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Research Article

A New Nonlinear Hybrid Technique with fixed and adaptive step-size approaches

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ABSTRACT

Linear and nonlinear numerical techniques are the most popular techniques for finding approximate solutions to initial value problems in numerous scientific fields. Due to the substantial importance of ordinary differential equations, an attempt has been made in the present research study to obtain a new nonlinear hybrid technique based upon contra-harmonic and harmonic means having fourth-order accuracy. Theoretical analysis in terms of consistency, stability, asymptotic errors (local and global truncation errors), and convergence has also been carried out. The newly formulated technique is compared with some existing techniques having the same characteristics and observed to be much better because of errors, CPU time, and stability region. The adaptive step-size approach improves the performance of the proposed technique, and strategies to control the errors are developed. Some numerical experiments for scalar and vector initial value problems, including logistic growth, sinusoidal and industrial Robot Arm systems, are presented to show better performance of the proposed technique.

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INTRODUCTION

Initial value problems in ordinary differential equations have always been significant in different scientific subjects like mathematics, physics, biology, fluid mechanics, and other various fields. Initial value problems play an essential

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role in modeling various phenomena in biological science and the physical laws of nature. The theory of ordinary differential equations is being used in physics to study the earth's rotation, studies about stars, electric current, gravitational laws, and electrical circuits. The laws of Newton about masses and forces are also expressed in terms of ordinary differential equations. Other studies include the vibratory motion of springs in mechanical engineering, transmission dynamics of diseases in epidemiology, and many other areas. In biology, study about blood pressure, sugar, medicines, DNA, and RNA molecules is also based upon models under the theory of ordinary differential equations. In computational fluid dynamics, measurement of flow of the different fluids, velocity profile of the fluids, share the stress and pressure distribution in the Poiseuille and Coutllie flow are all shown in terms of ordinary differential equations. In chemistry, synthetic response energy, study about different reactions of metals, and kinetic energy are all based upon ordinary differential equations. In population dynamics (Verhulst-Pearl model), in engineering (beats of a vibrating framework) and a lot more can be found in [1–8] for applications of ordinary differential equations. For applications of partial differential equations, one can consult the recently published works in [9–12]. The most recent applications can be found in [13–17].

Despite the frequent occurrence of these mathematical models in several interesting areas, it cannot be denied that most of these models are not exactly solvable, mainly due to the involvement of nonlinear terms and the stiffness of the problem in hand. In other words, their solutions can neither be represented in terms of elementary mathematical functions nor can they be found in analytical expressions. This is where numerical techniques come to our rescue. With the advent of digital computers, getting accurate approximate solutions to mathematical models once considered unfathomable has become straightforward. Instead of producing a closed-form solution for a model, numerical techniques generate a sequence of results in a discrete fashion which is easily tabulated for graphical interpretation of the solution.

One mathematical model is different from another in many respects, such as physical interpretation of the model, characteristics of parameters contained therein, and given conditions. Therefore, one numerical technique is not sufficient to serve the general purpose, thereby leading to have much research works on developing new numerical techniques [18–22]. These are the techniques suitable for a particular set of problems, and this search continues to the day. Various scholars have either devised new numerical techniques for solving initial value problems or improving existing ones in many aspects such as convergence rate, order of accuracy, stability, efficiency, computational cost, number of slope evaluations per integration step, speed, and implementation.

Standard numerical techniques to solve initial value problems in ordinary differential equations include linear explicit and implicit Runge-Kutta techniques, linear explicit and implicit Adams-Bashforth-Moulton techniques, exponential techniques, multi-derivative techniques, backward differentiation formulae, and a few others. The nonstandard techniques such as improved linear explicit Runge-Kutta schemes with reduced slope evaluations, accelerated Runge-Kutta schemes, singly implicit Runge-Kutta schemes, A-stable Runge-Kutta schemes, collocation schemes, two-derivative Runge-Kutta schemes, semi-implicit hybrid schemes, explicit and implicit block techniques have been developed in the available literature. Apart from these, non-linear/rational numerical techniques to solve mathematical models having characteristics of stiffness and singularity have also been developed.

A few strategies have been created utilizing the possibility of various techniques, for example, the mathematical mean, centroidal mean, symphonious mean, power means, Lehmer mean, and the heronian mean. The three-phase strategy (3-stages and third-order), depending upon the combined means, and multiderivative techniques in the average mean were developed in [23–32].

One of the difficulties in implementing the standard 4-stages fourth-order Runge-Kutta technique is the absence of control of errors' procedure in the computation of the numerical results. Several techniques have been adopted to overcome these weaknesses, specifically this is done by introducing the procedure that can control the errors in the results. Amongst them are techniques developed by Merson, Scraton, and Fehlberg. Related work was carried out by Sanugi, who introduced a fourth-order AGM (Arithmetic Geometric Mean) technique, which is based on geometric mean plus a fourth-order technique based on the arithmetic mean [33].

The present article is structured as follows: Section 2 contains formulation and derivation of the proposed non-linear fourth-order technique based upon contra-harmonic and harmonic means. Detailed analysis having stability, consistency, convergence, error control, and error bounds are given in Section 3. The proposed technique is also tested via an adaptive step-size approach as discussed in Section 4. The performance of the proposed technique is checked in Section 5 under numerical simulations, and finally, conclusion with future remarks are provided in section 6.

FORMULATION AND DERIVATION

In this section, we will derive a new four-stage fourth order nonlinear RK technique for solving initial value problems of the type

$$\frac{dy}{dt} = g\left(t,y(t)\right), \quad y\left(t_0\right) = y_0 \tag{1}$$

The general s-stage of the proposed non-linear technique can be written as follows:

Table 1. Butcher array

α_1	γ_{11}					
α_2	γ_{21}	γ_{22}				
α_3	γ_{31}	γ_{32}	γ_{33}			
$\vdots$	$\vdots$	$\vdots$	$\ddots$			
α_{s-1}	$\gamma_{s-1,1}$	$\gamma_{s-1,2}$	$\gamma_{s-3,3}$	$\cdots$	$\gamma_{s-1,s-1}$	
α_s	$\gamma_{s,1}$	$\gamma_{s,2}$	$\gamma_{s,3}$	$\cdots$	$\gamma_{s,s-1}$	$\gamma_{s,s}$
	ω_1	ω_2	ω_3	$\cdots$	ω_{s-1}	ω_s

$$y_{n+1} = y_n + \sum_{i=1}^{s} \omega_i k_i \tag{2}$$

where

$$k_i = g\left(t_n + \alpha_i h, y_n + h\sum_{j=1}^{s} \gamma_{ij} k_j\right) \tag{3}$$

and

$$\alpha_i = \sum_{j=1}^{s} \gamma_{ij}, \quad i=0,1,2,3,\ldots,s, \tag{4}$$

with α and ω are s-dimensional vectors and $A(\gamma_{ij})$ be the s × s matrix. Based upon above notations, the Butcher tableau given in the Table 1. Butcher array can be formulated in the following way:

Since we intend to formulate a four-stage fourth order technique, above equations are simplified as follows:

$$\begin{aligned}
&k1 = g(tn, yn),\\
&k2 = g(tn + \sigma 2h, yn + \gamma 21hk1),\\
&k3 = g(tn + \sigma 3h, yn + (\gamma 31k1 + \gamma 32k2)h),\\
&k4 = g(tn + \sigma 4h, yn + (\gamma 41k1 + \gamma 42k2 + \gamma 43k3)h),
\end{aligned} \tag{5}$$

$$y_{n+1} = y_n + h\left(\begin{array}{l}
\omega_1 \dfrac{k_1^2 + k_2^2 + k_3^2}{k_1 + k_2 + k_3} + \omega_2 \dfrac{k_2^2 + k_3^2 + k_4^2}{k_2 + k_3 + k_4}\\
+ \omega_3 \dfrac{k_1 k_2 k_3}{k_1 k_2 + k_1 k_3 + k_2 k_3}\\
+ \omega_4 \dfrac{k_2 k_3 k_4}{k_2 k_3 + k_2 k_4 + k_3 k_4}
\end{array}\right). \tag{6}$$

The unknown parameters ω_q (q = 1, 2, 3, 4) and γ_{ij} (i, j = 2, 3, 4) have to be calculated. Without losing complexity, and for the sake of clarity, the function g is taken to be the function of only the dependent variable, y. The true solution via the Taylor series expansion is expressed below up to fifth order:

$$y_{n+1} = y(t_n) + gh + \frac{g_y g h^2}{2} + \left(\frac{g_{y,y} g^2}{6} + \frac{g_y^2 g}{6}\right) h^3 + \left(\frac{g_{y,y,y} g^3}{24} + \frac{g_{y,y} g^2 g_y}{6} + \frac{g_y^3 g}{24}\right) h^4 + \left(\begin{array}{l}
\dfrac{g_{y,y,y,y} g^4}{120} + \dfrac{7\left(g_{y,y,y} g^3 g_y\right)}{120}\\
+ \dfrac{11}{120} g_{y,y} g^2 g_y^2 + \dfrac{1}{30} g_{y,y}^2 g^3\\
+ \dfrac{1}{120} g_y^4 g
\end{array}\right) h^5 + O(h^6). \tag{7}$$

Next, all the four slopes k_1, k_2, k_3, and k_4 have been expanded into the Taylor series about t_n and substituted in the equation (6) for comparing the coefficients as given in the equation (7). One can obtain a nonlinear system of algebraic equations (order conditions) based upon 10 unknowns and 7 equations.

$$-\omega_2 - \omega_1 - \frac{\omega_4}{3} - \frac{\omega_3}{3} + 1 = 0, \tag{8}$$

$$\begin{aligned}
&\frac{\left(-6\gamma_{4,3} - 6\gamma_{2,1} - 6\gamma_{3,1} - 6\gamma_{3,2} - 6\gamma_{4,1} - 6\gamma_{4,2}\right)\omega_2}{18}\\
&+\frac{\left(-2\gamma_{4,3} - 2\gamma_{2,1} - 2\gamma_{3,1} - 2\gamma_{3,2} - 2\gamma_{4,1} - 2\gamma_{4,2}\right)\omega_4}{18}\\
&+\frac{\left(-6\omega_1 - 2\omega_3\right)\gamma_{2,1}}{18} + \frac{\left(-6\omega_1 - 2\omega_3\right)\gamma_{3,1}}{18}\\
&+\frac{\left(-6\omega_1 - 2\omega_3\right)\gamma_{3,2}}{18} + \frac{1}{2} = 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\frac{\left(\begin{array}{l}
-3\gamma_{3,1}^2 - 6\gamma_{3,1}\gamma_{3,2} - 3\gamma_{3,2}^2 - 3\gamma_{4,1}^2\\
+\left(-6\gamma_{4,3} - 6\gamma_{4,2}\right)\gamma_{4,1} - 3\gamma_{4,3}^2\\
-6\gamma_{4,3}\gamma_{4,2} - 3\gamma_{2,1}^2 - 3\gamma_{4,2}^2
\end{array}\right)\omega_2}{18}\\
&+\frac{\left(\begin{array}{l}
-\gamma_{3,1}^2 - 2\gamma_{3,1}\gamma_{3,2} - \gamma_{3,2}^2 - \gamma_{4,1}^2\\
+\left(-2\gamma_{4,3} - 2\gamma_{4,2}\right)\gamma_{4,1}\\
-\gamma_{4,3}^2 - 2\gamma_{4,3}\gamma_{4,2} - \gamma_{2,1}^2 - \gamma_{4,2}^2
\end{array}\right)\omega_4}{18}\\
&+\frac{\left(-3\omega_1 - \omega_3\right)\gamma_{3,1}^2}{18} - \frac{\left(\gamma_{3,2}\left(\omega_1 + (\omega_3)/3\right)\right)\gamma_{3,1}}{3}\\
&+\frac{\left(-3\omega_1 - \omega_3\right)\gamma_{3,2}^2}{18} + 1/6 + \frac{\left(-3\omega_1 - \omega_3\right)\gamma_{2,1}^2}{18} = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\frac{1}{216}\left(-12\gamma_{3,1}^3 - 36\gamma_{3,1}^2\gamma_{3,2} - 36\gamma_{3,1}\gamma_{3,2}^2 - 12\gamma_{3,2}^3\right)\\
&-12\left(\gamma_{4,3} + \gamma_{2,1} + \gamma_{4,1} + \gamma_{4,2}\right)\left(\gamma_{4,1}^2 + \left(2\gamma_{4,3} - \gamma_{2,1}\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+2\gamma_{4,2})\gamma_{4,1}+\gamma_{4,2}^2+(2\gamma_{4,3}-\gamma_{2,1})\gamma_{4,2}+\gamma_{4,3}^2\\
&-\gamma_{4,3}\gamma_{2,1}+\gamma_{2,1}^2)\omega_2+\frac{1}{216}(-4\gamma_{3,1}^3-12\gamma_{3,1}^2\gamma_{3,2}\\
&-12\gamma_{3,1}\gamma_{3,2}^2-4\gamma_{3,2}^3-4(\gamma_{4,3}+\gamma_{2,1}+\gamma_{4,1}+\gamma_{4,2})\gamma_{4,1}^2\\
&+(2\gamma_{4,3}-\gamma_{2,1}+2\gamma_{4,2})\gamma_{4,1}+\gamma_{4,2}^2+(2\gamma_{4,3}-\gamma_{2,1})\gamma_{4,2}\\
&+\gamma_{4,3}^2-\gamma_{4,3}\gamma_{2,1}+\gamma_{2,1}^2))\omega_4+((-12\omega_1-4\omega_3)\gamma_{3,1}^3)\\
&\frac{1}{216}-\left(\omega_1+(\omega_3)\frac{1}{3}\right)\gamma_{3,2}\gamma_{3,1}^2)\frac{1}{6}-(\omega_1+\omega_3)\gamma_{3,2}^2\gamma_{3,1})\\
&\frac{1}{6}+(-12\omega-1-4\omega_3)\gamma_{3,2}^3)\frac{1}{216}-(\omega_1\gamma_{2,1}^3)\frac{1}{18}\\
&-(\omega_3)\gamma_{2,1}^3)\frac{1}{54}+\frac{1}{24}=0,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{1}{54}(-12\gamma_{2,1}^2+(12\gamma_{4,3}+12\gamma_{3,1}-6\gamma_{3,2}+12\gamma_{4,1}\\
&-6\gamma_{4,2})\gamma_{2,1}-12\gamma_{3,1}^2+(-6\gamma_{4,3}-24\gamma_{3,2}+12\gamma_{4,1}\\
&+12\gamma_{4,2})\gamma_{3,1}-12\gamma_{3,2}^2+(-6\gamma_{4,3}+12\gamma_{4,1}+12\gamma_{4,2})\\
&\gamma_{3,2}-12(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2})^2)\omega_2+\frac{1}{54}(4\gamma_{2,1}^2\\
&+(-4\gamma_{4,3}-4\gamma_{3,1}-10\gamma_{3,2}-4\gamma_{4,1}-10\gamma_{4,2})\gamma_{2,1}\\
&+4\gamma_{3,1}^2+(-10\gamma_{4,3}+8\gamma_{3,2}-4\gamma_{4,1}-4\gamma_{4,2})\gamma_{3,1}\\
&+4\gamma_{3,2}^2+(-10\gamma_{4,3}-4\gamma_{4,1}-4\gamma_{4,2})\gamma_{3,2}+4(\gamma_{4,3}\\
&+\gamma_{4,1}+\gamma_{4,2})^2)\omega_4+\frac{1}{54}(-12\omega_1+4\omega_3)\gamma_{2,1}^2\\
&+\frac{1}{54}((12\omega_1-4\omega_3)\gamma_{3,1}-\left(6\left(\omega_1+5\omega_3\frac{1}{3}\right)\right)\\
&\gamma_{3,2})\gamma_{2,1}+(1/54)(-12\omega_1+4\omega_3)\gamma_{3,1}^2-4\gamma_{3,2}\\
&(\omega_1-\frac{1}{3}\omega_3)\gamma_{3,1}(\frac{1}{9}+1/6+\frac{1}{54}(-12\omega_1+4\omega_3)\\
&\gamma_{3,2}^2=0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\frac{1}{648}(-32\gamma_{3,2}^3+(-24\gamma_{4,2}+120\gamma_{2,1}-96\gamma_{3,1}+24\gamma_{4,1}\\
&+24\gamma_{4,2})\gamma_{3,2}^2+(-24\gamma_{2,1}^2+(-216\gamma_{4,2}+144\gamma_{3,1}\\
&-96\gamma_{4,1}-144\gamma_{4,2})\gamma_{2,1}-96\gamma_{3,1}^2+(-48\gamma_{4,2}+48\gamma_{4,1}\\
&+48\gamma_{4,2})\gamma_{3,1}+120(\gamma_{4,2}+\gamma_{4,1}+\gamma_{4,2})(\gamma_{4,3})+\frac{1}{5}\gamma_{4,1}\\
&+\frac{1}{5}\gamma_{4,2}))\gamma_{3,2}-32\gamma_{2,1}^3+(24\gamma_{4,2}+24\gamma_{3,1}+24\gamma_{4,1}\\
&-24\gamma_{4,2})\gamma_{2,1}^2+(24\gamma_{3,1}^2+(-96\gamma_{4,2}+48\gamma_{4,1}-96\gamma_{4,2})\\
&\gamma_{3,1}+(24(\gamma_{4,2}+\gamma_{4,1})+5\gamma_{4,2}))(\gamma_{4,2}+\gamma_{4,1}-\gamma_{4,2}))\\
&\gamma_{2,1}-32\gamma_{3,1}^3+(-24\gamma_{4,2}+24\gamma_{4,1}+24\gamma_{4,2})\gamma_{3,1}^2+(120\\
&(\gamma_{4,2}+\gamma_{4,1}+\gamma_{4,2}))(\gamma_{4,2}+(1/5)\gamma_{4,1}+(1/5)\gamma_{4,2})\gamma_{3,1}\\
&-32(\gamma_{4,2}+\gamma_{4,1}+\gamma_{4,2})^3)\omega_4+\frac{1}{648}(48\omega_1+48\omega_2\\
&+32\omega_3)\gamma_{3,2}^3+\frac{1}{648}((-288\omega_1-288\omega_2+120\omega_3)\gamma_{2,1}\\
&+(144\omega_1+144\omega_2-96\omega_3)\gamma_{3,1}+144\omega_2\gamma_{4,2})\gamma_{3,2}^2\\
&+\frac{1}{648}((144\omega_1+144\omega_2-24\omega_3)\gamma_{2,1}^2+((-288\omega_1\\
&-288\omega_2+144\omega_3)\gamma_{3,1}-72\omega_2(\gamma_{4,2}-2\gamma_{4,2}))\gamma_{2,1}\\
&+(144\omega_1+144\omega_2-96\omega_3)\gamma_{3,1}^2+288\omega_2\gamma_{3,1}\gamma_{4,2}\\
&-288\omega_2\gamma_{4,2}(\gamma_{4,2}+\gamma_{4,1}+\gamma_{4,2}))\gamma_{3,2}+\frac{1}{648}(48\omega_1\\
&+48\omega_2+32\omega_3)\gamma_{3,1}^2+\frac{1}{648}(24\gamma_{3,1}\omega_3+144\gamma_{4,2}\omega_2)\\
&\gamma_{2,1}^2+\frac{1}{648}(24\omega_3\gamma_{3,1}^2-144\omega_2)\gamma_{3,1}\gamma_{4,1}-288\omega_2\gamma_{4,2}\\
&(\gamma_{4,2}+\gamma_{4,1}+\gamma_{4,2}))\gamma_{2,1}+\frac{1}{648}(48\omega_1+48\omega_2\\
&-32\omega_3)\gamma_{3,1}^3+2\omega_2\gamma_{3,1}^2\gamma_{4,2}\frac{1}{9}-4\omega_2\gamma_{4,2}(\gamma_{4,2}+\gamma_{4,1}\\
&+\gamma_{4,2})\gamma_{3,1}\frac{1}{9}+2\omega_2\gamma_{4,2}^3\frac{1}{27}+2\omega_2(\gamma_{4,1}+\gamma_{4,2})\gamma_{4,1}^2\frac{1}{9}\\
&+2\omega_2(\gamma_{4,1}+\gamma_{4,2})^2\gamma_{4,2}\frac{1}{9}+2\omega_2\gamma_{4,1}^3(1/27)2\omega_2\gamma_{4,1}^2\\
&\gamma_{4,2}\frac{1}{9}+\omega_2\gamma_{4,1}\gamma_{4,2}^2\frac{1}{9}+\omega_2\gamma_{4,2}^3\frac{1}{27}+\frac{1}{24}=0,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\frac{1}{54}(-12\gamma_{3,1}^3+(-3\gamma_{4,3}+6\gamma_{2,1}-36\gamma_{3,2}+6\gamma_{4,1}\\
&+6\gamma_{4,2})\gamma_{3,1}^2+(-36\gamma_{3,2}^2+(-6\gamma_{4,3}-6\gamma_{2,1}+12\gamma_{4,1}\\
&+12\gamma_{4,2})\gamma_{3,2}+6\gamma_{2,1}^2)-(12(\gamma_{4,3}-(1/2)\gamma_{4,1})\\
&-(1/2)\gamma_{4,2}))(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2}))\gamma_{3,1}-12\gamma_{3,2}^3\\
&+(-3\gamma_{4,3}-12\gamma_{2,1}+6\gamma_{4,1}+6\gamma_{4,2})\gamma_{3,2}^2+(-3\gamma_{2,1}^2\\
&-(12(\gamma_{4,3}-(1/2)\gamma_{4,1}-(1/2)\gamma_{4,2}))(\gamma_{4,3}+\gamma_{4,1}\\
&+\gamma_{4,2}))\gamma_{3,2}-12\gamma_{2,1}^3+(6\gamma_{4,3}+6\gamma_{4,1}+3\gamma_{4,2})\gamma_{2,1}^2\\
&+(6(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2}))(\gamma_{4,3}+\gamma_{4,1}+2\gamma_{4,2})\gamma_{2,1}\\
&-12(\gamma_{4,3}+\gamma_{4,1}-\gamma_{4,2})^3)\omega_2+(1/54)(4\gamma_{3,1}^3\\
&+(-5\gamma_{4,3}-2\gamma_{2,1}+12\gamma_{3,2}-2\gamma_{4,1}-2\gamma_{4,2})\gamma_{3,1}^2\\
&+(12\gamma_{3,2}^2+(-10\gamma_{4,3}-10\gamma_{2,1}-4\gamma_{4,1}-4\gamma_{4,2})\\
&\gamma_{3,2}-2\gamma_{2,1}^2-(8(\gamma_{4,3}+(1/4)\gamma_{4,1}+(1/4)\gamma_{4,2}))
\end{aligned}
$$

$$
\begin{aligned}
&\left(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2}\right)\big)\gamma_{3,1}+4\gamma_{3,2}^{3}+\big(-5\gamma_{4,3}-8\gamma_{2,1}\\
&-2\gamma_{4,1}-2\gamma_{4,2}\big)\gamma_{3,2}^{2}+\Big(-5\gamma_{2,1}^{2}-\Big(8\Big(\gamma_{4,3}+(1/4)\\
&\gamma_{4,1}+\frac{1}{4}\gamma_{4,2}\Big)\Big)\left(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2}\right)\gamma_{3,2}+4\gamma_{2,1}^{3}\\
&+\left(-2\gamma_{4,3}-2\gamma_{4,1}-5\gamma_{4,2}\right)\gamma_{2,1}^{2}-\Big(2\Big(\gamma_{4,3}+\gamma_{4,1}\\
&+4\gamma_{4,2}\Big)\Big)\left(\gamma_{4,3}+\gamma_{4,1}+\gamma_{4,2}\right)\gamma_{2,1}+4\Big(\gamma_{4,3}+\gamma_{4,1}\\
&+\gamma_{4,2}\Big)^{3}\Big)\omega_{4}+(1/54)\left(-12\omega_{1}+4\omega_{3}\right)\gamma_{3,1}^{3}\\
&+(1/9)\left(\omega_{1}-(1/3)\omega_{3}\right)\left(\gamma_{2,1}-6\gamma_{3,2}\right)\gamma_{3,1}^{2}\\
&+(1/54)\Big(\left(-36\omega_{1}+12\omega_{3}\right)\gamma_{3,2}^{2}-\Big(6\Big(\omega_{1}+5\omega_{3}\\
&(1/3)\Big)\Big)\gamma_{2,1}\gamma_{3,2}+6\gamma_{2,1}^{2}\left(\omega_{1}-(1/3)\omega_{3}\right)\Big)\gamma_{3,1}\\
&+(1/54)\left(-12\omega_{1}+4\omega_{3}\right)\gamma_{3,2}^{3}-(1/9)\Big(2\Big(\omega_{1}\\
&+2\omega_{3}(1/3)\Big)\Big)\gamma_{2,1}\gamma_{3,1}^{2}-\frac{1}{18}\left(\omega_{1}+5\omega_{3}(1/3)\right)\\
&\gamma_{2,1}^{2}\gamma_{3,2}+1/6+(1/54)\left(-12\omega_{1}+4\omega_{3}\right)\\
&\gamma_{2,1}^{3}=0.
\end{aligned}
\tag{14}
$$

Having solved the above nonlinear system (8–14), the unique real valued solution is obtained and has been arranged in the Butcher array shown in the Table 2:

After substituting all the computed values in the equations (5) and (6), we obtained the following four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic and Harmonic means:

$$
y_{n+1}=y_n+\frac{h}{4}\left[\begin{array}{l}\left(\dfrac{k_1^2+k_2^2+k_3^2}{k_1+k_2+k_3}+\dfrac{k_2^2+k_3^2+k_4^2}{k_2+k_3+k_4}\right)+3\\ \left(\dfrac{k_1k_2k_3}{k_1k_2+k_1k_3+k_2k_3}+\dfrac{k_2k_3k_4}{k_2k_3+k_2k_4+k_3k_4}\right)\end{array}\right],
\tag{15}
$$

Table 2. Butcher array for the proposed non-linear RK(4,4) technique.

0	0			
1/2	1/2	0		
1/2	0	1/2	0	
1	0	0	1	0
	1/4	1/4	3/4	3/4

where

$$
\begin{aligned}
&k_1=g\left(t_n, y_n\right),\\
&k2=g\left(t_n+h/2, y_n+h/2k_1\right),\\
&k3=g\left(t_n+h/2, y_n+h/2k_2\right),\\
&k4=g\left(t_n+h, y_n+hk_3\right).
\end{aligned}
\tag{16}
$$

It is also worth to be noted that the slopes (16) of the proposed technique have same parameters' values as that of found in the classical linear RK technique. However, despite having such simple slopes the proposed technique is computationally more accurate having smaller errors which is shown in the forthcoming sections. Moreover, a pseudo-code for the proposed technique has been provided in the algorithm 1 given below.

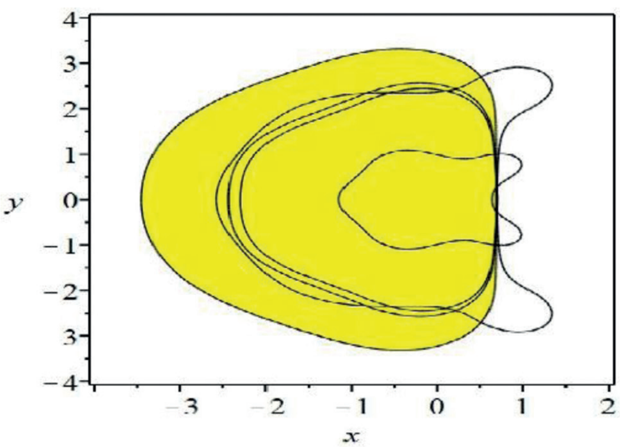

Figure 1. Comparison of stability regions (shaded in yellow color) for the four-stage fourth-order nonlinear Runge-Kutta techniques based on CoMHM, 4sHERK, MCHW, CoM, and HM.

Table 3. Ranges of the stability regions for the techniques under consideration.

	Real axis		Imaginary	
Technique	**Negative**	**Positive**	**Negative**	**Positive**
CoMHM	−3.472	0.749	−3.286	3.286
4sHERK	−2691	0.691	−2.887	−2.887
MCHW	−1.147	0.593	−0.139	0.139
CoM	−2.436	0.710	−2.649	2.649
HM	−2.539	0.612	−2.463	2.563

Algorithm 1: Pseudo-Code of the proposed CoMHM method.

```
procedure CoMHM(g, t, y, h, n)
integer j, n;   real k1, k2, k3, k4, h, t, t0, y
external function g
Output 0, t, y
t0 ← t
for(j = 1 to n)
k1 ← g(tn, yn)
k2 ← g(tn + h/2, yn + (h/2)k1)
k3 ← g(tn + h/2, yn + (h/2)k2)
k4 ← g(tn + h, yn + k3)
y ← y + (h/4)[((k1^2 + k2^2 + k3^2)/(k1 + k2 + k3) + (k2^2 + k3^2 + k4^2)/(k2 + k3 + k4))
   +3((k1k2k3)/(k1k2 + k1k3 + k2k3) + (k2k3k4)/(k2k3 + k2k4 + k3k4))]
t ← t0 + j * h
Output j, t, y
end for
end procedure CoMHM
```

THEORETICAL ANALYSIS
STABILITY ANALYSIS

In this section, we discuss and compare the stability region for the newly proposed four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic and Harmonic means with stability regions of some existing four-stage fourth-order nonlinear RK type techniques having different means. To evaluate stability polynomial, we use simple test equation $y'(t) = \beta y(t)$ where β is a complex constant. The stability polynomials for the techniques considered in the present investigation are as follows:

1. Stability polynomial for the newly proposed four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic and Harmonic means (CoMHM):

$$P(z) = 1 + z + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^4 + \frac{1}{864}z^5. \tag{17}$$

2. Stability polynomial for the four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic Explicit mean (4sHERK) [32]:

$$P(z) = 1 + z + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^4 + \frac{23}{960}z^5. \tag{18}$$

3. Stability polynomial for the four-stage fourth-order nonlinear weighted Runge-Kutta technique based on Contra-Harmonic mean (MCHW) [34]:

$$P(z) = 1 + z + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^4 + \frac{141}{124}z^5. \tag{19}$$

4. Stability polynomial for the four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic mean (CoM) [29]:

$$P(z) = 1 + z + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^4 + \frac{21}{1280}z^5. \tag{20}$$

5. Stability polynomial for the four-stage fourth-order nonlinear Runge-Kutta technique based on Harmonic mean (HM) [30]:

$$P(z) = 1 + z + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^4 + \frac{3}{280}z^5. \tag{21}$$

The condition must be satisfied in order to find the stability region for any nonlinear RK type techniques in the complex plane. With the help of stability polynomials, the stability regions for all the techniques under consideration have been obtained via Maple 20 wherein it can be observed in Figure 1 the largest region is given by the newly proposed four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic and Harmonic means and similarly ranges under real and imaginary axes for the proposed technique are much greater than other techniques, as depicted in Table 3. This is the more powerful feature of the technique proposed here. The graphic surface for the proposed technique is shown in Figure 2 whereas techniques taken for comparison have graphic surfaces shown in Figure 3.

$$\left|\frac{y_{n+1}}{y_n}\right| = P(z) < 1, \tag{22}$$

CONSISTENCY ANALYSIS

Definition 4.1. For a given initial value problem $y'(t) = g(t,y)$, $y(t_0) = y_0$ a numerical technique is said to be consistent if the following criteria is satisfied:

$$\lim_{h\to 0} \mu(tn,yn;\ h) = g(t,y). \tag{23}$$

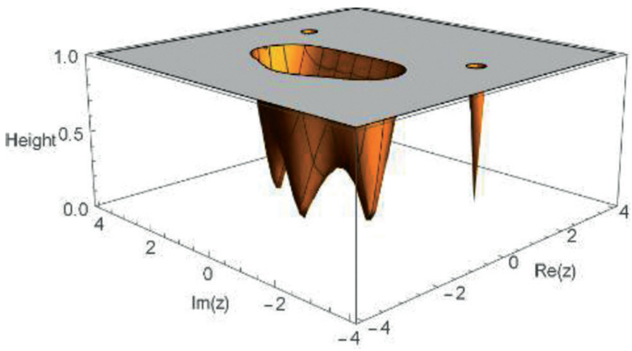

Figure 2. Graphic surface given by newly proposed four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic and Harmonic means (CoMHM).

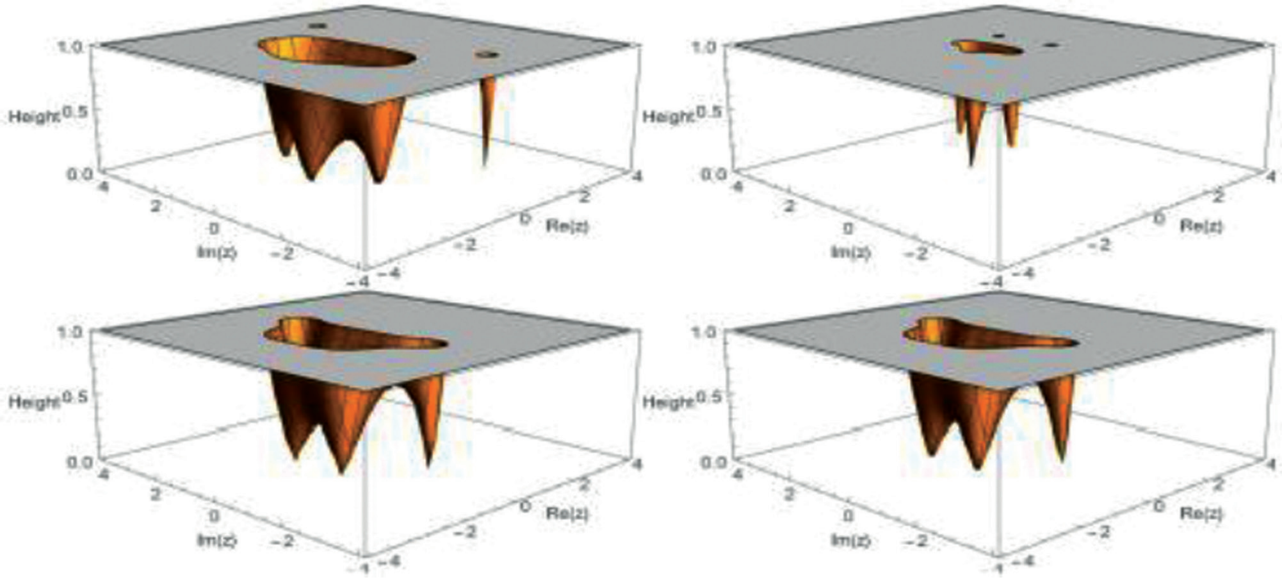

Figure 3. Graphic surfaces for the techniques (top left = 4sHERK, top right = MCHW, bottom left = CoM, bottom right = HM) under consideration for comparison purpose.

Based upon the above criteria, we have carried out the consistency analysis for the newly proposed technique in the following manner:

$$\lim_{h\to 0}\mu\left(\text{tn,yn; h}\right) = \frac{1}{4}\lim_{h\to 0}\left[\begin{array}{l}\left(\dfrac{k_1^2+k_2^2+k_3^2}{k_1+k_2+k_3}+\dfrac{k_2^2+k_3^2+k_3^2}{k_2+k_3+k_4}\right)+3\\ \left(\dfrac{k_1k_2k_3}{k_1k_2+k_1k_3+k_2k_3}+\dfrac{k_2k_3k_4}{k_2k_3+k_2k_4+k_3k_4}\right)\end{array}\right]. \tag{24}$$

Using slopes as defined in the Equations (16), we obtain the following:

$$\lim_{h\to 0}\mu\left(t_n, y_n;\ \text{h}\right) = \text{g}\left(\text{t,y}\right). \tag{25}$$

Thus, the newly proposed technique is proved to be consistent with fourth order accuracy.

ASYMPTOTIC ERROR ANALAYSIS

In four order to obtain the local truncation error and global truncation error of the newly proposed nonlinear technique (6), a usual functional associated to the technique has been considered as given below:

$$\text{L}(\sigma(\text{t}),\text{h}) = \sigma\left(\text{t+h}\right) - \text{y}_{\text{n+1}}, \tag{26}$$

where y_{n+1} is the proposed technique presented in (15–16) and σ(t) is an arbitrary function defined on the interval $[t_0,t_n]$ and is taken to be differentiable as many times as required. Having expanded $\sigma(t)$ by the Taylor series about t under the local assumption and collecting the terms containing powers of h, the local truncation error of (15–16) has been obtained below that ensures at least fourth order accuracy of the proposed technique.

$$\begin{aligned}\delta_{n+1} &= \delta_n + \text{h}\left\{\text{g}\left(t_n, y_n\right) - \text{g}\left(t_n, y\left(t_n\right)\right)\right\}\\ &+ \frac{h^2}{2!}\left\{g_y g\left(t_n, y_n\right) - g_y g\left(t_n, y\left(t_n\right)\right)\right\}\\ &+ \frac{h^3}{3!}\left\{g_y^2 g\left(t_n, y_n\right) - g_y^2\left(t_n, y\left(t_n\right)\right)\right\}\\ &+ \frac{h^4}{4!}\left\{\begin{array}{l} g_y^4 g^4\left(t_n, y_n\right) - g_y^4 g^4\left(t_n, y\left(t_n\right)\right)\\ + \dfrac{h^5}{8640} y^v)(\xi)\end{array}\right\}.\end{aligned} \tag{27}$$

Next, we will evaluate global truncation error (GTE) associated with the proposed technique (15–16). In order to do this, we first write

$$\delta_n = y_n - y\left(t_n\right), \tag{28}$$

and similarly

$$\delta_{n+1} = y_{n+1} - y\left(t_{n+1}\right), \tag{29}$$

The way we found LTE in (27), we will follow the same routine to evaluate the GTE as shown in the following computations:

Assuming the Lipchitz condition on g with an arbitrary constant κ and that $|y(t)^v| \le T, \forall t \in [t_0, t_n]$, we can write the above equation in the following manner:

$$|\delta_{n+1}| \le \left(1 + hk + \frac{h^2}{2!}k + \frac{h^3}{3!}k + \frac{h^4}{4!}k\right)|\delta_n| + \frac{h^5}{8640}T, |\delta_{n+1}| \le \left(1 + hk\sum_{r=1}^{4}\frac{h^{r-1}}{r!}\right)|\delta_n| + \frac{h^5}{8640}T,$$

Assuming

$$\mathrm{S} = \mathrm{hk}\sum_{r=0}^{4}\frac{h^{r-1}}{r!}, S_1 = 1 + S, \quad S_2 = \frac{h^5}{8640}T,$$

we can have the following

$$|\delta_{n+1}| \le S_1|\delta_n| + S_2. \quad (30)$$

Hence the total error δ_{n+1} consists of two parts. One part is the local truncation error, which is no more than S2 and which is present even if we start the step with no error at all, i.e. with $|\delta_n| = 0$. The other part is due to the combined error from all previous steps. At the beginning of step number n + 1, the combined error is $|\delta_n|$. During the step, this error gets magnified by no more than a factor of S1. The second half of the derivation is to repeatedly apply (30) n = 0, 1, 2 . . . , by definition $y(t_0) = y_0$ so that $|\delta_0| = 0$, so

$$|\delta_n| \le (1+S)^{n-1}S_2 + \cdots + (1+S)S_2 + S_2,$$
$$|\delta_n| = \sum_{i=0}^{n-1}(1+S)^i S_2.$$

Using the concept of geometric series, we obtain

Table 4. Comparison of techniques with respect to maximum error, final error, norm and CPU time for numerical experiment 1 with step-size h = $\mathbf{10^{-2}}$ over [0, 10].

Techniques	Maximum	Final	Norm	Time
CoMHM	1.5432e-04	3.2972e-05	1.9114e-03	1.8834e-03
4sHERK	3.4291e-04	7.6547e-05	5.0303e-03	2.4730e-04
MCHW	4.8578e-04	1.1031e-04	4.9255e-03	2.3210e-04
CoM	5.6414e-04	1.0956e-04	7.0232e-03	2.2880e-04
HM	5.6423e-04	1.0958e-04	7.0247e-03	2.3450e-04

Table 5. The performance of proposed technique with respect to number of steps, maximum error, final error, norm and CPU time for numerical experiment 1 with adaptive step-size over [0, 0.5] under different values of the tolerance.

tol	Steps	Maximum	Final	Norm	Time
1.0000e-02	38	2.7779e-02	1.6255e-02	8.4925e-02	7.3660e-04
1.0000e-03	69	8.5799e-03	3.1241e-03	3.6898e-02	1.6822e-03
1.0000e-04	139	8.5145e-04	3.7795e-04	5.0607e-03	6.3503e-03
1.0000e-05	289	1.3873e-04	5.5867e-05	1.1818e-03	9.6650e-03
1.0000e-06	622	2.3165e-05	6.0401e-06	2.5032e-04	1.3933e-02
1.0000e-07	1356	5.1783e-06	1.1697e-06	8.5147e-05	1.6271e-02

Table 6. Comparison of techniques with respect to absolute percent relative global truncation errors for numerical experiment 1 with step-size h = $\mathbf{10^{-2}}$ over [0, 10].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
2.0000e+00	8.0535e-04	6.2238e-03	4.7452e-04	6.3842e-04	6.3753e-04
4.0000e+00	8.0536e-04	8.0025e-03	1.0099e-03	9.4253e-04	9.4114e-04
6.0000e+00	4.3565e-03	1.4492e-02	1.6795e-03	1.7981e-02	1.7984e-02
8.0000e+00	5.6807e-03	1.0695e-02	1.8014e-02	1.9443e-02	1.9447e-02
1.0000e+01	5.6808e-03	1.3189e-02	1.9006e-02	1.8876e-02	1.8880e-02

Table 7. Comparison of techniques with respect to absolute percent relative local truncation errors for numerical experiment 1 with step-size h = $\mathbf{10^{-2}}$ over [0, 10].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
2.0000e+00	2.6619e-10	2.1728e-05	1.1986e-05	6.8898e-06	6.8827e-06
4.0000e+00	4.3317e-11	1.9598e-05	9.9287e-06	5.7121e-06	5.7027e-06
6.0000e+00	7.8250e-11	1.0108e-05	2.5088e-06	1.4631e-06	1.4667e-00
8.0000e+00	8.3207e-09	8.5859e-05	4.9493e-05	2.8545e-05	2.8535e-05
1.0000e+01	4.1317e-11	1.4289e-05	5.8749e-06	3.3743e-06	3.3672e-06

Table 8. Comparison of techniques with respect to maximum error, final error, norm and CPU time for numerical experiment 2 with step-size h = $\mathbf{10^{-2}}$ over [0, 0.5].

Techniques	Maximum	Final	Norm	Time
CoMHM	1.5539e-06	1.5539e-06	2.1734e-06	3.3820e-04
4sHERK	8.6035e-03	8.6035e-03	1.3352e-02	1.5100e-04
MCHW	8.1004e-04	8.1004e-04	1.0733e-03	1.1940e-04
CoM	7.9457e-05	7.9457e-05	1.0581e-04	9.2600e-05
HM	1.0303e-04	1.0303e-04	1.3664e-04	1.1350e-04

Table 9. Comparison of techniques with respect to absolute percent relative global truncation errors for numerical experiment 2 with step-size h = $\mathbf{10^{-2}}$ over [0, 0.5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	6.6613e-14	6.6613e-14	6.6613e-14	6.6613e-14	6.6613e-14
1.0000e-01	7.8352e-08	9.8149e-04	9.0105e-06	9.6458e-07	1.1142e-06
2.0000e-01	2.3320e-07	2.8483e-03	3.4056e-05	3.6036e-06	4.2633e-06
3.0000e-01	6.2642e-07	7.1615e-03	1.2557e-04	1.3099e-05	1.5898e-05
4.0000e-01	2.1220e-06	2.0410e-02	6.4840e-04	6.6292e-05	8.2732e-05
5.0000e-01	1.7200e-05	9.5234e-02	8.9664e-03	8.7952e-04	1.1405e-03

Table 10. Comparison of techniques with respect to absolute percent relative local truncation errors for numerical experiment 2 with step-size h = $\mathbf{10^{-2}}$ over [0, 0.5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	6.6613e-14	6.6613e-14	6.6613e-14	6.6613e-14	6.6613e-14
1.0000e-01	9.6838e-09	1.2022e-04	1.2737e-06	1.3547e-07	1.5867e-07
2.0000e-01	1.8994e-08	2.2194e-04	3.6068e-06	3.7759e-07	4.5614e-07
3.0000e-01	4.8745e-08	4.9328e-04	1.3857e-05	1.4268e-06	1.7691e-06
4.0000e-01	2.1025e-07	1.5307e-03	9.1370e-05	9.1997e-06	1.1692e-05
5.0000e-01	3.2537e-06	1.0454e-02	2.0990e-03	2.0151e-04	2.6595e-04

Table 11. Comparison of techniques with respect to maximum error, final error, norm and CPU time for numerical experiment 3 with step-size h = 10^{-2} over [0, 5].

Techniques	Maximum	Final	Norm	Time
CoMHM	4.2006e-11	4.2006e-11	4.9380e-10	5.4240e-04
4sHERK	6.1800e-06	6.1800e-06	7.3359e-05	1.9860e-04
MCHW	5.8989e-06	5.8989e-06	6.9010e-05	1.4030e-04
CoM	3.4158e-06	3.4158e-06	3.9949e-05	1.3890e-04
HM	3.4098e-06	3.4098e-06	3.9880e-05	1.3990e-04

Table 12. Comparison of techniques with respect to absolute percent relative global truncation errors for numerical experiment 3 with step-size h = $\mathbf{10^{-2}}$ over [0, 5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
1.0000e+00	4.9668e-10	9.8871e-05	8.2032e-05	4.7377e-05	4.7304e-05
2.0000e+00	5.8807e-10	9.3287e-05	8.3754e-05	4.8441e-05	4.8359e-05
3.0000e+00	5.8060e-10	8.4694e-05	7.9662e-05	4.6113e-05	4.6033e-05
4.0000e+00	5.6751e-10	8.2106e-05	7.8361e-05	4.5373e-05	4.5294e-05
5.0000e+01	5.5927e-10	8.2281e-05	7.8539e-05	4.5478e-05	4.5399e-05

Table 5. Comparison of techniques with respect to absolute percent relative local truncation errors for numerical experiment 3 with step-size h = $\mathbf{10^{-2}}$ over [0, 5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
1.0000e+00	7.8482e-12	1.2518e-06	1.1063e-06	6.3962e-07	6.3854e-07
2.0000e+00	6.6207e-12	9.1296e-07	8.7475e-07	5.0651e-07	5.0561e-07
3.0000e+00	6.0196e-12	8.4839e-07	8.1993e-07	4.7487e-07	4.7403e-07
4.0000e+00	5.7813e-12	8.4074e-07	8.0613e-07	4.6683e-07	4.6602e-07
5.0000e+01	5.6170e-12	8.4264e-07	8.0219e-07	4.6450e-07	4.6370e-07

Table 6. Comparison of techniques with respect to maximum error, final error, norm and CPU time for numerical experiment 4 with step-size h = $\mathbf{10^{-2}}$ over [0, 5].

Techniques	Maximum	Final	Norm	Time
CoMHM	6.3771e-13	6.3771e-13	7.0342e-12	1.2305e-02
4sHERK	6.3807e-07	6.3807e-07	6.9726e-06	1.0512e-02
MCHW	8.7970e-12	8.7970e-12	1.0384e-10	1.2422e-02
CoM	7.7760e-13	7.7760e-13	9.2229e-12	7.3507e-03
HM	1.9602e-12	1.9602e-12	2.2476e-11	6.8962e-03

Table 7. Comparison of techniques with respect to absolute percent relative global truncation errors for numerical experiment 4 with step size h = $\mathbf{10^{-2}}$ over [0, 5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
1.0000e+00	5.2791e-12	5.1248e-06	9.2059e-11	8.3308e-12	1.8643e-11
2.0000e+00	1.0067e-11	9.8356e-06	1.6734e-10	1.4989e-11	3.4567e-11
3.0000e+00	1.4264e-11	1.4041e-05	2.2439e-10	2.0023e-11	4.7421e-11
4.0000e+00	1.7794e-11	1.7645e-05	2.6286e-10	2.3329e-11	5.6912e-11
5.0000e+01	2.0546e-11	2.0557e-05	2.8342e-10	2.5053e-11	6.3154e-11

Table 8. Comparison of techniques with respect to absolute percent relative local truncation errors for numerical experiment 4 with step-size h = $\mathbf{10^{-2}}$ over [0, 5].

t	CoMHM	4sHERK	MCHW	CoM	HM
0	0	0	0	0	0
1.0000e+00	5.2615e-14	5.0147e-08	8.5938e-13	7.0154e-14	1.7538e-13
2.0000e+00	5.5618e-14	4.6716e-08	6.9523e-13	5.5618e-14	1.5295e-13
3.0000e+00	5.5618e-14	4.6716e-08	6.9523e-13	5.5618e-14	1.5295e-13
4.0000e+00	5.3222e-14	3.8264e-08	3.7256e-13	1.7741e-14	1.0644e-13
5.0000e+01	2.8615e-14	3.3386e-08	2.2892e-13	1.4308e-14	7.1538e-14

Table 17. The maximum error, final error, and norm for numerical experiment 5 obtained with each technique under discussion with the step-size h = $\mathbf{10^{-5}}$ over the interval [0, 10].

Techniques	Sate Variable	Maximum	Final	Norm
CoMHM	$y_1(t)$	8.0900e-10	3.8761e-11	6.1875e-08
	$y_2(t)$	5.4154e-09	3.9086e-11	6.3832e-08
4sHERK	$y_1(t)$	1.4001e-08	9.3529e-10	4.1217e-07
	$y_2(t)$	9.8184e-09	7.2907e-10	3.6646e-07
MCHW	$y_1(t)$	8.8268e-08	7.4676e-10	5.7611e-06
	$y_2(t)$	8.5049e-08	5.2329e-10	5.8573e-06
CoM	$y_1(t)$	4.2317e-09	2.1724e-10	1.2217e-07
	$y_2(t)$	3.5108e-09	8.1399e-11	1.3650e-07
HM	$y_1(t)$	4.2298e-09	2.1719e-10	1.2213e-07
	$y_2(t)$	6.8925e-09	8.5446e-11	1.3479e-07

Table 9. Maximum error, final error, and norm for numerical experiment 6 with step-size h = $\mathbf{10^{-2}}$ under the proposed technique CoMHM.

State Variable	Maximum	Final	Norm
$y_1(t)$	7.6783e-07	6.9957e-07	7.4564e-06
$y_2(t)$	1.4243e-07	1.4243e-07	8.1162e-07
$y_3(t)$	4.3382e-08	4.3347e-08	4.3354e-07
$y_4(t)$	7.7251e-11	7.7251e-11	4.5419e-10

Table 19. Maximum error, final error, and norm for numerical experiment 6 with step-size h = $\mathbf{10^{-2}}$ under the 4sHERK technique

State Variable	Maximum	Final	Norm
$y_1(t)$	2.0492e-05	2.0484e-05	1.8972e-04
$y_2(t)$	3.2230e-06	3.2230e-06	1.7524e-05
$y_3(t)$	1.1993e-06	1.1993e-06	1.0779e-05
$y_4(t)$	1.0327e-09	1.0327e-09	5.5501e-09

Table 20. Maximum error, final error, and norm for numerical experiment 6 with step-size h = $\mathbf{10}^{-2}$ under the MCHW technique.

State Variable	Maximum	Final	Norm
$y_1(t)$	9.1588e-06	9.1588e-06	8.1441e-05
$y_2(t)$	1.2348e-06	1.2348e-06	6.5814e-06
$y_3(t)$	5.3302e-07	5.3302e-07	4.6293e-06
$y_4(t)$	6.7066e-10	6.7066e-10	3.6747e-09

Table 21. Maximum error, final error, and norm for numerical experiment 6 with step-size h = $\mathbf{10}^{-2}$ under the CoM technique.

State Variable	Maximum	Final	Norm
$y_1(t)$	5.7741e-06	5.7741e-06	5.2138e-05
$y_2(t)$	8.1216e-07	8.1216e-07	4.3640e-06
$y_3(t)$	3.3801e-07	3.3801e-07	2.9722e-06
$y_4(t)$	4.4125e-10	4.4125e-10	2.4386e-09

Table 22. Maximum error, final error, and norm for numerical experiment 6 with step-size h = $\mathbf{10}^{-2}$ under the HM technique.

State Variable	Maximum	Final	Norm
$y_1(t)$	5.7762e-06	5.7762e-06	5.2155e-05
$y_2(t)$	8.1240e-07	8.1240e-07	4.3652e-06
$y_3(t)$	3.3794e-07	3.3794e-07	2.9717e-06
$y_4(t)$	4.4117e-10	4.4117e-10	2.4382e-09

$$|\delta_n| \le \left(\frac{S_1^n - 1}{S_1 - 1}\right) S_2. \tag{31}$$

If we use the inequality,503

$$(1+\mathrm{t}) \le e^t, \tag{32}$$

we obtain the following

$$S_1^n = (1+S)^n = \left(1 + \mathrm{hk}\left(\sum\nolimits_{r=0}^{4} \frac{h^{r-1}}{r!}\right)\right)^n \le e^{s_n} = e^{k(t_n - t_0)J}, \tag{33}$$

where $\mathrm{J} = \sum\nolimits_{r=0}^{4} \frac{h^{r-1}}{r!}$. Substituting (33) into the inequality (31), we obtain the following inequality for δ

$$|\delta_n| \le \left(\frac{e^{k(t_n - t_0)J}}{S_1 - 1}\right) S_2.$$

or

$$|\delta_n| \le \frac{h^4}{8640 J} T\left(e^{k(t_n - t_0)J} - 1\right). \tag{34}$$

Thus, the proposed technique possesses the GTE of $O(h^4)$

ERROR CONTROL AND BOUNDS

Using the idea of Lotkin bounds from the research paper [35], if the bounds given below for the function g(t, y) and its partial derivatives hold for $t \in [t_0, t_n]$ and $y \in (-\infty, \infty)$, we have

$$|g(t,y)| < Q, \left|\frac{\partial^{m+n} g(t,y)}{\partial t^m g y^n}\right| < \frac{P^{m+n}}{Q^{n-1}}, (\mathrm{m}, \mathrm{n}) \le \mathrm{r}, \tag{35}$$

where P and Q are positive arbitrary constants and r is order of the proposed technique (15–16). Here r = 4 is the order of accuracy for the proposed technique. Hence, using the equations (27) and (35) we have

$$g_{y,y,y,y} g^4 = g_{y,y,y} g^3 g_y = g_{y,y} g^2 g_y^2 = g_{y,y}^2 g^3 = g_y^4 g = P^4 Q. \tag{36}$$

Taking coefficients from the equation (4.16), we get

$$|LTE| < \frac{227}{8640} P^4 Q h^5. \tag{37}$$

Now we denote the tolerance as τ and taking |LTE| ≤ τ, the error is controlled where the step-size h is chosen from the Equation (37) to obtain the following formula:

$$(2.6273\mathrm{e}-02) P^4 Q h^5 < \tau, \tag{38}$$

giving the following error bound on the step-size h:

$$h < \left[\frac{(3.8062\mathrm{e}+01)\,\tau}{P^4 Q}\right]^{\frac{1}{5}}. \tag{39}$$

ADAPTIVE STEP-SIZE APPROACH

During formulation and theoretical analysis of the proposed technique (15–16), constant step-size h has been used. However, a numerical technique must be suitable enough for step-size construction in order to be effective, as suggested by many researchers [36]. In this regard, we have carried a possible step-size approach for the proposed technique (15–16) by considering a third order technique given in [33]:

$$y_{n+1} = y_n + \frac{h}{2}\left(\frac{k_1^2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_3^2}{k_2 + k_3}\right), \tag{40}$$

where

$$\begin{aligned} k_1 &= g(t_n, y_n), \\ k_2 &= g\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_1\right), \\ k_3 &= g\left(t_n + \frac{2h}{3}, y_n + \frac{2h}{3}k_2\right), \end{aligned} \tag{41}$$

and employing both techniques as embedded type techniques. Same sort of approaches have also been used in [37, 38]. No additional computational cost will be needed since the values required by lower order technique are also those to be employed by the higher order technique. The procedure as adopted in [39] will also be followed by us in the following way. Generally, suppose that the local error while using a technique of order r to obtain y_{n+1} be given by

$$L_n = y(t_n + \mathrm{h}) - y_{n+1}, \tag{42}$$

where y(t) is the true solution for the initial value problem (1). Now applying a technique of order r + 1 for computing a result y^*_{n+1} on this step, we obtain the following

$$\begin{aligned} comp_{est} &= y^*_{n+1} - y_{n+1} = \left(y(t_n + h) - y_{n+1}\right) \\ &\quad - \left(y(t_n + h) - y^*_{n+1}\right), \\ comp_{est} &= L_n + O\left(h^{r+2}\right). \end{aligned} \tag{43}$$

Since L_n is $O(h^{r+1})$ and dominating in the above equation for sufficiently small values of h. The above Eq.is taken as commutable estimate of the local error of the lower order technique. One has to look for a possible pair of techniques (a pair which shares as many slope evaluations as possible) when it comes to embedded type techniques. In such a situation, the lower order technique is employed to estimate the local error and the technique with higher order is employed to advance the integration procedure. The approach of using the more accurate result y^*_{n+1} is known as local extrapolation. A local error tolerance $\in$(tol) must be framed and, if the estimated error is too large relative to this tolerance, the step is rejected and another attempt is made with a smaller step-size.

Having explained the entire process, we discuss the adaptive step-size approach in the following way. From (43), we obtain:

$$y(t_n + h) - y_{n+1} = h^{r+1}\Lambda(t_n) + O\left(h^{r+2}\right). \tag{44}$$

Now, if we take a step from t_n with a new step-size θh, the following would be the error

$$\begin{aligned} &(\theta h)^{r+1}\Lambda(t_n) + O\left(h^{r+2}\right) = (\theta)^{r+1}h(p) + 1\Lambda(t_n) \\ &+ O\left(h^{r+2}\right) = \theta^{r+1}comp_{est} + O\left(h^{r+2}\right). \end{aligned} \tag{45}$$

The prediction of largest step-size passing the error test would correspond to selecting phi so that

$$\left|\varnothing^{r+1}comp_{est}\right| \approx \in(tol). \tag{46}$$

This type of new step-size becomes

$$h\left[\frac{\varepsilon(tol)}{comp_{est}}\right]^{\frac{1}{r+1}}. \tag{47}$$

Inclusion of a safety factor ρ in (47) is highly suggested by various researchers on the basis of extensive numerical experiments. Thus, we also follow the strategy and obtain the result as follows:

$$h_{new} = \rho h\left[\frac{\varepsilon(tol)}{comp_{est}}\right]^{\frac{1}{r+1}}, \tag{48}$$

where ρ is a suitable safety factor ρ ≈ 0.9 [40]. The sole purpose of this safety factor is to avoid failed integration steps and r stands for the order of the lower order technique. In our present scenario, r = 3. This approach has been applied successively to predict the step-size for the next step after a successful step is achieved, that is, when $|comp_{est}|$ < (tol).

NUMERICAL DYNAMICS WITH RESULTS AND DISCUSSION

This section is all about the testing of the proposed technique while using some numerical experiments. We have chosen different types of initial value problems to check the performance, including autonomous, non-autonomous, linear, and nonlinear scalar and vector versions. Constant and adaptive step-size approaches are employed, and errors (maximum absolute error, error at final mesh point of the time interval, norm, absolute percent relative local and global truncation errors) are computed. For comparison, we have taken four techniques called 4sHERK [32], MCHW [34], CoM [29], and HM [30]; each having four stages and fourth-order of accuracy as that of the proposed technique. CPU time in seconds is also determined under the MATLAB environment of version 9.8.0.1323502 (R2020a), using processor Intel(R) Core(TM) i7-1065G7 CPU @1.50

GHz with installed memory (RAM) 24.0GB having system type of 64-bit OS, x64-based processor.

In each Table 4, 8, 11, and 14 for scalar numerical experiments 1–4, one can notice that the maximum absolute errors, absolute errors at final mesh point of the integration interval, and the error norms are all smaller in case of the proposed technique while CPU time is also comparable. As observed in Table 5 for the first numerical experiment that the adaptive step-size approach for the proposed technique utilizes fewer steps than the fixed step-size to yield absolute errors under consideration. Since the better performance with the adaptive strategy is obvious, we have solved the remaining problems via a fixed step-size approach.

Moreover, the absolute percent relative global and local truncation errors are observed to be much smaller than errors produced by the techniques taken for comparison for the scalar numerical experiments 1–4 as depicted in the Tables 6, 9, 10, 12, 13, 15, and 16. Finally, two linear differential systems describing sinusoidal behavior [42] and dynamics of Robot Arm [41] have also been considered for comparison purpose whereupon, once again, the proposed technique has outperformed other techniques as observed in Tables 17, 18, 19, 20, 21 and 22 for the computation of maximum absolute errors, errors at final mesh point of the integration interval, and the error norms. Moreover, the notations in terms of abbreviations used throughout the article are listed in the appendix A provided at end of the conclusion section.

Problem 1. Consider the following first-order linear IVP of oscillatory behavior:

$$y'(t) = y(t)\cos(t), y(0) = 1, t \in [0, 10], \quad (49)$$

with the exact solution $y(t) = e^{\sin(t)}$

Problem 2. Consider the following first-order IVP of nonlinear and autonomous nature:

$$y'(t) = 1 + y(t) + y^2(t), y(0) = 1, t \in [0, 0.5], \quad (50)$$

with the exact solution $y(t) = -\frac{1}{2} + \frac{\sqrt{3}\tan\left(\frac{\pi}{3} + t\frac{\sqrt{3}}{2}\right)}{2}$.

Problem 3. Consider the following first-order IVP of nonlinear and non-autonomous nature:

$$y'(t) = \frac{\exp(t)}{1 + y^2(t)}, y(0) = 1, t \in [0,5], \quad (51)$$

with the exact solution

$$y(t) = \frac{\left(4 + 12\exp(t) + 4\sqrt{5 + 6\exp(t) + 9\exp(2t)}\right)^{\frac{2}{3}} - 4}{2\left(\left(4 + 12\exp(t) + 4\sqrt{5 + 6\exp(t) + 9\exp(2t)}\right)^{\frac{1}{3}}\right)}.$$

Problem 4. Consider the following nonlinear IVP for the Logistic growth [43]

$$y'(t) = \frac{y(t)}{4}\left(1 - \frac{y(t)}{20}\right), y(0) = 1, t \in [0,5], \quad (52)$$

with exact solution $y(t) = \frac{20}{1 + 19\exp\left(-\frac{t}{4}\right)}$.

Problem 5. Consider the following sinusoidal problem given in [42],in the range $0 \le t \le 10$.

$$\begin{aligned} y_1'(t) &= -2y_1(t) + y_2(t) + 2\sin(t), \\ y_2' &= 998y_1(t) - 999y_2(t) + 999\cos(t) - 999\sin(t), \end{aligned} \quad (53)$$

subject to the following initial conditions:
$y_1(0) = 2$, $y_2(0) = 3$, with the following exact solution:

$$\begin{aligned} y_1(t) &= 2\exp(-t) + \sin(t), \\ y_2(t) &= 2\exp(-t) + \cos(t). \end{aligned}$$

Problem 6. Consider the following autonomous linear problem for the Robot Arm system [41] with four state variables:

$$\begin{aligned} y_1'(t) &= y_2(t), \\ y_2'(t) &= 0.2140y_2(t) - 0.1730y_1(t) + 0.0265, \\ y_3' &= y_4, \\ y_4'(t) &= -0.130321y_4(t) - 0.00191844y_3(t) \\ &\quad + 0.00935089, \end{aligned} \quad (54)$$

subject to the following initial conditions:

$$y_1(0) = -1, y_2(0) = 0, y_3(0) = -1, y_4(0) = 0,$$

with the following exact solution:

$$y_1(t) = \frac{42693\exp\left(\frac{107t}{1000}\right)\sqrt{559}\sin\left(\frac{17\sqrt{559}t}{1000}\right)}{3288038} - \frac{399\exp\left(\frac{107t}{1000}\right)\cos\left(\frac{17\sqrt{559}t}{1000}\right)}{346} + \frac{53}{346},$$

$$y_2(t) = \frac{399\exp\left(\frac{107t}{1000}\right)\sqrt{559}\sin\left(\frac{17\sqrt{559}t}{1000}\right)}{19006},$$

$$\begin{pmatrix} -146863035493\sqrt{76940521} \\ -953774933673023 \end{pmatrix}$$

$$y_3(t)=\frac{\exp\left(\frac{11t\sqrt{76940521}}{2000000}\right)\left(\frac{130321t}{2000000}\right)}{324732700835928}\begin{pmatrix}146863035493\sqrt{76940521}\\-953774933673023\end{pmatrix}+\left(\frac{935089}{191844}\right)+\frac{\exp\left(\frac{11t\sqrt{76940521}}{2000000}\right)-\left(\frac{130321t}{2000000}\right)}{324732700835928},\quad(56)$$

$$y_4(t)=\frac{1126933\left(\exp\left(\frac{11t\sqrt{76940521}}{2000000}\right)-\left(\frac{130321t}{2000000}\right)-\exp\left(\frac{-11t\sqrt{76940521}}{2000000}\right)-\left(\frac{130321t}{2000000}\right)\right)\sqrt{76940521}}{84634573100}.$$

CONCLUSION AND FUTURE REMARKS

A new nonlinear hybrid technique under Contra-Harmonic and Harmonic Means has been developed with fourth-order accuracy and four stages. The proposed technique has smaller local and global truncation errors despite using a simple slope structure as used in the classical linear Runge-Kutta technique. Theoretical analysis for the stability, consistency, asymptotic errors, step-size control, and the error bound is also carried out. Four techniques having similar characteristics as that of the proposed one do not perform better than the proposed when it comes to stability, errors, and bound imposed on the step-size h. Moreover, unlike other research works, we have employed an adaptive step-size approach to show the improved performance of the proposed technique. Various numerical experiments with scalar and vector versions of initial value problems have also confirmed the better performance of the proposed technique. Future studies would include the possible extension of the proposed nonlinear numerical technique in the realm of fractional numerical dynamics. Moreover, a nonlinear numerical technique based on rational approximation of functions will be utilized so that we can possibly obtain stronger stability features including A– and L– stability.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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APPENDIX A

RK	Classical Runge-Kutta Method.
h	Step-size Tolerance
LTE	Local truncation error
GTE	Global truncation error
est	Estimate Error
len	Local Error
CoMHM	Contra-Harmonic and Harmonic means
4sHERK	Four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic Explicit mean
MCHW	Four-stage fourth-order nonlinear weighted Runge-Kutta technique based on Contra-Harmonic mean
CoM	Four-stage fourth-order nonlinear Runge-Kutta technique based on Contra-Harmonic mean
HM	Four-stage fourth-order nonlinear Runge-Kutta technique based on Harmonic mean