilbert scales. Choosing an optimal value for β is still an open problem.

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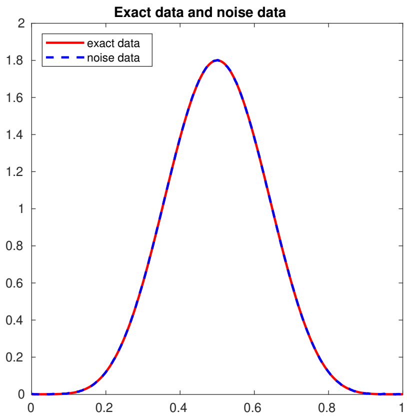

Fig:9 Noise data and exact data with $\delta = 0.001$ and $n = 300$ for the Phillips example.

obtain

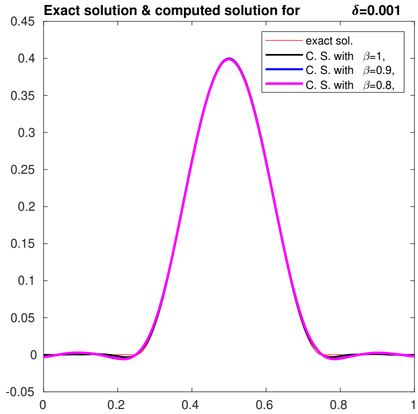

Fig:10 Solutions when $\delta = 0.001$ and $n = 300$ in the Phillips example.

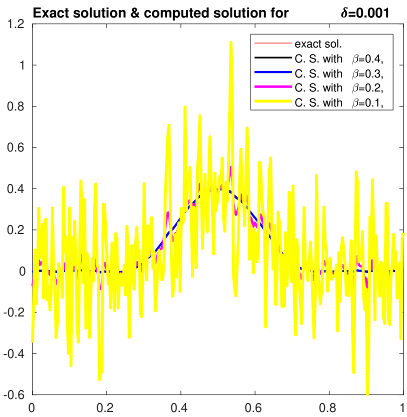

Fig:11 Solutions when $\delta = 0.001$ and $n = 300$ in the Phillips example.

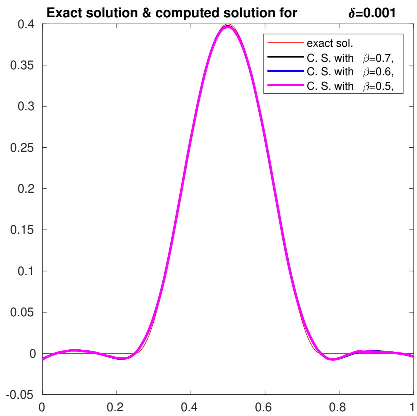

Fig:12 Solutions when $\delta = 0.001$ and $n = 300$ in the Phillips example.

Appendix A. Estimates for G_1

Using the fomula (3.10) and Proposition (2.5), we have

$$
\begin{aligned}
G_1 &\leq \frac{1}{\bar{f}(-\beta)} \|(T_h^* T_h)^{\frac{-\beta}{2}} [(T_h^* T_h)^{\frac{\beta}{2}} - (T^* T)^{\frac{\beta}{2}}] y\| \\
&= \frac{\sin \pi (\frac{\beta}{2})}{\bar{f}(-\beta)\pi} \| \int_0^\infty \lambda^{\frac{\beta}{2}} (T_h^* T_h)^{\frac{-\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [(T^* T) - (T_h^* T_h)] (T^* T + \lambda I)^{-1} y d\lambda \| \\
&\leq \frac{\sin \pi (\frac{\beta}{2})}{\bar{f}(-\beta)\pi} \| \int_0^\infty \lambda^{\frac{\beta}{2}} (T_h^* T_h)^{\frac{-\beta}{2}} (T_h^* T_h + \lambda I)^{-1} \, [T_h^* (T - T_h) \\
&\quad + (T^* - T_h^*) T] \, (T^* T + \lambda I)^{-1} T \hat{x} d\lambda \| \\
&\leq \frac{\sin \pi (\frac{\beta}{2})}{\bar{f}(-\beta)\pi} \| \int_0^\infty \lambda^{\frac{\beta}{2}} (T_h^* T_h)^{\frac{-\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [T_h^* (T - T_h)] (T^* T + \lambda I)^{-1} T \hat{x} d\lambda \|,
\end{aligned}
$$

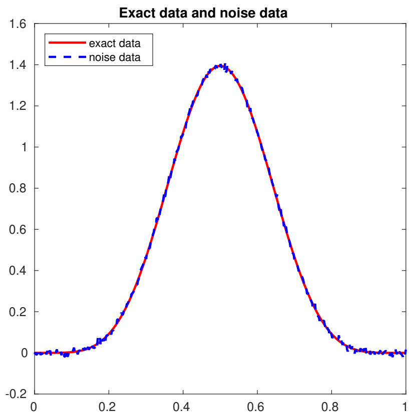

Fig:13 Noise data and exact data with $\delta = 0.01$ and $n = 500$ for the Phillips example.

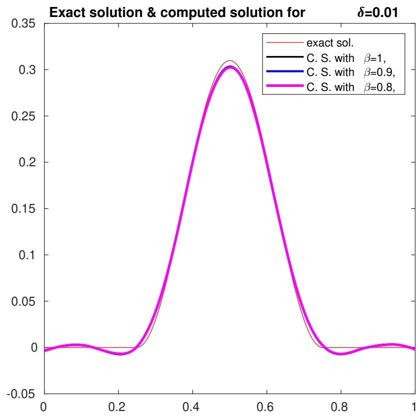

Fig:14 Solutions when $\delta = 0.01$ and $n = 500$ in the Phillips example.

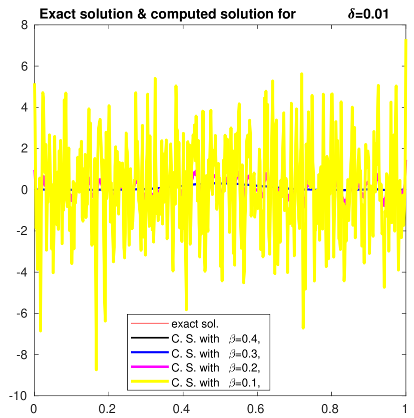

Fig:15 Solutions when $\delta = 0.01$ and $n = 500$ in the Phillips example.

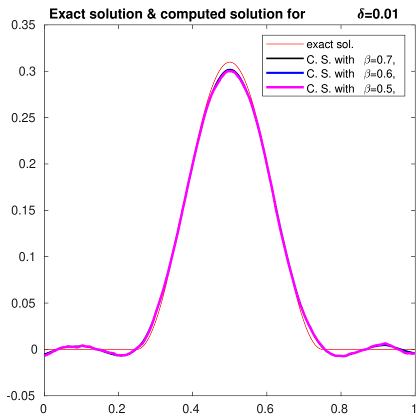

Fig:16 Solutions when $\delta = 0.01$ and $n = 500$ in the Phillips example.

where (here and below) we used $(T_h^*T_h + \lambda I)^{-1} \equiv (T_h^*T_h + \lambda P_h)^{-1} \equiv (T_h^*T_h + \lambda)^{-1}P_h$ and $(T_h^*T_h + \lambda)^{-1}P_h[T^* - T_h^*] = (T_h^*T_h + \lambda I)^{-1}P_h[I - P_h]T^* = 0$. Observe that

$$\begin{aligned}
&\| \int_0^\infty \lambda^{\frac{\beta}{2}} (T_h^*T_h)^{\frac{-\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^*[T - T_h](T^*T + \lambda I)^{-1} T\hat{x} d\lambda \| \\
&\leq \| \int_0^1 \lambda^{\frac{\beta}{2}} (T_h^*T_h)^{\frac{-\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^*[T - T_h](T^*T + \lambda I)^{-1} T\hat{x} d\lambda \| \\
&+ \| \int_1^\infty \lambda^{\frac{\beta}{2}} (T_h^*T_h)^{\frac{-\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^*[T - T_h](T^*T + \lambda I)^{-1} T\hat{x} d\lambda \| \\
&= \Gamma_1 + \Gamma_2
\end{aligned} \tag{A.1}$$

Table 2. Relative errors for different values of β for the Phillips example.

β		$n=300$		$n=500$	
		$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$
1	α	3.391855e-01	1.015508e-02	3.464049e-01	1.059873e-02
	$E_{\alpha,\beta}$	4.194623e-02	1.223043e-02	3.771075e-02	1.037706e-02
	E_{ratio}	1.258356e+00	1.160252e+00	1.131313e+00	9.844453e-01
0.9	α	3.343474e-01	5.401894e-02	3.474458e-01	5.500000e-02
	$E_{\alpha,\beta}$	4.222346e-02	1.951024e-02	4.039446e-02	1.929680e-02
	E_{ratio}	1.266673e+00	1.850858e+00	1.211823e+00	1.830639e+00
0.8	α	3.257190e-01	7.727778e-02	3.396219e-01	8.178772e-02
	$E_{\alpha,\beta}$	4.248689e-02	2.070888e-02	4.050663e-02	2.095765e-02
	E_{ratio}	1.274576e+00	1.964569e+00	1.215188e+00	1.988200e+00
0.7	α	3.188334e-01	8.382101e-02	3.294550e-01	8.490842e-02
	$E_{\alpha,\beta}$	3.953119e-02	2.070800e-02	4.064611e-02	2.040183e-02
	E_{ratio}	1.185907e+00	1.964485e+00	1.219373e+00	1.935471e+00
0.6	α	3.059524e-01	8.077767e-02	3.146328e-01	8.447804e-02
	$E_{\alpha,\beta}$	4.007452e-02	1.776854e-02	3.855526e-02	2.013184e-02
	E_{ratio}	1.202206e+00	1.685631e+00	1.156648e+00	1.909857e+00
0.5	α	2.895052e-01	7.868060e-02	2.975993e-01	8.142263e-02
	$E_{\alpha,\beta}$	4.101069e-02	1.684125e-02	3.888572e-02	1.918115e-02
	E_{ratio}	1.230291e+00	1.597662e+00	1.166561e+00	1.819667e+00
0.4	α	2.522316e-01	7.432798e-02	2.713859e-01	7.632939e-02
	$E_{\alpha,\beta}$	4.367092e-02	1.596039e-02	4.016585e-02	1.836948e-02
	E_{ratio}	1.310096e+00	1.514099e+00	1.204965e+00	1.742666e+00
0.3	α	1.641012e-01	6.011976e-02	1.498880e-01	5.430934e-02
	$E_{\alpha,\beta}$	5.882925e-02	1.458577e-02	6.554305e-02	1.869007e-02
	E_{ratio}	1.764835e+00	1.383694e+00	1.966274e+00	1.773081e+00
0.2	α	6.623901e-03	7.510152e-03	4.981844e-03	4.752450e-03
	$E_{\alpha,\beta}$	1.871649e+00	1.596788e-01	2.831882e+00	2.981823e-01
	E_{ratio}	5.614810e+01	1.514809e+01	8.495572e+01	2.828781e+01
0.1	α	2.061885e-03	2.064347e-03	1.991443e-03	1.877291e-03
	$E_{\alpha,\beta}$	1.404229e+01	1.367919e+00	1.862952e+01	1.838851e+00
	E_{ratio}	4.212583e+02	1.297690e+02	5.588807e+02	1.744472e+02

where $\Gamma_1 = \| \int_0^1 \lambda^{\frac{\beta}{2}} (T_h^*T_h)^{\frac{-\beta}{2}} (T_h^*T_h+\lambda I)^{-1} T_h^*[T-T_h](T^*T+\lambda I)^{-1} T\hat{x} d\lambda\|$ and $\Gamma_2 = \| \int_1^\infty \lambda^{\frac{\beta}{2}} (T_h^*T_h)^{\frac{-\beta}{2}} (T_h^*T_h+\lambda I)^{-1} T_h^*[T-T_h](T^*T+\lambda I)^{-1} T\hat{x} d\lambda\|$. Further, note that

$$\begin{aligned}
\Gamma_1 &\leq \int_0^1 \lambda^{\frac{\beta}{2}} \|(T_h^*T_h)^{\frac{1-\beta}{2}} (T_h^*T_h+\lambda I)^{-1}\| \|[T-T_h]\| \\
&\quad \|(T^*T+\lambda I)^{-1}(T^*T)^{(\frac{t}{2a}+\frac{1}{2})}\| \|(T^*T)^{\frac{-t}{2a}}\hat{x}\| d\lambda \\
&\leq g(\frac{-t}{a})\epsilon_h \int_0^1 \lambda^{\frac{t}{2a}-1} \|\hat{x}\|_t d\lambda \\
&= g(\frac{-t}{a})\frac{2a}{t} E\epsilon_h \qquad \text{(A.2)}
\end{aligned}$$

and

$$\begin{aligned}\Gamma_2 &\leq \int_1^\infty \lambda^{\frac{\beta}{2}} \|(T_h^*T_h)^{\frac{1-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}\| \\ &\quad \|[T-T_h]\| \|(T^*T+\lambda I)^{-1}\| \|(T^*T)^{\frac{1}{2}}\hat{x}\| d\lambda \\ &\leq \int_1^\infty \frac{\lambda^{\frac{\beta}{2}}}{\lambda^{2-\frac{1}{2}+\frac{\beta}{2}}}\epsilon_h \|y\| d\lambda \\ &\leq 2\|y\|\epsilon_h. \end{aligned} \tag{A.3}$$

Therefore, by (A.1), (A.2) and (A.3), we have

$$G_1 \leq \tilde{C}_1 \epsilon_h. \tag{A.4}$$

Appendix B. Estimates for G_2

Again, we have

$$\begin{aligned} G_2 &\leq \alpha^2 \|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}[(A_{s,\beta,h}+\alpha I)^{-2}-(A_{s,\beta}+\alpha I)^{-2}]L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\ &\quad +\alpha^2\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|. \\ &=: G_{21}+G_{22}, \end{aligned} \tag{B.1}$$

where $G_{21} = \alpha^2 \|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}[(A_{s,\beta,h}+\alpha I)^{-2}-(A_{s,\beta}+\alpha I)^{-2}]L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$ and $G_{22} = \alpha^2\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$. Note that

$$\begin{aligned} G_{21} &= \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}[A_{s,\beta}^2 - A_{s,\beta,h}^2 + 2\alpha(A_{s,\beta}-A_{s,\beta,h})] \\ &\quad \times (A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\ &\leq \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}(A_{s,\beta}-A_{s,\beta,h})(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\ &\quad +\|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}(A_{s,\beta}-A_{s,\beta,h})(A_{s,\beta}+\alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\ &=: \Gamma_3+\Gamma_4 \end{aligned} \tag{B.2}$$

where $\Gamma_3 = \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}(A_{s,\beta}-A_{s,\beta,h})(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$ and $\Gamma_4 = \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}(A_{s,\beta}-A_{s,\beta,h})(A_{s,\beta}+\alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$.

Let $Z = (A_{s,\beta} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y$. Then

$$\begin{aligned}
\Gamma_3 &= \alpha^2 \|(A_{s,\beta,h} + \alpha I)^{-1} A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta} - A_{s,\beta,h}) Z\| \\
&= \alpha^2 \|(A_{s,\beta,h} + \alpha I)^{-1} A_{s,\beta,h}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} [(T^*T)^{\frac{1+\beta}{2}} - (T_h^*T_h)^{\frac{1+\beta}{2}}] L^{-s} Z\| \\
&\leq \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \frac{\sin \pi(\frac{1+\beta}{2})}{\pi} \\
&\quad \times \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \int_0^\infty \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} [T^*T - T_h^*T_h] (T^*T + \lambda I)^{-1} L^{-s} Z\| \\
&\leq \frac{\sin \pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \int_0^1 \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^* [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda\| \\
&+ \frac{\sin \pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \\
&\quad \int_1^\infty \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^* [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda\| \\
&= \frac{\sin \pi(\frac{1+\beta}{2})}{\pi} (\Gamma_5 + \Gamma_6) \qquad \text{(B.3)}
\end{aligned}$$

where
$\Gamma_5 = \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \int_0^1 \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^* [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda\|$
and
$\Gamma_6 = \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \int_1^\infty \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} T_h^* [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda\|$.
Using Proposition 2.5 and Proposition 2.6, we have

$$\begin{aligned}
\Gamma_5 &\leq \bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \|\int_0^1 \lambda^{\frac{1+\beta}{2}} (T_h^*T_h)^{\frac{1}{2}} (T_h^*T_h + \lambda I)^{-1} [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda\|_a \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \times \int_0^1 \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^{2-\frac{t}{2a}}} \epsilon_h \|(T^*T)^{\frac{-t}{2a}} L^{-s} (A_{s,\beta} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} T \hat{x}\| d\lambda \qquad \text{(B.4)} \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) g(\frac{-t}{a})}{\bar{f}(-1) F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)} \frac{2a}{(\beta-1)a+t} \epsilon_h \|A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\| \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) g(\frac{-t}{a}) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{\bar{f}(-1) F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)} \frac{2a}{(\beta-1)a+t} c_{(\beta-1)a,0} \epsilon_h \|\hat{x}\|_t \qquad \text{(B.5)}
\end{aligned}$$

and

$$\begin{aligned}\Gamma_6 &\leq \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}\\&\int_1^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\|\\&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\\&\int_1^\infty \lambda^{\frac{1+\beta}{2}}\|(T_h^*T_h+\lambda I)^{-1}\|\|[T-T_h]\|\|(T^*T+\lambda I)^{-1}\|\|L^{-s}Z\|d\lambda.\end{aligned}$$

$$\begin{aligned}&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)F\left(\frac{2s}{(1+\beta)a+2s}\right)}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\\&\int_1^\infty \lambda^{\frac{-3}{2}+\frac{\beta}{2}}\epsilon_h\|A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta+1}{2}}L^{-s}L^s\hat{x}\|d\lambda\\&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2s}{(1+\beta)a+2s}\right)}{\bar{f}(-1)F\left(\frac{2s}{(1+\beta)a+2s}\right)}\left(\frac{2}{1-\beta}\right)c_{(\beta-1)a,0}\|\hat{x}\|\epsilon_h.\end{aligned}\tag{B.6}$$

Therefore, by (B.3),(B.4) and (B.6) we have

$$\Gamma_3 \leq {}_2\epsilon_h. \tag{B.7}$$

Similarly, we have for $Z_1 = (A_{s,\beta}+\alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y$,

$$\begin{aligned}\Gamma_4 &= \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}[(T^*T)^{(\frac{1+\beta}{2})}-(T_h^*T_h)^{(\frac{1+\beta}{2})}]L^{-s}Z_1\|\\&\leq \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{(1-\beta)a}{(1+\beta)a+2s}}A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}[(T_h^*T_h)^{(\frac{1+\beta}{2})}-(T^*T)^{(\frac{1+\beta}{2})}]L^{-s}Z_1\|\\&\leq \bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}}\\&\times\|\frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T^*T-T_h^*T_h](T^*T+\lambda I)^{-1}L^{-s}Z_1d\lambda\|_a\\&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)}\frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}}\\&\times\|\int_0^1 \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Z_1d\lambda\|\\&+\frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)}\frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}}\\&\times\|\int_1^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_0^1 \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^{2-\frac{t}{2a}}} \epsilon_h \|(T^*T)^{\frac{-t}{2a}} L^{-s}(A_{s,\beta}+\alpha I)^{-1} L^{-s}(T^*T)^{\frac{\beta+1}{2}} L^{-s} L^s \hat{x}\| d\lambda \\
&\quad + \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_1^\infty \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^{2}} \epsilon_h \|L^{-s}(A_{s,\beta}+\alpha I)^{-1} L^{-s}(T^*T)^{\frac{\beta}{2}}(T^*T)^{\frac{1}{2}} L^{-s} L^s \hat{x}\| d\lambda \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2})}{\bar{f}(-1) F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_0^1 \lambda^{\frac{-3+\beta}{2}+\frac{t}{2a}} \epsilon_h \|(A_{s,\beta}+\alpha I)^{-1} A_{s,\beta}^{\frac{2\beta a+2s}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} L^s \hat{x}\| d\lambda \\
&\quad + \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2})}{\bar{f}(-1) F\left(\frac{2s}{(1+\beta)a+2s}\right)\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_1^\infty \lambda^{\frac{-3+\beta}{2}} \epsilon_h \|(A_{s,\beta}+\alpha I)^{-1} A_{s,\beta}^{\frac{2\beta a+2s}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} L^s \hat{x}\| d\lambda \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2}) G\left(\frac{2(1-\beta)a}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)\pi} c_{(\beta-1)a,0} \\
&\quad \times (c_3+c_4)\epsilon_h \\
&= {}_3\epsilon_h. \qquad \text{(B.8)}
\end{aligned}
$$

Therefore, by (B.6), (B.7) and (B.8), we have

$$G_{21} \leq ({}_2 + {}_3)\epsilon_h. \qquad \text{(B.9)}$$

Also, we have

$$
\begin{aligned}
G_{22} &\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) \alpha^2 \|(A_{s,\beta}+\alpha I)^{-2} L^{-s}(T^*T)^{\frac{\beta}{2}} T\hat{x}\|_{\beta a+s} \\
&= \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} \alpha^2 \|(A_{s,\beta}+\alpha I)^{-2} A_{s,\beta,h}^{\frac{a+s}{(1+\beta)a+2s}} L^s \hat{x}\| \\
&= \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} \alpha^2 \|(A_{s,\beta}+\alpha I)^{-2} A_{s,\beta,h}^{\frac{a+t}{(1+\beta)a+2s}} A_{s,\beta,h}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\| \qquad \text{(B.10)} \\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} \alpha^{\frac{a+t}{(1+\beta)a+2s}} \|\hat{x}\|_t. \qquad \text{(B.11)}
\end{aligned}
$$

Thus, we have

$$G_2 \leq ({}_2 + {}_3)\epsilon_h + \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} \alpha^{\frac{a+t}{(1+\beta)a+2s}} E.$$

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