Sigma J Eng Nat Sci, Vol. 40, No. 1, pp. 9-26, March, 2022

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2022.00002

Research Article

Bayesian estimation for topp-leone distribution using progressively censored data and asymmetric loss functions

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ARTICLE INFO

Article history
Received: 31 March 2020
Accepted: 09 May 2021

Key words:
Topp-Leone distribution; progressive type-II censored data; bayesian estimation, asymmetric loss functions; non-informative prior; lndley's approximation

ABSTRACT

In this study, based on progressive type-II censored data, Bayes estimators of the unknown parameter of the Topp-Leone distribution are derived by using informative and non-informative, priors under square error (symmetric), and linex, general entropy, and precautionary (asymmetric) loss functions. The Bayes estimators cannot be obtained in closed-forms, for this reason, Lindley's approximation method is used to compute the approximate estimates. The asymptotic confidence and the highest posterior density credible intervals for the unknown parameter are obtained. The performances of the proposed Bayes estimators are compared with the corresponding maximum likelihood estimator for different sample sizes in terms of average estimate and mean squared error through an extensive simulation study. Finally, a real data set is provided to illustrate the results.

Cite this article as: Usta I. Bayesian estimation for topp-leone distribution using progressively censored data and asymmetric loss functions. Sigma J Eng Nat Sci 2022;40(1):9–26.

INTRODUCTION

The Topp–Leone (TL) distribution is a univariate continuous distribution with a finite support set, which was originally proposed and used as a model for some failure data by Topp and Leone [1]. The probability density function (pdf) of the TL distribution is given by

$$f(x;\alpha)=2\alpha x^{\alpha-1}(1-x)(2-x)^{\alpha-1},\quad 0<x\langle 1,\ \alpha\rangle 0 \quad (1)$$

where α is the shape parameter. Its cumulative distribution function (cdf) is also given as

$$F(x;\alpha)=\begin{cases}0 & , \quad x\le 0\\ x^{\alpha}(2-x)^{\alpha}, & 0<x<1\\ 1 & , \quad x\ge 1\end{cases} \quad (2)$$

MirMostafaee et al. [2] indicated that the TL distribution can be used as an alternative distribution to the beta and Kumaraswamy distributions which have the same support set. It can also be deduced from Nadarajah and Kotz [3] that the TL distribution has two advantages: (i) the cdf of the TL can be written in closed form; thus, it simplifies

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This paper was recommended for publication in revised form by Regional Editor Gülhayat Gölbaşı Şimşek

Published by Yıldız Technical University Press, İstanbul, Turkey

the computation of the likelihood function for censored data; (ii) the TL distribution has a bathtub-shaped hazard rate function for the full range of parameter values.

On the other hand, most lifetime distributions are known to have an infinite support set (0, ∞) as the failure time of the component can be infinite. However, in practice, we often encounter physical constraints that are the inherent consequence of real-world phenomena such as limited power supply, restricted maintenance resources, and the design life of component or product. In such cases, the distribution with a finite support set may be more suitable to model the lifetimes. Thus, the TL distribution provides the experimenter or researcher with an alternative model for representing the lifetimes.

In recent years, the TL distribution has started to attract great interest in the literature. For example, Nadarajah and Kotz [3] studied the structural properties of the TL distribution and obtained explicit expressions for its moments and characteristic function. Ghitany et al. [4] discussed some reliability measures of the TL distribution, such as the hazard rate, mean residual lifetime, reversed hazard rate, expected inactivity time, and their stochastic orderings. Ghitany [5] also obtained asymptotic distribution of order statistics of the TL distribution. The moments of order statistics from the TL distribution were derived by Genc [6] and MirMostafaee [7]. Moreover, some of the recent studies on the statistical inference of the TL distribution can be provided from MirMostafaee et al. [2], El-Sayed et al. [8], Sindhu et al. [9], Feroze and Aslam [10], Bayoud [11], and Arora et al [12].

It is well known that in lifetime testing experiments, the experimenter may not always observe the failure times of all units placed on the test. Samples that result from such cases are called censored samples. There are several types of censoring schemes. Since progressive censoring schemes allow the experimenter to remove units before the termination of the experiment, these schemes in the last few years have been recently studied more extensively than others by many authors. Due to the flexibility feature, progressive censoring schemes are commonly used in reliability experiments, clinical trials, and life-testing experiments, etc. For further details about progressive censoring, the readers can refer to Balakrishnan and Aggarwala [13].

The progressive type-II censoring scheme firstly introduced by Cohen [14] can be defined as follows: The experimenter places n identical units on a life test at time zero and decides to observe only m ($< n$) failures. When the first failure is observed, R_1 of the remaining $n - 1$ surviving units are randomly selected and removed. Then after the second observed failure, R_2 of the remaining $n - 1 - R_1$ surviving units are randomly selected and removed, and so on. Finally, the experiment terminates until the m^{th} failure is observed and remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units all removed. The values of R_1, R_2,... R_m and n, m are all prefixed. It is to be noted that if $R_1 = R_{2,}\ R_{m-1}$, then $R_m = n - m$ that corresponds usual type-II censoring. If $R_1 = ... = R_m = 0$ then $n = m$ that represents the complete sample.

In the literature, the statistical inference on different lifetime distributions under progressive type-II censoring has been studied by many authors, including Wingo [15], Musleh and Helu [16], Usta and Gezer [17] , and Mahto et al. [18]. In these and many similar studies, the estimates of the unknown parameters of the distribution are obtained based on the maximum likelihood estimators (MLE) using Newton-Rapson (NR) algorithm. It is also known that significant properties of MLE are provided only when the sample size is large enough and when the lifetimes are slightly censored Bousqet [19]. However, it is often used data with small sample sizes in the lifetime analysis. In such cases, it has been emphasized by many authors that using the Bayes estimators is more appropriate when prior knowledge is available Robert [20].

Although classical and Bayesian estimation methods have been applied by some authors to estimate the parameter of the TL distribution in recent years, the estimation of the parameter of the TL distribution under progressive type-II censored data has still not received much attention in the literature. For instance, Bayoud [21] studied the classical and Bayesian estimations for the shape parameter of the TL distribution when the sample is progressive type-II censored. In this study, the Bayes estimator of the parameter was obtained by using the exponential prior and the squared error loss function. Feroze et al. [22] considered the Bayesian estimation for the parameters and reliability characteristics of the two parametric TL distribution based on progressive type-II censored data. They applied the Bayesian estimation under different loss functions with the assumption of the independent gamma and beta priors for the scale and shape parameters of the TL distribution. An important point to note about these two studies is that both of them have assumed the informative prior(s) for the parameter(s) of the TL distribution. However, in many real cases it is very difficult to know in advance information about the parameters. For this reason, assuming different informative and non-informative priors, Bayesian estimation of the unknown parameter of the TL distribution under different loss functions (symmetrical and asymmetrical) in the presence of progressive type-II censoring still attracts the attention of researchers.

Therefore, in this study, we aim to examine the performance of Bayes estimators of the parameter of the TL distribution under different loss function assuming different prior distribution for different sample sizes when the sample is progressive type-II censored. In accordance with this object, the Bayes estimators of the parameter are derived by using a non-informative (Jeffrey's) prior and an informative (gamma) prior under different loss functions, namely, squared error (symmetric), linear-exponential, general entropy, and precautionary (asymmetric) loss functions. The Bayes estimators cannot be obtained in closed-forms, therefore Lindley's approximation method is used for computing the approximate Bayes estimates. Furthermore, an extensive simulation study is performed to compare the performances of the Bayes estimators with corresponding MLE for different sample sizes, and a real data set has been analyzed for illustrative purposes.

The remainder of this study is organized as follows. The MLE is presented in Section 2. Section 3 provides the Bayesian estimation. The asymptotic confidence and the highest posterior density credible intervals are constructed in Section 4. The results of simulation study are given in Section 5. A real data set is analyzed in Section 6, followed by conclusions in Section 7.

MAXIMUM LIKELIHOOD ESTIMATION

Let $\boldsymbol{X} = (X_1, X_2,..., X_m)$ with $X_1 \le X_2 \le ... \le X_m$ denote the progressively type-II censored sample of size m from a sample of size n with progressively censoring scheme $\boldsymbol{R} = (R_1, R_2,..., R_m)$, drawn from the TL distribution whose pdf given by (1). The likelihood function for the shape parameter a based on a progressively type-II censored sample can be written as

$$l(\alpha|\boldsymbol{x}) = A\prod_{i=1}^{m} f(x_i;\alpha)\left[1-F(x_i;\alpha)\right]^{R_i} \quad (3)$$

where $A = n(n-1-R_1)...\left(n-\sum_{i=1}^{m-1}(R_i+1)\right)$. The log-likelihood function can be derived by using Equations 1–3 as follows:

$$\ln l(\alpha|\boldsymbol{x}) \propto m\ln(\alpha) + \alpha\sum_{i=1}^{m}\ln\left(2x_i - x_i^2\right) + \sum_{i=1}^{m} R_i \ln\left(1-\left(2x_i - x_i^2\right)^{\alpha}\right) \quad (4)$$

The MLE of the shape parameter α (say $\hat{\alpha}_{MLE}$) can be obtained by equating the partial derivative of $\ln l(\alpha|x)$ function given in Equation 4 to zero as the following equation:

$$\frac{\partial \ln l(\alpha|\boldsymbol{x})}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln\left(2x_i - x_i^2\right) - \sum_{i=1}^{m} R_i\left(\frac{\left(2x_i - x_i^2\right)^{\alpha}}{1-\left(2x_i - x_i^2\right)^{\alpha}}\right)\ln\left(2x_i - x_i^2\right) = 0 \quad (5)$$

It is clear that an explicit form to the solution of Equation 5 does not exist, and hence the numerical methods, such as the Newton–Raphson, are required to compute the MLE of the parameter a.

MAXIMUM LIKELIHOOD ESTIMATION

In this section, the Bayes estimators of the shape parameter α have been derived by using a non-informative (Jeffrey's) prior and an informative (gamma) prior under different loss functions, namely, squared error, linear-exponential, general entropy, and precautionary loss functions.

One of the most commonly used loss functions is the squared error loss function (SELF) given as follows

$$L\left(\theta,\hat{\theta}\right) = \left(\theta-\hat{\theta}\right)^2 \quad (6)$$

It is a symmetric loss function that assigns equal weight to overestimation as well as underestimation. The Bayes estimator of q under Equation 6 is the posterior mean, which is given by

$$\hat{\theta}_{BS} = E_{\pi}(\theta) \quad (7)$$

Here, π is the posterior pdf of θ. The SELF is commonly used in the Bayesian estimation because it does not need extensive numerical computation. However, the use of a symmetric loss function might be inappropriate for different estimation problems as emphasized by Zellner [23], Basu and Ebrahimi [24]. Due to this reason, the use of asymmetric loss functions may be more appropriate.

A useful alternative to the SELF is an asymmetric loss function, called the linear-exponential (LINEX) loss function, introduced by Varian [25] and defined as

$$L\left(\theta,\hat{\theta}\right) = e^{c\left(\hat{\theta}-\theta\right)} - c\left(\hat{\theta}-\theta\right) - 1, \quad c \neq 0 \quad (8)$$

The magnitude and sign of the loss parameter c in Equation 8 represents the degree of symmetry and direction, respectively. When the value of parameter c closes to zero, the LINEX loss function becomes approximately the SELF and therefore it becomes almost symmetric. According to Zellner [23], the Bayes estimator of q under Equation 8 is given as

$$\hat{\theta}_{BL} = -\frac{1}{c}\ln E_{\pi}\left(e^{-c\theta}\right) \quad (9)$$

provided that E_{π} $(e^{-c\theta})$ exists and is finite. Recently, the LINEX loss function has been used by various authors; see e.g. Muslehv and Helu [16], and Usta and Akdede [26].

Another useful asymmetric loss function is the general entropy loss function (GELF) suggested by Calabria and Pulcini [27]:

$$L\left(\theta,\hat{\theta}\right) = \left(\frac{\hat{\theta}}{\theta}\right)^c - c\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \ c \neq 0 \quad (10)$$

The loss parameter c given in Equation 10 reflects the departure from symmetry. The Bayes estimator of q under GELF is given by

$$\hat{\theta}_{BG} = \left[E_\pi\left(\theta^{-c}\right)\right]^{-1/c} \tag{11}$$

provided that $E_\pi(\theta^{-c})$ exists and is finite. Note that if $c = -1$ in Equation (11), the Bayes estimator $\hat{\theta}_{BG}$ coincides with the Bayes estimator $\hat{\theta}_{BS}$. Many authors have used this loss function in different estimation problems, see e.g. Usta and Akdede [26] and Pandey and Rao [28].

Norstorm [29] introduced the precautionary loss function (PLF) as an alternative asymmetric loss function as follows:

$$L\left(\theta,\hat{\theta}\right) = \frac{\left(\theta-\hat{\theta}\right)^2}{\hat{\theta}} \tag{12}$$

The PLF approaches infinitely near the origin to prevent underestimation, hence it gives conservative estimators. The Bayes estimator of θ under PLF is given as

$$\hat{\theta}_{BP} = \left[E_\pi\left(\theta^2\right)\right]^{1/2} \tag{13}$$

provided that $E_\pi(\theta^2)$ exists and is finite. Note that if $c = -2$ in Equation 11, the Bayes estimator $\hat{\theta}_{BG}$ is the same with the Bayes estimator $\hat{\theta}_{BP}$. This loss function has been recently applied by several authors, including, Sindhu et al. [9], Pandey and Rao [28].

It is well known that prior distribution of the unknown parameters need to be assumed for the Bayesian inference. In this article, the Bayes estimators for the shape parameter are derived under the assumption of the non-informative Jeffrey's prior and the informative gamma prior, separately. It is assumed that a follows

a) the non-informative Jeffrey's prior: $v_1(\alpha) \propto \frac{1}{\alpha},\ \alpha > 0$,

b) the informative gamma prior: $v_2(\alpha) \propto \alpha^{a-1}e^{-b\alpha},\ \alpha>0,\ a>0, b>0.$

If $\boldsymbol{X} = (X_1, X_2,..., X_m)$ is a progressively type-II censored sample from TL distribution, then, we can derive the posterior pdfs of α under the prior pdfs $v_j(\alpha)$, $j = 1,2$, as follows:

$$\pi_j(\alpha|x) = \frac{v_j(\alpha) l(\alpha|\underset{\sim}{x}) v_j(\alpha)}{\int_0^\infty v_j(\alpha) l(\alpha|\underset{\sim}{x}) d\alpha} = \frac{v_j(\alpha)\prod_{i=1}^m \alpha\left(2x_i - x_i^2\right)^{\alpha-1}\left(1-\left(2x_i-x_i^2\right)^{\alpha}\right)^{R_i}}{\int_0^\infty v_j(\alpha)\prod_{i=1}^m \alpha\left(2x_i - x_i^2\right)^{\alpha-1}\left(1-\left(2x_i-x_i^2\right)^{\alpha}\right)^{R_i} d\alpha},\ j=1,2 \tag{14}$$

On the other hand, as mentioned before, the Bayes estimator of any function of α, say $u(\alpha)$, is the expected value of $u(\alpha)$ under the posterior pdf. The expected value of $u(\alpha)$ under the posterior pdfs given in Equation 14 is obtained by

$$E_{\pi_j}\left(u(\alpha)|x\right) = \frac{\int_0^\infty u(\alpha) v_j(\alpha)\prod_{i=1}^m \alpha\left(2x_i - x_i^2\right)^{\alpha-1}\left(1-\left(2x_i-x_i^2\right)^{\alpha}\right)^{R_i} d\alpha}{\int_0^\infty v_j(\alpha)\prod_{i=1}^m \alpha\left(2x_i - x_i^2\right)^{\alpha-1}\left(1-\left(2x_i-x_i^2\right)^{\alpha}\right)^{R_i} d\alpha},\ j=1,2 \tag{15}$$

It can be observed from Equation 15 that the expected value of $u(\alpha)$ is expressed as a ratio of two integrals which cannot be solved analytically. Hence, Lindley's approximation method is employed to obtain the Bayes estimates of α.

Lindley's Approximation Method

Lindley [30] proposed an approximation method for evaluating the posterior expectation of an arbitrary function $u(\boldsymbol{\theta})$ as

$$E\left(u(\theta)|\boldsymbol{x}\right) = \frac{\int u(\theta)v(\theta)e^{\ln l(\theta)}d\theta}{\int v(\theta)e^{\ln l(\theta)}d\theta} \tag{16}$$

which can be asymptotically approximated by

$$E\left(u(\theta)|\boldsymbol{x}\right) \approx u + \frac{1}{2}\sum_i\sum_j\left(u_{ij}+2u_i\rho_j\right)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_s L_{ijk}\sigma_{ij}\sigma_{ks}u_s \tag{17}$$

where $i, j, k, s = 1,2...,r$, $\boldsymbol{\theta} = (\theta_1, \theta_2,..., \theta_r)$.

For a single parameter, $\theta = (\alpha)$, Equation 17 reduces to the following expression:

$$E\left(u(\alpha)|\boldsymbol{x}\right) \approx u\left(\hat{\alpha}\right) + 0.5\left(\hat{u}_{11}+2\hat{u}_1\hat{\rho}_1\right)\hat{\sigma}_{11} + 0.5\left(\hat{L}_{111}\hat{u}_1\hat{\sigma}_{11}^2\right) \tag{18}$$

where $\hat{\alpha}$ is the MLE of α, $\hat{u}_1 = \partial u/\partial\alpha\big|_{\alpha=\hat{\alpha}}$, $\hat{u}_{11} = \partial^2 u/\partial\alpha^2\big|_{\alpha=\hat{\alpha}}$, $\hat{\rho}_1 = \partial\ln(v)/\partial\alpha\big|_{\alpha=\hat{\alpha}}$, $\hat{L}_{11} = \partial^2\ln(l)/\partial\alpha^2\big|_{\alpha=\hat{\alpha}}$, $\hat{L}_{111} = \partial^3\ln(l)/\partial\alpha^3\big|_{\alpha=\hat{\alpha}}$, $\hat{\sigma}_{11} = -1/\hat{L}_{11}$.

Lindley's method has been applied by many authors to obtain the approximate Bayes estimates, see for example, Muslehv and Helu [16], and Usta and Akdede [26].

Bayes estimates using non-informative Jeffrey's prior distribution

In this subsection, the approximate Bayes estimates of the shape parameter a using the Jeffrey's prior under different loss functions are obtained by applying Lindley's method.

From Equation 7, it is known that the Bayes estimator of a based on the SELF using the Jeffrey's prior is

$$\hat{\alpha}_{BS} = E_{\pi_1}(\alpha) = \int_0^\infty \alpha \pi_1(\alpha|\underset{\sim}{x})d\alpha \tag{19}$$

After substituting the value of $\pi_1(\alpha|\underset{\sim}{x})$ from Equation 14, it can be written as

$$\hat{\alpha}_{BS} = E_{\pi_1}(u(\alpha)) = \frac{\int_0^\infty u(\alpha)e^{lnl(\alpha|\underset{\sim}{x})+\rho(\alpha)}d\alpha}{\int_0^\infty e^{lnl(\alpha|\underset{\sim}{x})+\rho(\alpha)}d\alpha} \tag{20}$$

where $u(\alpha) = \alpha$, $\rho = \ln(v_1) = -\ln(\alpha)$ and $\ln l(\alpha|\boldsymbol{x}) \propto m\ln(\alpha) + \alpha\sum_{i=1}^m \ln(2x_i - x_i^2) + \sum_{i=1}^m R_i \ln\left(1-(2x_i - x_i^2)^\alpha\right)$.

It can easily be verified that $u = 1$, $u_{11} = 0$, $\rho_1 = -\frac{1}{\alpha}$. Thus, using Lindley's approach given in Equation 18, the approximate Bayes estimate of α under the SELF and the Jeffrey's prior, denoted $\hat{\alpha}_{BS}^J$, , can be derived as follows

$$\hat{\alpha}_{BS}^J = E_{\pi_1}(\alpha|\boldsymbol{x}) \approx \hat{\alpha}_{MLE} + \frac{1}{\hat{\alpha}_{MLE}}\frac{1}{\hat{L}_{11}} + \frac{1}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2} \tag{21}$$

where $\hat{\alpha}_{MLE}$ is MLE of α, $\hat{L}_{11} = \left.\frac{\partial^2 \ln l(\alpha|\boldsymbol{x})}{\partial\alpha^2}\right|_{\hat{\alpha}_{MLE}}$

$$= -\frac{m}{\hat{\alpha}_{MLE}^2} - \sum_{i=1}^m R_i \left(\frac{(2x_i - x_i^2)^{\hat{\alpha}_{MLE}}}{\left[1-(2x_i - x_i^2)^{\hat{\alpha}_{MLE}}\right]^2}\right)\left[\ln(2x_i - x_i^2)\right]^2,$$

$$\hat{L}_{111} = \left.\frac{\partial^3 \ln l(\alpha|\boldsymbol{x})}{\partial\alpha^3}\right|_{\hat{\alpha}_{MLE}} = \frac{2m}{\alpha_{MLE}^3} - \sum_{i=1}^m R_i \left(\frac{(2x_i - x_i^2)^{\hat{\alpha}_{MLE}}\left[1+(2x_i - x_i^2)^{\hat{\alpha}_{MLE}}\right]}{\left[1-(2x_i - x_i^2)^{\hat{\alpha}_{MLE}}\right]^3}\right)\left[\ln(2x_i - x_i^2)\right]^3$$

If $u(\alpha)$ is taken as $e^{-c\alpha}$ and similar process in the SELF is followed, then $u_1 = -ce^{-c\alpha}$, $u_{11} = c^2e^{-c\alpha}$ and the approximate Bayes estimate for α under the LINEX loss using the Jeffrey's prior, denoted $\hat{\alpha}_{BL}^J$, can be given by

$$\hat{\alpha}_{BL}^J = -\frac{1}{c}\ln\left(E_{\pi_1}\left(e^{-c\alpha}|\boldsymbol{x}\right)\right) \tag{22}$$

where $E_{\pi_1}\left(e^{-c\alpha}|\boldsymbol{x}\right) \approx \hat{\alpha}_{MLE} - \frac{ce^{-c\hat{\alpha}_{MLE}}}{2}\frac{1}{\hat{L}_{11}}\left(c + 2\hat{\alpha}_{MLE}^{-1}\right) - \frac{ce^{-c\hat{\alpha}_{MLE}}}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2}$.

If $u(\alpha) = \alpha^{-c}$, then $u_1 = -c\alpha^{-c-1}$, $u_{11} = c(c+1)\alpha^{-c-1}$ and approximate Bayes estimate of α under GELF, denoted as $\hat{\alpha}_{BG}^J$, can be written as follows

$$\hat{\alpha}_{BG}^J = \left[E_{\pi_1}\left(\alpha^{-c}\right)\right]^{-1/c} \approx \left[\hat{\alpha}_{MLE} - \frac{c(c+3)\hat{\alpha}_{MLE}^{-c-2}}{2}\frac{1}{\hat{L}_{11}} - \frac{c\hat{\alpha}_{MLE}^{-c-1}}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2}\right]^{-\frac{1}{c}} \tag{23}$$

If $u(\alpha) = \alpha^2$, then $u_1 = 2\alpha$, $u_{11} = 2$ and the approximate Bayes estimate of a under PLF, say $\hat{\alpha}_{BP}^J$, can be obtained by

$$\hat{\alpha}_{BP}^J = \left[E_{\pi_1}\left(\alpha^2|\boldsymbol{x}\right)\right]^{1/2} \approx \left[\hat{\alpha}_{MLE} + \frac{1}{\hat{L}_{11}} + \frac{\hat{\alpha}_{MLE}\hat{L}_{111}}{\hat{L}_{11}^2}\right]^{1/2} \tag{24}$$

Bayes estimates using informative gamma prior distribution

In this subsection, the approximate Bayes estimates of a *under different loss functions using* the gamma prior are also obtained by applying Lindley's approximation into our case where we get the first derivative of $\rho = (v_2) = (a-1)\ln(\alpha) - b\alpha$ according to α as $\rho_1 = \frac{a-1}{\alpha} - b$.

Hence, the approximate Bayes estimates of α under the gamma prior, denoted by $\hat{\alpha}_{BS}^J$ for the SELF, $\hat{\alpha}_{BL}^J$ for the LINEX loss, $\hat{\alpha}_{BG}^J$ for the GELF, $\hat{\alpha}_{BP}^J$ for the PLF, respectively, can be obtain as

$$\hat{\alpha}_{BS}^G = E_{\pi_2}(\alpha|\boldsymbol{x}) \approx \hat{\alpha}_{MLE} - \frac{1}{\hat{L}_{11}}\left(\frac{(a-1)}{\hat{\alpha}_{MLE}} - b\right) + \frac{1}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2} \tag{25}$$

$$\hat{\alpha}_{BL}^G = -\frac{1}{c}\ln\left(E_{\pi_2}\left(e^{-c\alpha}|\boldsymbol{x}\right)\right) \approx -\frac{1}{c}\ln\left(\hat{\alpha}_{MLE} - \frac{ce^{-c\hat{\alpha}_{MLE}}}{2}\frac{1}{\hat{L}_{11}}\left[c - 2\left(\frac{(a-1)}{\hat{\alpha}_{MLE}} - b\right)\right] - \frac{ce^{-c\hat{\alpha}_{MLE}}}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2}\right) \tag{26}$$

$$\hat{\alpha}_{BG}^G = \left[E_{\pi_2}\left(\alpha^{-c}|\boldsymbol{x}\right)\right]^{-1/c} \approx \left[\hat{\alpha}_{MLE} - \frac{c\hat{\alpha}_{MLE}^{-c-2}}{2}\frac{1}{\hat{L}_{11}}\left(c + 3 - 2a + 2b\hat{\alpha}_{MLE}\right) - \frac{c\hat{\alpha}_{MLE}^{-c-1}}{2}\frac{\hat{L}_{111}}{\hat{L}_{11}^2}\right]^{-1/c} \tag{27}$$

$$\hat{\alpha}_{BP}^{G} = \left[E_{\pi_2}\left(\alpha^2 | \boldsymbol{x}\right)\right]^{1/2}$$
$$\approx \left[\hat{\alpha}_{MLE} + \frac{1}{\hat{L}_{11}}\left(1-2a+2b\hat{\alpha}_{MLE}\right) + \frac{\hat{\alpha}_{MLE}\hat{L}_{111}}{\hat{L}_{11}^2}\right]^{1/2} \quad (28)$$

INTERVAL ESTIMATIONS

In this section, the construction of asymptotic confidence and the highest posterior density (HPD) intervals is considered for the shape parameter.

Asymptotic Interval Estimation

In this subsection, the asymptotic confidence interval for the parameter α is constructed by using the asymptotic distribution of MLE of the parameter. It is well known that the MLE of the parameter α ($\hat{\alpha}_{MLE}$) is asymptotically normally distributed with the mean α and the variance $\hat{\sigma}_{11} = -1/\hat{L}_{11}$, which is given in Equation (18). For further details see Ferguson [31].

Thereby, the (1−γ) 100% asymptotic confidence interval for the parameter α is obtained as follows

$$\hat{\alpha}_{MLE} \pm z_{\gamma/2}\sqrt{\hat{\sigma}_{11}} \quad (29)$$

Where $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail $\gamma/2$.

Highest posterior density credible interval

In this subsection, the construction of the highest posterior density (HPD) credible interval is considered for the parameter α. Due to the flexibility and simplicity of the Bayesian approach, in recent years, it has been more preferable to the frequentist approach in interval estimation. The HPD credible interval is more desirable among Bayesian credible intervals because of its two main properties: first, the density for every point inside the credible interval is greater than that for every point outside of it; and second, the credible interval has the shortest length for a given probability content (1−γ). The (1−γ) 100% HPD credible interval [θ_U, θ_L] for a random quantity θ is defined as

$$P\left(\theta_U \le \theta \le \theta_L\right) = \int_{\theta_L}^{\theta_U} \pi\left(\theta | \boldsymbol{x}\right) d\theta = 1-\gamma \quad (30)$$

where $\pi(\theta|\boldsymbol{x})$ is the posterior pdf of θ. For more details about credible intervals, the readers may refer [32].

In order to obtain the (1−γ) 100% HPD interval for the parameter α, we use the posterior samples by following the method described in detail by Bayoud [21], Chen and Shao [33].

SIMULATION STUDY

In this section, a Monte Carlo simulation study is conducted to compare the performance of the MLE and the Bayesian estimators based on the symmetric and asymmetric loss functions using the non-informative and informative priors for the shape α parameter. We first use the following algorithm given by Balakrishnan and Sandhu [34] to generate a progressively type-II censored sample from the TL distribution for given values of n, m and $\boldsymbol{R} = (R_1, R_2,..., R_m)$:

1. Generate m independent uniform $U(0,1)$, random variables $W_1, W_2,..., W_m$.
2. For given values of the progressive scheme $\boldsymbol{R}$, set $V_i = W_i^{\left(i+\sum_{j=m-i+1}^{m} R_j\right)^{-1}}$ for $i = 1,2,..., m$.
3. Set $U_i = 1 - V_m \times V_{m-1} \times ... \times V_{m-i+1}$, $i = 1,2,...,m$ then, $U_1, U_2,..., U_m$, is a progressive type-II censored sample of size m from $U(0,1)$.
4. Finally, for given values of parameter α, set $X_i = F^{-1}\left(U_i\right) = 1-\sqrt{1-U_i^{1/\alpha}}$, $i = 1,2,...,m$. Then, $X_1, X_2,..., X_m$ is the progressive type-II censored sample from the TL distribution with the shape parameter α.

Table 1. Censoring scheme

n = 20, m = 5		**n = 40, m = 10**		**n = 80, m = 20**	
C.S 1	$(0^{*4},15)$	**C.S 9**	$(0^{*9},30)$	**C.S 17**	$(0^{*19},60)$
C.S 2	(3^{*5})	**C.S 10**	(3^{*10})	**C.S 18**	(3^{*20})
C.S 3	$(0^{*2},15, 0^{*2})$	**C.S 11**	$(0^{*4},30, 0^{*5})$	**C.S 19**	$(0^{*9},60, 0^{*10})$
C.S 4	$(15, 0^{*4})$	**C.S 12**	$(30, 0^{*5})$	**C.S 20**	$(60, 0^{*1})$
n = 20, m = 10		**n = 40, m = 20**		**n = 80, m = 40**	
C.S 5	$(0^{*9},10)$	**C.S 13**	$(0^{*19},20)$	**C.S 21**	$(0^{*39},40)$
C.S 6	(1^{*10})	**C.S 14**	(1^{*20})	**C.S 22**	(1^{*40})
C.S 7	$(0^{*4},10, 0^{*5})$	**C.S 15**	$(0^{*9},20, 0^{*10})$	**C.S 23**	$(0^{*19},40, 0^{*20})$
C.S 8	$(10, 0^{*9})$	**C.S 16**	$(20, 0^{*19})$	**C.S 24**	$(40, 0^{*39})$

The simulation study is carried out for different sample sizes; $n = 20, 40, 80$, and for different choices of the effective sample size m in each case; m such that $(m/n) = 0.25, 0.5$. Different choices of the progressive type-II censoring schemes, presented in Table 1, are considered as: (i) items are removed at the end of the experiment (usual type-II censoring), (ii) units are uniformly removed, (iii) units are removed at the middle of the experiment, (iv) units are removed at the beginning of the experiment. In Table 1, for instance, the censoring scheme (C.S) with $(0^{*2}, 15, 0^{*2})$ represents a censoring scheme with specification $\boldsymbol{R} = (R_1, R_2, \ldots, R_m) = (0,0,15,0,0)$.

In each case, the progressively censored samples with the shape parameters $\alpha = 0.5, 1, 2, 5$ are generated and the MLEs and the approximate Bayes estimates, using Lindley's approximation method, are obtained as described in Sections 2-3. For computing approximate Bayes estimates, the informative gamma prior with $a = 4$, $b = 4/\alpha$ is selected to provide $E(\alpha) = a/b$ is chosen. The values for the loss parameter are taken to be $c = \pm 0.75$.

For 5000 repetitions, the performance of all estimators is measured with different criteria such as average estimates (AE) and mean square error (MSE) which are given as follows:

$$AE = \frac{1}{5000}\sum_{i=1}^{5000}\hat{\alpha}_i \qquad (31)$$

$$MSE = \frac{1}{5000}\sum_{i=1}^{5000}\left(\hat{\alpha}_i - \alpha\right)^2 \qquad (32)$$

where a_1 is the estimate of a for the ith simulated sample [35].

The simulation results are reported in Tables 2–10. Tables 2–10 show the AE and the MSE values for the MLE and the proposed Bayes estimates of α.

Table 2. AE and MSE of the estimators of when $\alpha = 0.5$ for $n = 20$, $m = 5,10$

	$n = 20, m = 5$				$n = 20, m = 10$			
	C. S 1	**C. S 2**	**C. S 3**	**C. S 4**	**C. S 5**	**C. S 6**	**C. S 7**	**C. S 8**
$\hat{\alpha}_{MLE}$	0.5221	0.5267	0.5256	0.5336	0.5229	0.5301	0.5323	0.5219
	0.0170	0.0202	0.0212	0.0279	0.0163	0.0178	0.0183	0.0201
$\hat{\alpha}^J_{BS}$	0.5261	0.5333	0.5334	0.5560	0.5236	0.5323	0.5347	0.5305
	0.0175	0.0211	0.0223	0.0323	0.0164	0.0181	0.0186	0.0212
$\hat{\alpha}^J_{BP}$	0.5407	0.5496	0.5497	0.5785	0.5368	0.5467	0.549	0.5487
	0.0194	0.0237	0.0250	0.0378	0.018	0.0202	0.0208	0.0241
$\hat{\alpha}^J_{BG}$ (−0.75)	0.5223	0.5290	0.5291	0.5499	0.5202	0.5287	0.5311	0.5258
	0.0170	0.0206	0.0217	0.0310	0.016	0.0177	0.0181	0.0206
$\hat{\alpha}^J_{BG}$ (0.75)	0.5006	0.5045	0.5046	0.5138	0.5009	0.5076	0.5101	0.4985
	0.0152	0.0179	0.0190	0.0251	0.0145	0.0156	0.0159	0.0179
$\hat{\alpha}^J_{BL}$ (−0.75)	0.5323	0.5403	0.5404	0.5662	0.5291	0.5385	0.5409	0.5383
	0.0187	0.0228	0.0241	0.0358	0.0174	0.0194	0.0199	0.023
$\hat{\alpha}^J_{BL}$ (0.75)	0.5199	0.5261	0.5262	0.5453	0.518	0.5262	0.5286	0.5226
	0.0163	0.0196	0.0207	0.0289	0.0154	0.0169	0.0174	0.0195
$\hat{\alpha}^G_{BS}$	0.5132	0.5160	0.5158	0.5231	0.5122	0.5178	0.5197	0.5124
	0.0084	0.0089	0.0094	0.0089	0.0086	0.0087	0.0089	0.008
$\hat{\alpha}^G_{BP}$	0.5270	0.5311	0.5307	0.5417	0.5247	0.5313	0.5331	0.5287
	0.0095	0.0101	0.0107	0.0105	0.0095	0.0098	0.0101	0.0094
$\hat{\alpha}^G_{BG}$ (−0.75)	0.5098	0.5123	0.5121	0.5183	0.5091	0.5145	0.5164	0.5083
	0.0083	0.0087	0.0092	0.0084	0.0084	0.0086	0.0088	0.0079
$\hat{\alpha}^G_{BG}$ (0.75)	0.4904	0.4913	0.4912	0.4908	0.4918	0.4959	0.4979	0.4852
	0.0082	0.0087	0.0092	0.0082	0.0084	0.0084	0.0085	0.0081
$\hat{\alpha}^G_{BL}$ (−0.75)	0.5189	0.5223	0.5220	0.5310	0.5173	0.5234	0.5252	0.5191
	0.0090	0.0095	0.0101	0.0094	0.0091	0.0093	0.0095	0.0087
$\hat{\alpha}^G_{BL}$ (0.75)	0.5077	0.5100	0.5099	0.5155	0.5072	0.5124	0.5143	0.5059
	0.0080	0.0084	0.0089	0.0082	0.0082	0.0083	0.0085	0.0076

Table 3. AE and MSE of the estimators of α when $\alpha = 0.5$ for $n = 40$, $m = 10,20$

	$n = 40, m = 10$				$n = 40, m = 20$			
	C. S 9	**C. S 10**	**C. S 11**	**C. S 12**	**C. S 13**	**C. S 14**	**C. S 15**	**C. S 16**
$\hat{\alpha}_{MLE}$	0.5096	0.5172	0.5166	0.5103	0.5097	0.5145	0.5081	0.5169
	0.0076	0.0094	0.0102	0.0146	0.0075	0.0079	0.0067	0.0111
$\hat{\alpha}_{BS}^{J}$	0.5114	0.5204	0.5208	0.524	0.51	0.5156	0.5091	0.5215
	0.0077	0.0097	0.0105	0.0159	0.0075	0.0079	0.0068	0.0114
$\hat{\alpha}_{BP}^{J}$	0.5185	0.5286	0.5291	0.5372	0.5165	0.5226	0.516	0.5314
	0.0081	0.0104	0.0113	0.0176	0.0079	0.0084	0.0071	0.0124
$\hat{\alpha}_{BG}^{J}$ (−0.75)	0.5096	0.5183	0.5187	0.5206	0.5084	0.5138	0.5074	0.519
	0.0076	0.0095	0.0104	0.0156	0.0074	0.0078	0.0067	0.0112
$\hat{\alpha}_{BG}^{J}$ (0.75)	0.4989	0.5061	0.5061	0.4999	0.4988	0.5033	0.4973	0.5042
	0.0072	0.0088	0.0096	0.0139	0.0071	0.0073	0.0064	0.0103
$\hat{\alpha}_{BL}^{J}$ (−0.75)	0.5142	0.5237	0.5242	0.5295	0.5126	0.5184	0.5118	0.5256
	0.008	0.01	0.011	0.0169	0.0077	0.0082	0.007	0.012
$\hat{\alpha}_{BL}^{J}$ (0.75)	0.5085	0.5171	0.5174	0.5184	0.5075	0.5127	0.5064	0.5175
	0.0075	0.0093	0.0101	0.015	0.0073	0.0077	0.0066	0.0109
$\hat{\alpha}_{BS}^{G}$	0.5086	0.5158	0.5159	0.5151	0.5075	0.5122	0.5068	0.5154
	0.0059	0.0069	0.0074	0.0087	0.0058	0.006	0.0052	0.0074
$\hat{\alpha}_{BP}^{G}$	0.5156	0.5239	0.5242	0.5279	0.5139	0.5192	0.5136	0.5252
	0.0062	0.0075	0.0081	0.01	0.0061	0.0064	0.0055	0.0081
$\hat{\alpha}_{BG}^{G}$ (−0.75)	0.5068	0.5137	0.5139	0.5118	0.5059	0.5105	0.5051	0.513
	0.0058	0.0068	0.0073	0.0083	0.0058	0.0059	0.0051	0.0073
$\hat{\alpha}_{BG}^{G}$ (0.75)	0.4964	0.5019	0.5018	0.4927	0.4966	0.5003	0.4952	0.4989
	0.0056	0.0064	0.0069	0.008	0.0056	0.0056	0.005	0.0069
$\hat{\alpha}_{BL}^{G}$ (−0.75)	0.5114	0.519	0.5193	0.5204	0.51	0.515	0.5095	0.5194
	0.0061	0.0072	0.0078	0.0093	0.006	0.0062	0.0053	0.0078
$\hat{\alpha}_{BL}^{G}$ (0.75)	0.5058	0.5125	0.5126	0.5099	0.505	0.5095	0.5042	0.5116
	0.0057	0.0067	0.0072	0.0078	0.0057	0.0058	0.005	0.0071

Table 4. AE and MSE of the estimators of α when $\alpha = 0.5$ for $n = 80$, $m = 20,40$

	$n = 80, m = 20$				**$n = 80, m = 40$**			
	***C. S* 17**	***C. S* 18**	***C. S* 19**	***C. S* 20**	***C. S* 21**	***C. S* 22**	***C. S* 23**	***C. S* 24**
$\hat{\alpha}_{MLE}$	0.5048	0.5049	0.5051	0.5083	0.5066	0.5063	0.5072	0.5105
	0.0037	0.0043	0.0042	0.0083	0.0035	0.0033	0.0035	0.0061
$\hat{\alpha}^J_{BS}$	0.5057	0.5064	0.507	0.5163	0.5068	0.5068	0.5077	0.5127
	0.0037	0.0044	0.0043	0.0088	0.0036	0.0033	0.0036	0.0062
$\hat{\alpha}^J_{BP}$	0.5092	0.5104	0.511	0.5241	0.51	0.5103	0.5111	0.518
	0.0038	0.0045	0.0044	0.0094	0.0037	0.0034	0.0037	0.0065
$\hat{\alpha}^J_{BG}$ (−0.75)	0.5048	0.5054	0.506	0.5142	0.506	0.506	0.5068	0.5114
	0.0037	0.0043	0.0043	0.0087	0.0035	0.0033	0.0035	0.0062
$\hat{\alpha}^J_{BG}$ (0.75)	0.4995	0.4994	0.4999	0.5021	0.5012	0.5008	0.5017	0.5035
	0.0036	0.0042	0.0041	0.0081	0.0034	0.0032	0.0034	0.0059
$\hat{\alpha}^J_{BL}$ (−0.75)	0.507	0.508	0.5086	0.5194	0.508	0.5082	0.509	0.5148
	0.0038	0.0044	0.0044	0.0091	0.0036	0.0033	0.0036	0.0064
$\hat{\alpha}^J_{BL}$ (0.75)	0.5043	0.5048	0.5054	0.5131	0.5055	0.5055	0.5063	0.5106
	0.0037	0.0043	0.0042	0.0085	0.0035	0.0032	0.0035	0.0061
	0.0094	0.008	0.0056	0.0085	0.0052	0.0039	0.004	0.0061
$\hat{\alpha}^G_{BS}$	0.505	0.5055	0.5061	0.513	0.5061	0.5061	0.5069	0.5108
	0.0033	0.0038	0.0037	0.0064	0.0032	0.0029	0.0032	0.0051
$\hat{\alpha}^G_{BP}$	0.5085	0.5096	0.5102	0.5209	0.5093	0.5096	0.5103	0.5161
	0.0034	0.0039	0.0038	0.007	0.0033	0.003	0.0033	0.0054
$\hat{\alpha}^G_{BG}$ (−0.75)	0.5041	0.5045	0.5051	0.5111	0.5053	0.5052	0.506	0.5095
	0.0032	0.0037	0.0037	0.0063	0.0031	0.0029	0.0031	0.0051
$\hat{\alpha}^G_{BG}$ (0.75)	0.4988	0.4985	0.4991	0.4992	0.5005	0.5001	0.501	0.5017
	0.0032	0.0036	0.0036	0.006	0.0031	0.0028	0.003	0.0048
$\hat{\alpha}^G_{BL}$ (−0.75)	0.5063	0.5071	0.5077	0.5162	0.5073	0.5075	0.5082	0.5129
	0.0033	0.0038	0.0038	0.0067	0.0032	0.003	0.0032	0.0053
$\hat{\alpha}^G_{BL}$ (0.75)	0.5036	0.504	0.5046	0.5099	0.5048	0.5048	0.5056	0.5088
	0.0032	0.0037	0.0036	0.0062	0.0031	0.0029	0.0031	0.005

Table 5. AE and MSE of the estimators of when $\alpha = 1$ for $n = 20$, $m = 5{,}10$

	n = 20, m = 5				**n = 20, m = 4**			
	C. S 1	**C. S 2**	**C. S 3**	**C. S 4**	**C. S 5**	**C. S 6**	**C. S 7**	**C. S 8**
$\hat{\alpha}_{MLE}$	1.0523	1.0611	1.0535	1.0451	1.0487	1.0593	1.061	1.0621
	0.0699	0.0855	0.0781	0.0983	0.0692	0.0656	0.0715	0.086
$\hat{\alpha}^{J}_{BS}$	1.0601	1.0743	1.069	1.0886	1.0501	1.0638	1.0659	1.0796
	0.0718	0.0893	0.0822	0.1122	0.0695	0.0667	0.0727	0.0912
$\hat{\alpha}^{J}_{BP}$	1.0896	1.1073	1.1017	1.1326	1.0766	1.0926	1.0944	1.1166
	0.0801	0.1006	0.0926	0.1306	0.0763	0.0747	0.081	0.1045
$\hat{\alpha}^{J}_{BG}$ (−0.75)	1.0526	1.0658	1.0606	1.0767	1.0434	1.0565	1.0586	1.07
	0.07	0.0868	0.0799	0.108	0.068	0.065	0.0709	0.0883
$\hat{\alpha}^{J}_{BG}$ (0.75)	1.009	1.0165	1.0114	1.0062	1.0047	1.0143	1.0168	1.0144
	0.0619	0.0753	0.0694	0.0893	0.0614	0.0572	0.0625	0.075
$\hat{\alpha}^{J}_{BL}$ (−0.75)	1.0849	1.1026	1.0967	1.1262	1.0723	1.088	1.09	1.1113
	0.0821	0.1036	0.0952	0.1351	0.0784	0.0763	0.083	0.1077
$\hat{\alpha}^{J}_{BL}$ (0.75)	1.035	1.0454	1.0406	1.0479	1.0279	1.0394	1.0416	1.0469
	0.063	0.0768	0.0708	0.0912	0.0619	0.0585	0.0638	0.0769
$\hat{\alpha}^{G}_{BS}$	1.0321	1.0368	1.0352	1.0366	1.0259	1.0362	1.037	1.0364
	0.0351	0.0377	0.0356	0.033	0.0351	0.034	0.0357	0.035
$\hat{\alpha}^{G}_{BP}$	1.06	1.0671	1.0657	1.0753	1.0509	1.0636	1.0638	1.0703
	0.0398	0.0431	0.0411	0.0416	0.0386	0.0388	0.0402	0.0415
$\hat{\alpha}^{G}_{BG}$ (−0.75)	1.0252	1.0292	1.0277	1.0268	1.0198	1.0294	1.0303	1.0281
	0.0343	0.0369	0.0347	0.0318	0.0346	0.0332	0.0349	0.0341
$\hat{\alpha}^{G}_{BG}$ (0.75)	0.9866	0.9873	0.9851	0.9704	0.9853	0.9918	0.9934	0.9817
	0.0334	0.0364	0.0341	0.0323	0.0345	0.0319	0.0339	0.0338
$\hat{\alpha}^{G}_{BL}$ (−0.75)	1.0555	1.0625	1.0609	1.07	1.0466	1.0592	1.0596	1.0655
	0.0408	0.0444	0.0422	0.0428	0.0397	0.0397	0.0413	0.0429
$\hat{\alpha}^{G}_{BL}$ (0.75)	1.0104	1.0132	1.0115	1.006	1.0068	1.0148	1.0161	1.0101
	0.0319	0.0343	0.0321	0.0296	0.0328	0.0307	0.0325	0.0312

Table 6. AE and MSE of the estimators of α when $\alpha = 1$ for $n = 40$, $m = 10,20$

	$n = 40, m = 10$				$n = 40, m = 10$			
	***C. S* 9**	***C. S* 10**	***C. S* 11**	***C. S* 12**	***C. S* 13**	***C. S* 14**	***C. S* 15**	***C. S* 16**
$\hat{\alpha}_{MLE}$	1.0294	1.022	1.0244	1.021	1.0331	1.0244	1.0262	1.0377
	0.0361	0.0335	0.0376	0.0613	0.0309	0.0301	0.0316	0.0444
$\hat{\alpha}^{J}_{BS}$	1.033	1.0282	1.0327	1.0488	1.0337	1.0266	1.0284	1.047
	0.0365	0.0342	0.0388	0.0667	0.031	0.0304	0.0318	0.0459
$\hat{\alpha}^{J}_{BP}$	1.0474	1.0444	1.0493	1.0752	1.0468	1.0406	1.0422	1.0669
	0.0387	0.0365	0.0413	0.0734	0.0328	0.0321	0.0336	0.0499
$\hat{\alpha}^{J}_{BG}$ (−0.75)	1.0293	1.0241	1.0285	1.0418	1.0304	1.023	1.0249	1.0419
	0.0361	0.0337	0.0382	0.0652	0.0306	0.03	0.0314	0.0451
$\hat{\alpha}^{J}_{BG}$ (0.75)	1.0078	0.9999	1.0037	1.0002	1.011	1.0022	1.0044	1.0121
	0.0338	0.0316	0.0356	0.0584	0.0287	0.0283	0.0296	0.041
$\hat{\alpha}^{J}_{BL}$ (−0.75)	1.0445	1.041	1.0459	1.0703	1.0441	1.0377	1.0394	1.0631
	0.0391	0.0368	0.0418	0.075	0.0331	0.0324	0.0339	0.0506
$\hat{\alpha}^{J}_{BL}$ (0.75)	1.0213	1.0153	1.0194	1.0261	1.0232	1.0154	1.0174	1.0306
	0.0343	0.032	0.0361	0.0592	0.0292	0.0286	0.03	0.0419
$\hat{\alpha}^{G}_{BS}$	1.0255	1.0211	1.0245	1.0304	1.0271	1.0206	1.0221	1.034
	0.0272	0.0247	0.0278	0.0328	0.0239	0.0229	0.0242	0.0298
$\hat{\alpha}^{G}_{BP}$	1.0398	1.0371	1.0409	1.0559	1.0401	1.0345	1.0358	1.0536
	0.029	0.0266	0.03	0.0403	0.0254	0.0244	0.0258	0.0329
$\hat{\alpha}^{G}_{BG}$ (−0.75)	1.0219	1.0171	1.0204	1.0239	1.0239	1.0171	1.0187	1.0291
	0.0268	0.0244	0.0274	0.0307	0.0235	0.0226	0.0239	0.0292
$\hat{\alpha}^{G}_{BG}$ (0.75)	1.0011	0.9936	0.9964	0.9857	1.005	0.9968	0.9988	1.0008
	0.0255	0.0233	0.026	0.0318	0.0224	0.0217	0.0229	0.0274
$\hat{\alpha}^{G}_{BL}$ (−0.75)	1.037	1.0338	1.0377	1.0514	1.0375	1.0316	1.033	1.05
	0.0293	0.0269	0.0304	0.0411	0.0256	0.0246	0.026	0.0333
$\hat{\alpha}^{G}_{BL}$ (0.75)	1.0143	1.0085	1.0116	1.0099	1.0169	1.0097	1.0114	1.0185
	0.0256	0.0231	0.026	0.0279	0.0225	0.0216	0.0229	0.0273

Table 7. AE and MSE of the estimators of a when $\alpha = 1$ for $n = 80$, $m = 20,40$

	***n* = 80, *m* = 20**				***n* = 80, *m* = 40**			
	***C. S* 17**	***C. S* 18**	***C. S* 19**	***C. S* 20**	***C. S* 21**	***C. S* 22**	***C. S* 23**	***C. S* 24**
$\hat{\alpha}_{MLE}$	1.0135	1.008	1.0185	1.0188	1.0146	1.016	1.0095	1.0198
	0.0145	0.0172	0.0171	0.0347	0.0131	0.0147	0.0139	0.0235
$\hat{\alpha}^{J}_{BS}$	1.0152	1.011	1.0224	1.0347	1.0149	1.017	1.0105	1.0243
	0.0146	0.0174	0.0174	0.0366	0.0131	0.0147	0.014	0.0239
$\hat{\alpha}^{J}_{BP}$	1.0223	1.0191	1.0306	1.0504	1.0213	1.024	1.0173	1.0349
	0.0151	0.0179	0.0181	0.0391	0.0135	0.0152	0.0143	0.025
$\hat{\alpha}^{J}_{BG}$ (−0.75)	1.0134	1.009	1.0204	1.0306	1.0133	1.0152	1.0088	1.0216
	0.0145	0.0173	0.0173	0.0361	0.013	0.0146	0.0139	0.0237
$\hat{\alpha}^{J}_{BG}$ (0.75)	1.0028	0.997	1.0081	1.0064	1.0037	1.0048	0.9987	1.0057
	0.014	0.0168	0.0165	0.0335	0.0126	0.0141	0.0135	0.0226
$\hat{\alpha}^{J}_{BL}$ (−0.75)	1.0207	1.0172	1.0288	1.0472	1.0199	1.0224	1.0158	1.0326
	0.0151	0.018	0.0182	0.0395	0.0135	0.0153	0.0144	0.0252
$\hat{\alpha}^{J}_{BL}$ (0.75)	1.0097	1.0048	1.016	1.0218	1.0099	1.0116	1.0053	1.0159
	0.0141	0.0168	0.0168	0.0341	0.0127	0.0143	0.0136	0.0228
$\hat{\alpha}^{G}_{BS}$	1.0136	1.0094	1.0201	1.0277	1.0135	1.0153	1.0093	1.0207
	0.0129	0.015	0.0151	0.0265	0.0117	0.013	0.0124	0.0196
$\hat{\alpha}^{G}_{BP}$	1.0207	1.0174	1.0283	1.0433	1.0199	1.0223	1.0161	1.0313
	0.0133	0.0155	0.0157	0.0287	0.0121	0.0135	0.0127	0.0206
$\hat{\alpha}^{G}_{BG}$ (−0.75)	1.0118	1.0074	1.0181	1.0237	1.0119	1.0136	1.0076	1.0181
	0.0128	0.0149	0.0149	0.0261	0.0116	0.0129	0.0123	0.0194
$\hat{\alpha}^{G}_{BG}$ (0.75)	1.0013	0.9955	1.0059	1.0001	1.0023	1.0032	0.9975	1.0024
	0.0124	0.0146	0.0143	0.0245	0.0113	0.0125	0.012	0.0186
$\hat{\alpha}^{G}_{BL}$ (−0.75)	1.0191	1.0156	1.0265	1.0402	1.0184	1.0207	1.0145	1.029
	0.0133	0.0156	0.0158	0.029	0.0121	0.0135	0.0128	0.0207
$\hat{\alpha}^{G}_{BL}$ (0.75)	1.0081	1.0033	1.0138	1.0152	1.0085	1.0099	1.0041	1.0125
	0.0125	0.0146	0.0145	0.0246	0.0113	0.0126	0.0121	0.0187

Table 8. AE and MSE of the estimators of α when $\alpha = 2$ for $n = 20$, $m = 5,10$

	$n = 20, m = 5$				$n = 20, m = 10$			
	***C. S* 1**	***C. S* 2**	***C. S* 3**	***C. S* 4**	***C. S* 5**	***C. S* 6**	***C. S* 7**	***C. S* 8**
$\hat{\alpha}_{MLE}$	2.0939	2.1109	2.0921	2.0917	2.1119	2.0878	2.1072	2.1112
	0.2727	0.3166	0.309	0.3974	0.2671	0.2847	0.2716	0.3361
$\hat{\alpha}^{J}_{BS}$	2.1097	2.1372	2.1228	2.1792	2.1146	2.0966	2.1169	2.1458
	0.2798	0.3308	0.3247	0.4557	0.2684	0.2887	0.2762	0.356
$\hat{\alpha}^{J}_{BP}$	2.1684	2.2029	2.1877	2.2675	2.168	2.1533	2.1735	2.219
	0.3112	0.3727	0.3643	0.531	0.2965	0.3182	0.3069	0.4069
$\hat{\alpha}^{J}_{BG}$ (−0.75)	2.0947	2.1203	2.106	2.1555	2.1011	2.0822	2.1025	2.1268
	0.2729	0.3215	0.316	0.4383	0.2622	0.2823	0.2695	0.3447
$\hat{\alpha}^{J}_{BG}$ (0.75)	2.0078	2.0221	2.0084	2.0139	2.0232	1.9991	2.0193	2.0167
	0.2426	0.2797	0.277	0.3608	0.2342	0.2539	0.2392	0.2947
$\hat{\alpha}^{J}_{BL}$ (−0.75)	2.2039	2.2434	2.2266	2.3175	2.2014	2.1878	2.2085	2.2637
	0.359	0.4343	0.4239	0.6295	0.3395	0.366	0.3528	0.4801
$\hat{\alpha}^{J}_{BL}$ (0.75)	2.013	2.0265	2.0137	2.0191	2.0274	2.0041	2.0238	2.021
	0.2209	0.252	0.2485	0.3101	0.2156	0.2314	0.2193	0.262
$\hat{\alpha}^{G}_{BS}$	2.0569	2.0681	2.0593	2.0719	2.0643	2.0457	2.0637	2.0638
	0.1351	0.1444	0.1387	0.1365	0.1399	0.14	0.1398	0.1373
$\hat{\alpha}^{G}_{BP}$	2.1123	2.1293	2.1195	2.1499	2.1148	2.0986	2.1175	2.1305
	0.1525	0.1665	0.1588	0.1728	0.1556	0.1543	0.158	0.1617
$\hat{\alpha}^{G}_{BG}$ (−0.75)	2.0432	2.0529	2.0443	2.0521	2.0518	2.0326	2.0504	2.0472
	0.1325	0.1411	0.1358	0.1314	0.1374	0.138	0.137	0.1339
$\hat{\alpha}^{G}_{BG}$ (0.75)	1.966	1.9682	1.9596	1.9395	1.9822	1.9589	1.9763	1.9551
	0.1306	0.1387	0.1355	0.1322	0.1338	0.1386	0.133	0.1343
$\hat{\alpha}^{G}_{BL}$ (−0.75)	2.1503	2.1737	2.1615	2.2105	2.1485	2.1339	2.1552	2.1797
	0.1889	0.213	0.2023	0.2465	0.1867	0.1864	0.1948	0.2131
$\hat{\alpha}^{G}_{BL}$ (0.75)	1.9754	1.9784	1.9708	1.9563	1.99	1.9685	1.9847	1.9668
	0.1204	0.1267	0.1238	0.1155	0.1249	0.1291	0.1225	0.1214

Table 9. AE and MSE of the estimators of α when $\alpha = 2$ for $n = 40$, $m = 10,20$

	$n = 40, m = 10$				$n = 40, m = 20$			
	***C. S* 9**	***C. S* 10**	***C. S* 11**	***C. S* 12**	***C. S* 13**	***C. S* 14**	***C. S* 15**	***C. S* 16**
$\hat{\alpha}_{MLE}$	2.0409	2.0565	2.0192	2.0503	2.051	2.0503	2.0581	2.0547
	0.1332	0.1321	0.146	0.2303	0.1164	0.1302	0.1219	0.1653
$\hat{\alpha}_{BS}^{J}$	2.0479	2.0691	2.0357	2.106	2.0522	2.0546	2.0625	2.073
	0.1347	0.1353	0.1494	0.2522	0.1167	0.1312	0.1229	0.1706
$\hat{\alpha}_{BP}^{J}$	2.0766	2.1017	2.0682	2.1589	2.0782	2.0827	2.0902	2.1122
	0.142	0.145	0.1576	0.2793	0.123	0.1386	0.1304	0.1841
$\hat{\alpha}_{BG}^{J}$ (−0.75)	2.0407	2.0609	2.0274	2.0921	2.0456	2.0475	2.0555	2.063
	0.1332	0.1332	0.1477	0.2461	0.1153	0.1296	0.1213	0.1677
$\hat{\alpha}_{BG}^{J}$ (0.75)	1.9981	2.0122	1.9784	2.0088	2.0071	2.0058	2.0144	2.0041
	0.1261	0.1236	0.1403	0.2183	0.1091	0.1222	0.1137	0.1547
$\hat{\alpha}_{BL}^{J}$ (−0.75)	2.0925	2.1199	2.0858	2.1881	2.0929	2.0985	2.1059	2.1342
	0.1529	0.1579	0.1704	0.3157	0.1319	0.1492	0.1405	0.2032
$\hat{\alpha}_{BL}^{J}$ (0.75)	2.0028	2.0173	1.9842	2.0159	2.0114	2.0104	2.0188	2.01
	0.1209	0.1181	0.1337	0.1999	0.1051	0.1173	0.1094	0.1458
$\hat{\alpha}_{BS}^{G}$	2.0358	2.0534	2.0235	2.0687	2.041	2.0418	2.0495	2.0518
	0.1011	0.098	0.1086	0.1346	0.0901	0.0986	0.0934	0.1124
$\hat{\alpha}_{BP}^{G}$	2.0641	2.0857	2.0556	2.1201	2.0668	2.0696	2.077	2.0904
	0.1072	0.1062	0.1156	0.1546	0.0955	0.1048	0.0998	0.1231
$\hat{\alpha}_{BG}^{G}$ (−0.75)	2.0287	2.0453	2.0154	2.0556	2.0346	2.0349	2.0427	2.0421
	0.1	0.0964	0.1073	0.131	0.0891	0.0975	0.0922	0.1104
$\hat{\alpha}_{BG}^{G}$ (0.75)	1.9872	1.9981	1.9678	1.9788	1.997	1.9943	2.0026	1.9858
	0.0962	0.091	0.104	0.1226	0.0854	0.0934	0.0876	0.1051
$\hat{\alpha}_{BL}^{G}$ (−0.75)	2.0805	2.1046	2.0739	2.1527	2.0818	2.0859	2.0931	2.1136
	0.1173	0.1182	0.1275	0.1876	0.1037	0.1147	0.1092	0.1406
$\hat{\alpha}_{BL}^{G}$ (0.75)	1.9924	2.0037	1.9742	1.9886	2.0016	1.9993	2.0074	1.9927
	0.0921	0.0866	0.0986	0.1125	0.0821	0.0895	0.084	0.0987

Table 10. AE and MSE of the estimators of α when $\alpha = 2$ for $n = 80$, $m = 20,40$

	$n = 80, m = 20$				$n = 80, m = 40$			
	C. S 17	**C. S 18**	**C. S 19**	**C. S 20**	**C. S 21**	**C. S 22**	**C. S 23**	**C. S 24**
$\hat{\alpha}_{MLE}$	2.0199	2.0254	2.0399	2.0419	2.0211	2.028	2.0369	2.0254
	0.0596	0.0711	0.0703	0.1382	0.0521	0.0611	0.0569	0.0892
$\hat{\alpha}^{J}_{BS}$	2.0232	2.0316	2.0478	2.0738	2.0216	2.0301	2.039	2.0344
	0.0599	0.0718	0.0715	0.1465	0.0521	0.0614	0.0571	0.0905
$\hat{\alpha}^{J}_{BP}$	2.0374	2.0477	2.0642	2.1056	2.0344	2.044	2.0527	2.0554
	0.0616	0.0743	0.0745	0.1567	0.0535	0.0632	0.0591	0.0942
$\hat{\alpha}^{J}_{BG}$ (−0.75)	2.0196	2.0275	2.0437	2.0656	2.0184	2.0266	2.0355	2.0292
	0.0596	0.0713	0.0709	0.1442	0.0518	0.061	0.0567	0.0897
$\hat{\alpha}^{J}_{BG}$ (0.75)	1.9985	2.0033	2.019	2.0166	1.9992	2.0058	2.0151	1.9978
	0.0579	0.0689	0.0677	0.1334	0.0505	0.059	0.0545	0.0861
$\hat{\alpha}^{J}_{BL}$ (−0.75)	2.0449	2.0563	2.0731	2.123	2.0412	2.0515	2.0601	2.0666
	0.064	0.0775	0.078	0.17	0.0554	0.0657	0.0615	0.0995
$\hat{\alpha}^{J}_{BL}$ (0.75)	2.0015	2.0066	2.0222	2.022	2.0019	2.0086	2.0178	2.0019
	0.0568	0.0675	0.0663	0.1274	0.0496	0.058	0.0536	0.0836
$\hat{\alpha}^{G}_{BS}$	2.0204	2.0277	2.043	2.0592	2.0192	2.0268	2.0355	2.0285
	0.0528	0.0623	0.0617	0.1061	0.0465	0.0541	0.0506	0.0745
$\hat{\alpha}^{G}_{BP}$	2.0345	2.0438	2.0594	2.0908	2.032	2.0408	2.0492	2.0494
	0.0544	0.0646	0.0641	0.1153	0.0478	0.0559	0.0521	0.0779
$\hat{\alpha}^{G}_{BG}$ (−0.75)	2.0169	2.0236	2.0388	2.0512	2.016	2.0234	2.032	2.0233
	0.0525	0.0618	0.0612	0.1043	0.0462	0.0537	0.0502	0.0738
$\hat{\alpha}^{G}_{BG}$ (0.75)	1.9958	1.9996	2.0144	2.0035	1.9969	2.0027	2.0117	1.9923
	0.0512	0.0599	0.0585	0.0976	0.0452	0.0522	0.0484	0.0713
$\hat{\alpha}^{G}_{BL}$ (−0.75)	2.0421	2.0525	2.0684	2.1091	2.0388	2.0483	2.0566	2.0608
	0.0567	0.0677	0.0669	0.128	0.0496	0.0582	0.0538	0.083
$\hat{\alpha}^{G}_{BL}$ (0.75)	1.9988	2.0029	2.0176	2.0094	1.9997	2.0056	2.0145	1.9965
	0.0502	0.0586	0.0573	0.0929	0.0444	0.0512	0.0475	0.0691

Table 11. Progressively type-II censored samples

n	m	Scheme	Censoring Scheme	Censored Data
23	11	1	$(0^{*5},12)$	0.0060 0.0080 0.0110 0.0140 0.0210 0.0290 0.0360 0.0480 0.0590 0.0700 0.0920
23	11	2	$(0^{*8},8,0^{*8})$	0.0060 0.0080 0.0110 0.0140 0.0210 0.0290 0.7170 0.7590 0.8090 0.8530 0.8660
23	11	3	$(1^{*5},2,1^{*5})$	0.0060 0.0110 0.0210 0.0360 0.0590 0.0920 0.2130 0.4030 0.5440 0.7590 0.8530

Table 12. Results for a real data set

Methods	Complete	Scheme 1	Scheme 2	Scheme 3
$\hat{\alpha}_{MLE}$	0.5890	0.5511	0.6724	0.8709
$\hat{\alpha}^{J}_{BS,L}$	0.5890	0.5520	0.6768	0.8748
$\hat{\alpha}^{J}_{BP,L}$	0.6016	0.5643	0.6933	0.8959
$\hat{\alpha}^{J}_{BG,L}$ (-0.75)	0.5858	0.5489	0.6725	0.8694
$\hat{\alpha}^{J}_{BG,L}$ (0.75)	0.5673	0.5309	0.6480	0.8383
$\hat{\alpha}^{J}_{BL,L}$ (-0.75)	0.5946	0.5572	0.6852	0.8887
$\hat{\alpha}^{J}_{BL,L}$ (0.75)	0.5833	0.5469	0.6682	0.8608
%95 Asymptotic Intervals	(0.3514, 0.8372)	(0.3214, 0.7807)	(0.3769, 0.9678)	(0.4920, 1.2498)
%95 Credible Intervals	(0.3768, 0.8550)	(0.2373, 0.6899)	(0.3570, 0.8384)	(0.3708, 0.9903)

Some of the points are quite clear from Tables 2–10. For instance, for fixed effective sample size m as the sample size n increases MSEs of all estimates decrease. Similarly for fixed n as m increases MSEs decrease. Therefore, it can be deduced that the MLE and the proposed Bayes estimators are consistent. Moreover, as expected, the Bayes estimates based on the informative gamma prior perform better than those based on the non-informative Jeffrey's prior in terms of MSE.

On the other hand, in comparison of the MLE and the Bayesian estimators, for fixed sample size n and for any censoring scheme the proposed Bayes estimators obtained using Lindley's approximation of α under the gamma prior, except PLF, exhibit the best performance in terms of AE and MSE. Among the Bayes estimates under the gamma prior, $\hat{\alpha}^{G}_{BL}$ for loss parameter $c = 0.75$ has the smallest MSE values, followed by $\hat{\alpha}^{G}_{BG}$ with $c = 0.75$ for most of the cases considered, or vice versa. On the other hand, in case of the non-informative Jeffrey's prior, the approximate Bayes estimates obtained using Lindley's method of a under LINEX and GELF ($\hat{\alpha}^{J}_{BL}$ and $\hat{\alpha}^{J}_{BG}$) with $c = 0.75$ perform better than the MLE in most cases.

It should also be noted that the proposed Bayes estimators outperform the MLE, especially for small sample sizes. This result will be quite attractive to researchers in the lifetime analysis.

In addition, from Table 2–10, it can be observed that for fixed n and m, the MSEs of the considered estimators for the usual type-II censoring scheme are usually less than that of any other censoring schemes.

APPLICATION TO A REAL DATA SET

In this section, a real data set is investigated to verify the applicability of the proposed estimators. The dataset taken from Mazumdar and Gaver [36] represents the numerical values of the capacity factors for different units within the system. The dataset with 23 observations is as follows: 0.853, 0.759, 0.866, 0.809, 0.717, 0.544, 0.492, 0.403, 0.344, 0.213, 0.116, 0.116, 0.092, 0.070, 0.059, 0.048, 0.036, 0.029, 0.021, 0.014, 0.011, 0.008, and 0.006. This dataset was also studied by Arora et al. [12] under the type- 2 censoring scheme. They confirmed that the dataset follows the TL distribution using the Kolmogorov–Smirnov test.

We consider three different censoring schemes to generate the progressive type-II censored samples from the data, which is give in Table 11.

For the real dataset, the non-informative Jeffrey's prior is used for the shape parameter a. The MLE, the proposed Bayes estimates and the corresponding confidence intervals for a under the complete and progressive type-II censored samples are obtained and summarized in the following table.

It can be observed from Table 12 that all the Bayes estimates of the shape parameter a are nearly the same. Another point worth mentioning is the fact that the estimates for a based on progressively type-II censored sample according to Scheme 1 are close to that of the complete sample. The corresponding confidence intervals for the parameter are satisfactory in all the cases. The HPD credible intervals perform better than asymptotic confidence intervals in terms of interval lengths.

CONCLUSIONS

In this study, we consider the problem of Bayesian estimation for the shape parameter of the Topp-Leone (TL) distribution based on progressively type-II censored data. The Bayes estimators are derived by using a non-informative (Jeffrey's) prior and an informative (gamma) prior under both the symmetric (Squared Error) and asymmetric (LINEX, General Entropy and Precautionary) loss functions. The proposed Bayes estimators cannot be obtained in closed-forms and, for this reason, Lindley's approximation method is used to compute the estimates. Furthermore, the maximum likelihood estimator (MLE) is obtained to estimate the shape parameter of the TL distribution.

The performances of the proposed Bayes estimators are compared with the corresponding MLE through an extensive simulation study. From the results, it is observed that for all of the cases, the Bayes estimates under the informative prior are better than the MLE and the Bayes estimates under non-informative prior in terms of the MSE and AE. In addition, the Bayes estimates under the LINEX or the general entropy loss functions with the loss parameter c = 0.75 have the smallest MSE. Therefore, it can be said that the LINEX and the general entropy loss functions are better than other loss functions. It should also be emphasized that the proposed Bayes estimators outperform the MLE, especially for small sample sizes.

Consequently, the Bayes estimators under LINEX and general entropy loss functions can be suggested for estimating the shape parameter of the Topp-Leone distribution based on progressively type-II censoring sample.

ACKNOWLEDGMENTS

We would like to thank to the Editors and anonymous referees for their constructive comments on an earlier version of this manuscript, which resulted in this improved version.

FUNDING

This study was supported by Eskisehir Technical University Scientific Research Projects Coordination Unit under grant number 20ADP104 for the foundation support required for the realization of this study.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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