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Research Article

S-cotorsion modules and dimensions

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Abstract

Let R be a ring, S a multiplicative subset of R. An R-module M is said to be u-S-flat (u-always abbreviates uniformly) if $\mathrm{Tor}_1^R(M,N)$ is u-S-torsion R-module for all R-modules N. In this paper, we introduce and study the concept of S-cotorsion module which is in some way a generalization of the notion of cotorsion module. An R-module M is said to be S-cotorsion if $\mathrm{Ext}_R^1(F,M) = 0$ for any u-S-flat module F. This new class of modules will be used to characterize u-S-von Neumann regular rings. Hence, we introduce the S-cotorsion dimensions of modules and rings. The relations between the introduced dimensions and other (classical) homological dimensions are discussed. As applications, we give a new upper bound on the global dimension of rings.

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1. Introduction

Throughout, all rings considered are commutative with unity, all modules are unital and S always is a multiplicative subset of R, that is, $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$. Let R be a ring and M an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension, and flat dimension of M, and $\mathrm{wdim}(R)$ and $\mathrm{gldim}(R)$ to denote, respectively, the weak and global homological dimensions of R.

Recall from [1], a ring R is called S-Noetherian if any ideal of R is S-finite. An R-module M is called S-finite provided that $sM \subseteq F$ for some $s \in S$ and some finitely generated submodule F of M. In [10], X. Zhang defined an R-module M is said to be a u-S-torsion if $sT = 0$ for some $s \in S$. So an R-module M is S-finite if and only if M/F is u-S-torsion for some finitely generated submodule F of M. Also, the author of this paper introduced the class of u-S-flat modules F for which the functor $F \otimes_R -$ preserves u-S-exact sequences. The class of u-S-flat modules can be seen as a uniform generalization of that of flat modules, since an R-module F is u-S-flat if and only if $\mathrm{Tor}_1^R(F,M)$ is

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u-S-torsion for any R-module M. The class of u-S-flat modules owns the following u-S-hereditary property: let $0 \to A \to B \to C \to 0$ be a u-S-exact sequence, if B and C are u-S-flat so is A (see [[10], Proposition 3.4]). So it is worth to study the u-S-analogue of flat dimensions of R-modules and u-S-analogue of weak global dimension of commutative rings.

Recall from [10] that an R-sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ is called u-S-exact (at N) provided that there is an element $s \in S$ such that $s\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ and $s\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$. We say a long R-sequence $\cdots \longrightarrow A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \longrightarrow \dots$ is u-S-exact, if for any n there is an element $s \in S$ such that $s\mathrm{Ker}(f_{n+1}) \subseteq \mathrm{Im}(f_n)$ and $s\mathrm{Im}(f_n) \subseteq \mathrm{Ker}(f_{n+1})$. A u-S-exact sequence $0 \to A \to B \to C \to 0$ is called a short u-S-exact sequence. An R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) provided $0 \to M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \to 0$, $0 \to M \xrightarrow{f} N \to 0$) is u-S-exact. It is easy to verify an R-homomorphism $f : M \to N$ is a u-S-monomorphism (resp., u-S-epimorphism, u-S-isomorphism) if and only if $\mathrm{Ker}(f)$ (resp., $\mathrm{Coker}(f)$, both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$) is a u-S-torsion module.

In [11], author introduced the u-S-flat dimensions of modules and rings. Let R be a ring, S a multiplicative subset of R and n be a positive integer. We say that an R-module has a u-S-flat dimension less than or equal to n, u-S-$\mathrm{fd}_R(M) \leq n$, if $\mathrm{Tor}^R_{n+1}(M, N)$ is u-S-torsion R-module for all R-modules N. Hence, the u-S-weak global dimension of R is defined to be

$$u\text{-}S\text{-w.gl.dim}(R) = \sup\{u\text{-}S\text{-fd}_R(M) \mid \ M \text{ is an } R\text{-module}\}.$$

The class $\mathfrak{X}$ is usually called the right orthogonal complement (relative to the functor $\mathrm{Ext}^1_R(-,-)$) of the class $\mathfrak{X}$. Set $\mathfrak{F}$ the class of all flat R-modules. The modules M with $M \in \mathfrak{F}(R)^{\perp}$, called cotorsion modules [5], have been investigated and successfully used in the progress of settling the "flat cover conjecture ", which was conjectured by Enochs in [4]. It is now well known that all R-modules have flat covers for any ring R [2]. The class of cotorsion modules contains all pure injective (hence, injective) modules. In [3], Ding and Mao introduced the cotorsion dimensions of modules and rings; the cotorsion dimension of an R-module M, denoted by $\mathrm{cd}_R(M)$, is the least positive integer n for which $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for all flat R-modules F, and the global cotorsion dimension of R, denoted by $\mathrm{cot.D}(R)$, is defined as the supremum of the cotorsion dimensions of R-modules. The global cotorsion dimension of rings measures how far away a ring is from being perfect. The global cotorsion dimension of rings is also used to give an upper bound on the global dimension of rings as follows [[3], Theorem 7.2.11]: For any ring R, we have the inequality

$$\mathrm{gl.dim}(R) \leqslant \mathrm{w.gl.dim}(R) + \mathrm{cot.D}(R).$$

In Section 2, we study the elements of the right orthogonal complement of the class of all u-S-flat modules, called the class of S-cotorsion modules, and prove that this new class is injectively resolving and it is strictly contained in the class of cotorsion modules. We also prove that a ring is strongly S-perfect if and only if every module is S-cotorsion if and only if every u-S-flat module is S-cotorsion. Hence, we give a new description of the u-S-weak global dimension of rings.

In section 3 we introduce and characterize a dimension, called the S-cotorsion dimension, for modules and rings. The relations between the S-cotorsion dimension and the other dimensions are discussed. Moreover, we use the global S-cotorsion dimension to give an upper bound on the global dimension of rings. Many illustrative examples are given.

2. S-cotorsion modules

In this section, we study the right orthogonal complement relative to the functor $\mathrm{Ext}^1_R(-,-)$ of the class of all u-S-flat modules. We begin with the following definition.

Definition 2.1. Let R be a ring and S a multiplicative subset of R. An R-module M is called S-cotorsion if $\operatorname{Ext}_R^1(F, M) = 0$ for any u-S-flat R-module F.

Since every flat module is u-S-flat, we have the following inclusions:

$$\{\text{injective modules}\} \subseteq \{S\text{-cotorsion modules}\} \subseteq \{\text{cotorsion modules}\}$$

If any element in S is a unit, so every cotorsion module is S-cotorsion (since every u-S-flat module is flat (see [10])). Moreover, using [6, Corollary 2.11], it is easy to see that over a von Neumann regular ring, the three classes of modules above coincide.

Recall from [8], let S be a multiplicatively closed set of R. An R-module M is said to be S-torsion-free if $sx = 0$, for $s \in S$ and $x \in M$, implies $x = 0$.

Proposition 2.2. *Let R be a ring and S a multiplicative subset of R. The R-module $\operatorname{Hom}_R(M, E(N))$ is S-cotorsion for any R-module M and any S-torsion-free R-module N. In particular, if R is a domain, $\operatorname{Hom}_R(M, Q)$ is an S-cotorsion R-module for any R-module M with Q is the quotient field of R.*

Proof. By [[8], Theorem 3.4.11], for any u-S-flat R-module F we have the isomorphism

$$\operatorname{Ext}_R^1(F, \operatorname{Hom}_R(M, E(N))) \cong \operatorname{Hom}_R(\operatorname{Tor}_1^R(F, M), E(N))$$

On the other hand, by [[8], Exercise 2.34], $E(N)$ is an S-torsion-free R-modul and by [[10], Theorem 3.2], $\operatorname{Tor}_1^R(F, M)$ is a u-S-torsion R-module. Thus, by [[10], Proposition 2.5], $\operatorname{Hom}_R(\operatorname{Tor}_1^R(F, M), E(N)) = 0$, and so $\operatorname{Ext}_R^1(F, \operatorname{Hom}_R(M, E(N))) = 0$. Hence, $\operatorname{Hom}_R(M, E(N))$ is an S-cotorsion R-module. The particular case follows from the facts that Q is an S-torsion-free R-module (by [[8], Example 1.6.12]) and is an injective R-module (since R is a domain). □

Recall from [[8], Definition 1.6.10] that an R-module M is called an S-torsion module if for any $m \in M$, there is an $s \in S$ such that $sm = 0$. It is known (from [[9], Theorem 3.3.2]) that the weak global dimension of a ring R is determined by the injective dimensions of its cotorsion modules. Next, we give an analogue result for the u-S-weak global dimension of rings.

Theorem 2.3. *Let R be a ring and S a multiplicative subset of R consisting of finite elements. The following conditions are equivalent for any integer $n \geq 0$:*

1. $\operatorname{id}_R(C) \leqslant n$ *for any S-cotorsion R-module C.*
2. $\operatorname{id}_R(\operatorname{Hom}_R(M, E(N))) \leqslant n$ *for any R-module M and any S-torsion-free R-module N.*
3. *u-S-w.gl.dim$(R) \leqslant n$*

Proof. (1) $\Rightarrow$ (2). Follows from Proposition 2.2.
(2) $\Rightarrow$ (3). Let K be an R-module. Using [[8], Theorem 3.4.11], for any R-module M and any S-torsion-free R-module N, we have the isomorphism

$$0 = \operatorname{Ext}_R^{n+1}(K, \operatorname{Hom}_R(M, E(N))) \cong \operatorname{Hom}_R(\operatorname{Tor}_{n+1}^R(K, M), E(N)).$$

Now, using the monomorphism

$$0 \to \operatorname{Hom}_R(Tor_{n+1}^R(K, M), N) \to \operatorname{Hom}_R(\operatorname{Tor}_{n+1}^R(K, M), E(N))$$

we deduce that $\operatorname{Hom}_R(\operatorname{Tor}_{n+1}^R(K, M), N) = 0$. Hence, $\operatorname{Tor}_{n+1}^R(K, M)$ is S-torsion, and so u-S-torsion R-module by [[10], Proposition 2.3]. Thus, u-S-$\operatorname{fd}_R(K) \leqslant n$. Consequently, u-S-w.gl.dim$(R) \leqslant n$ by [[11], Proposition 3.2].
(3) $\Rightarrow$ (1). Let C be an arbitrary S-cotorsion R-module. For any R-module M, consider an exact sequence

$$0 \to F \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where $P_0, \ldots, P_{n-1}$ are projective and so, by [[11], Proposition 3.2], F is u-S-flat. Then,

$$\mathrm{Ext}_R^{n+1}(M, C) = \mathrm{Ext}_R^1(F, C) = 0$$

Thus, $\mathrm{id}_R(C) \leqslant n$. □

Let R be a ring. Recall that a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called pure exact if $0 \to X \otimes A \to X \otimes B \to X \otimes C \to 0$ is exact for every R-module X. In this case we also say that A is a pure sub-module of B. A module M is called pure injective if the sequence $\mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M) \to 0$ is exact whenever A is a pure sub-module of B. It is easily seen that pure injective modules are cotorsion (see [9] for more details). However, there are pure injective modules which are not S-cotorsion. Otherwise, kipping in mind [[9], Theorem 3.3.2] and Theorem 2.3 and the fact that u-S-w.gl.dim$(R) \leqslant$ w.gl.dim(R) for any ring R, we will have u-S-w.gl.dim$(R) =$ w.gl.dim(R) (for any ring R), which is not true in general (see, [[11], Example 3.11]).

X. Zhang in [10], defined the u-S-von Neumann regular ring as follows: Let R be a ring and S a multiplicative subset of R. R is called a u-S-von Neumann regular ring provided there exists an element $s \in S$ satisfies that for any $a \in R$ there exists $r \in R$ such that $sa = ra^2$. Thus by [[10], Theorem 3.13], R is a u-S-von Neumann regular ring if and only if every R-module is u-S-flat

Corollary 2.4. *Let R be a ring and S a multiplicative subset of R consisting of finite elements. The following conditions are equivalent for any ring R.*

1. *R is a u-S-von Neumann regular ring.*
2. *Every S-cotorsion R-module is injective.*
3. *For any R-module M and any S-torsion-free R-module N, $\mathrm{Hom}_R(M, E(N))$ is an injective R-module.*
4. *u-S-w.gl.dim$(R) = 0$*

Proof. Follows from Theorem 2.3 and [[11], Corollary 3.8]. □

Proposition 2.5. *Let R be a ring, S a multiplicative subset of R and M an R-module. Then, M is S-cotorsion if and only if $\mathrm{Ext}_R^{i+1}(F, M) = 0$ for each u-S-flat module F and for each positive integer i.*

Proof. Suppose M is an S-cotorsion R-module and let F be a u-S-flat R-module. For any positive integer $n > 0$, consider an exact sequence with the form

$$0 \to F' \to P_{n-1} \to \cdots \to P_0 \to F \to 0$$

where $P_0, \ldots, P_{n-1}$ are projective, and so necessarily F' is u-S-flat. Thus, $\mathrm{Ext}_R^{n+1}(F, M) = \mathrm{Ext}_R^1(F', M) = 0$. The case $i = 0$ is just the definition of S-cotorsion modules. The other implication is obvious. □

Proposition 2.6. *If B is a submodule of an S-cotorsion module C such that $\mathrm{id}_R(B) \leq n$, then C/B is S-cotorsion.*

Proof. Let $0 \to B \to C \to C/B \to 0$ be an exact sequence, Let F be a u-S-flat module. We have the exact sequence

$$\mathrm{Ext}_R^n(F, C) \to \mathrm{Ext}_R^n(F, C/B) \to \mathrm{Ext}_R^{n+1}(F, B).$$

Then we have $\mathrm{Ext}_R^n(F, C) = 0$ (since C is S-cotorsion and by Proposition 2.5) and $\mathrm{Ext}_R^{n+1}(F, B) = 0$ (since $\mathrm{id}_R(B) \leq n$). Then $\mathrm{Ext}_R^n(F, C/B) = 0$, which implies that C/B is S-cotorsion by Proposition 2.5 again. □

Let $\mathcal{X}$ be a class of R-modules. $\mathcal{X}$ is injectively resolving provided that $\mathcal{X}$ contains all injective modules, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{X}$ the conditions $X \in \mathcal{X}$ and $X'' \in \mathcal{X}$ are equivalent.

Proposition 2.7. *The class of all S-cotorsion modules is injectively resolving. Furthermore, it is closed under arbitrary direct products and under direct summands.*

Proof. The facts that the class of all S-cotorsion modules is closed under arbitrary direct products and under direct summands follow easily from [[8], Theorem 3.3.9]. Now, let $0 \to C' \to C \to C'' \to 0$ be an exact sequence where C' is S-cotorsion. Then, for each u-S-flat R-module F, we have the exact sequence

$$0 = \mathrm{Ext}^1_R(F, C') \to \mathrm{Ext}^1_R(F, C) \to \mathrm{Ext}^1_R(F, C'') \to \mathrm{Ext}^2_R(F, C') = 0$$

Hence, we deduce that C is S-cotorsion if and only if C'' is S-cotorsion. □

Lemma 2.8. *Let R be a ring and S a multiplicative subset of R. If A is a flat R-module and B a u-S-flat R-module. Then, $A \otimes_R B$ is u-S-flat R-module.*

Proof. Let F be an R-module. Hence, by [[8], Theorem 3.4.10], we have the isomorphism

$$\mathrm{Tor}^R_1(F, A \otimes_R B) \cong A \otimes_R \mathrm{Tor}^R_1(F, B).$$

For some $s \in S$ we have

$$s\mathrm{Tor}^R_1(F, A \otimes_R B) \cong s(A \otimes_R \mathrm{Tor}^R_1(F, B)) = A \otimes_R s\mathrm{Tor}^R_1(F, B).$$

Since B is a u-S-flat, $\mathrm{Tor}^R_1(F, B)$ is a u-S-torsion, so $s\mathrm{Tor}^R_1(F, B) = 0$ for some s. Thus,

$$s\mathrm{Tor}^R_1(F, A \otimes_R B) \cong A \otimes_R 0.$$

Hence, $\mathrm{Tor}^R_1(F, A \otimes_R B)$ is a u-S-torsion. Then, $A \otimes_R B$ is a u-S-flat. □

Next, we give some characterizations of S-cotorsion modules.

Proposition 2.9. *Let R be a ring, S a multiplicative subset of R and M an R-module. The following conditions are equivalent:*

(1) *M is S-cotorsion.*
(2) *$\mathrm{Hom}_R(F, M)$ is an S-cotorsion R-module for any flat R-module F.*
(3) *$\mathrm{Hom}_R(P, M)$ is an S-cotorsion R-module for any projective R-module P.*
(4) *Every exact sequence of R-modules $0 \to M \to B \to F \to 0$ with F is u-S-flat is splits.*
Moreover, if the class of S-cotorsion R-modules is closed under direct sums, then the above conditions are also equivalent to
(5) *$P \otimes_R M$ is an S-cotorsion R-module for any projective R-module P.*

Proof. (1) ⇒ (2). Let F be a flat R-module. For any u-S-flat R-module N, $N \otimes_R F$ is u-S-flat by Lemma 2.8.
Now, there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective (and so K is u-S-flat by [[10], Proposition 3.4]), which yields the exactness of the sequence

$$0 \to K \otimes_R F \to P \otimes_R F \to N \otimes_R F \to 0$$

We have the following exact sequence

$$\mathrm{Hom}_R(P \otimes_R F, M) \to \mathrm{Hom}_R(K \otimes_R F, M) \to \mathrm{Ext}^1_R(N \otimes_R F, M) = 0$$

which gives rise to the exactness of the sequence

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(F, M)) \to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M)) \to 0$$

On the other hand, the following sequence

$$\begin{aligned}\mathrm{Hom}_R(P, \mathrm{Hom}_R(F, M)) &\to \mathrm{Hom}_R(K, \mathrm{Hom}_R(F, M))\\ &\to \mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M))\\ &\to \mathrm{Ext}^1_R(P, \mathrm{Hom}_R(F, M)) = 0\end{aligned}$$

is exact. Hence, $\mathrm{Ext}^1_R(N, \mathrm{Hom}_R(F, M)) = 0$, and (2) follows.
(2) ⇒ (3). Trivial.
(3) ⇒ (1) and (5) ⇒ (1). Follows by letting $P = R$.
(1) ⇔ (4). Follows from [[8], Theorem 3.3.5].
(1) ⇒ (5). Let F be a u-S-flat R-module and P a projective R-module. As P projective, there exists a projective module P' such that $R^{(I)} = P \oplus P'$ for some index set I. Now we have $\mathrm{Ext}^1_R(F, M) = 0$ so $\mathrm{Ext}^1_R(F, R \otimes M) = 0$. Hence

$$\begin{aligned} \oplus_I \mathrm{Ext}^1_R(F, R \otimes M) &\cong \mathrm{Ext}^1_R(F, (P \oplus P') \otimes M) \\ &\cong \mathrm{Ext}^1_R(F, (P \otimes M) \oplus (P' \otimes M)) \\ &\cong \mathrm{Ext}^1_R(F, P \otimes M) \oplus \mathrm{Ext}^1_R(F, P' \otimes M) = 0 \end{aligned}$$

That is $\mathrm{Ext}^1_R(F, P \otimes M) = 0$ and $P \otimes M$ is S-cotorsion. □

Recall that a ring is called a perfect if every flat module is projective. It is proved in [[3], Corollary 7.2.7] that a ring is perfect if and only if every module is cotorsion. Next, we introduce a new class of rings, which is a u-S-version of perfect rings.

Definition 2.10. Let R be a ring and S a multiplicative subset of R. A ring R is called a strongly S-perfect ring if every u-S-flat is projective.

We have the following inclusions:

$$\{\text{semisimple rings}\} \subseteq \{\text{strongly } S\text{-perfect rings}\} \subseteq \{\text{perfect rings}\}.$$

If S is composed of units, then perfect rings and strongly S-perfect rings coincide. If $0 \in S$, then every R-module is u-S-flat, so strongly S-perfect rings and semi-simple rings coincide. And if R is a von Neumann regular ring, the three classes of rings above coincide.

In the following result, we give a characterization of strongly S-perfect rings by using S-cotorsion modules.

Proposition 2.11. *Let R be a ring and S a multiplicative subset of R. The following conditions are equivalent:*

(1) *Every R-module is S-cotorsion.*
(2) *Every u-S-flat R-module is S-cotorsion.*
(3) *R is strongly S-perfect.*

Proof. (1) ⇒ (2). Obvious.
(2) ⇒ (3). Let M be a u-S-flat R-module and pick a short exact sequence $0 \to N \to P \to M \to 0$, where P is projective, and so N is also u-S-flat by [[10], Proposition 3.4]. Thus, by hypothesis, $\mathrm{Ext}^1_R(M, N) = 0$. Hence, M is a direct summand of P, and so projective. Hence, R is strongly S-perfect.
(3) ⇒ (1). Let M be an R-module. For any u-S-flat R-module F, we have $\mathrm{Ext}^1_R(F, M) = 0$ since F is projective by (3). Hence, M is S-cotorsion. □

We know every S-cotorsion module is cotorsion. In the next Proposition, we prove that every cotorsion module is S-cotorsion if R is strongly S-perfect.

Proposition 2.12. *Let R be a ring and S a multiplicative subset of R. The following conditions are equivalent:*

(1) *Every cotorsion module is S-cotorsion.*
(2) *Every u-S-flat module is flat*
Moreover, if R is perfect ring, the above conditions are also equivalent to
(3) *R is strongly S-perfect.*

Proof. (1) $\Rightarrow$ (2). Recall that it is proved now that all R-modules have flat covers for any ring R (see, [2]). Let M be a u-S-flat R-module and let $f : F \to M$ be a flat cover of M which is certainly an epimorphism. Using [[9], Lemma 2.1.1], $K = \ker(f)$ is a cotorsion R-module, and so an S-cotorsion R-module. Thus, $\mathrm{Ext}^1_R(M, K) = 0$. Therefore, M is a direct summand of F, and so it is flat.
(2) $\Rightarrow$ (1). Let M be a cotorsion R-module. For any u-S-flat R-module F, we have $\mathrm{Ext}^1_R(F, M) = 0$, since F is flat by (2). Hence, M is an S-cotorsion.
(3) $\Leftrightarrow$ (2). By definition of strongly S-perfect ring and since R is perfect. □

Corollary 2.13. *Let R be a ring and S a multiplicative subset of R containing the zero. Then, R is strongly S-perfect ring if and only if every u-S-absolutely pure R-module is S-cotorsion.*

Proof. ($\Rightarrow$). Follows from Proposition 2.12.
($\Leftarrow$). Let M be a u-S-flat R-module, for any R-module N, we have $\mathrm{Ext}^1_R(M, N) = 0$ since N is a u-S-absolutely pure (since, $0 \in S$) and so N is a S-cotorsion R-module by hypothesis. Thus, M is projective which implies that R is strongly S-perfect ring. □

3. The S-cotorsion dimensions of modules and rings

In this section, we introduce and investigate the S-cotorsion dimensions of modules and rings and we study its properties and we give its characterization. We begin with the following definition.

Definition 3.1. Let R be a ring and S a multiplicative subset of R. For any R-module M, the S-cotorsion dimension of M, denoted by $S\text{-cd}_R(M)$, is the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for any u-S-flat R-module F. If no such integer exists, set $S\text{-cd}_R(M) = \infty$.
The global S-cotorsion dimension of R is defined by:

$$S\text{-cot.D}(R) = \sup\{S\text{-cd}_R(M) : \text{ M is an } R\text{-module}\}$$

Remark 3.2. Then following statements hold.

(1) $S\text{-cd}_R(M) = 0$ if and only if M is S-cotorsion.
(2) $\mathrm{cd}(M) \leq S\text{-cd}(M) \leq \mathrm{id}_R(M)$, where $\mathrm{cd}(M)$ is the cotorsion dimension of M. The equality if R is von Neumann regular ring by [6, Corollary 2.11].

We need the following lemma.

Lemma 3.3. *Consider an exact sequence*

$$0 \to M \to C_0 \to C_1 \to \ldots \to C_{n-1} \to K_n \to 0$$

where $C_0, \ldots, C_{n-1}$ are S-cotorsion, then

$$\mathrm{Ext}^{i+n}_R(F, M) \cong \mathrm{Ext}^i_R(F, K_n)$$

for all u-S-flat R-module F and all integers $i > 0$.

Proof. We proceed by induction on $n \geq 1$. If $n = 1$, for each u-S-flat R-module F, we have the exact sequence

$$0 = \mathrm{Ext}^i_R(F, C_0) \to \mathrm{Ext}^i_R(F, K_1) \to \mathrm{Ext}^{i+1}_R(F, M) \to \mathrm{Ext}^{i+1}_R(F, C_0) = 0$$

Hence, $\mathrm{Ext}^i_R(F, K_1) \cong \mathrm{Ext}^{i+1}_R(F, M)$.
Next we assume that $n > 1$, and set $K_{n-1} = \ker(C_{n-1} \to K_n)$. Applying the induction hypothesis to the exact sequences

$$0 \to M \to C_0 \to C_1 \to \cdots \to C_{n-2} \to K_{n-1} \to 0$$

and

$$0 \to K_{n-1} \to C_{n-1} \to K_n \to 0,$$

we have

$$\mathrm{Ext}_R^i(F, K_n) \cong \mathrm{Ext}_R^{i+1}(F, K_{n-1}) \cong \mathrm{Ext}_R^{i+n}(F, M).$$

Hence, we have the desired result. □

Next, we give a description of the S-cotorsion dimension of modules.

Proposition 3.4. *Let R be a ring and S a multiplicative subset of R. For any R-module M and integer $n \geq 0$, the following are equivalent:*

(1) *S-cd$(M) \leqslant n$.*
(2) *$\mathrm{Ext}_R^{n+1}(F, M) = 0$ for any u-S-flat R-module F.*
(3) *$\mathrm{Ext}_R^{n+j}(F, M) = 0$ for any u-S-flat R-module F and $j \geq 1$.*
(4) *If the sequence $0 \to M \to C_0 \to C_1 \to \cdots \to C_{n-1} \to C_n \to 0$ is exact with $C_0, \ldots, C_{n-1}$ S-cotorsion, then C_n is also S-cotorsion.*
(5) *There exists an exact sequence $0 \to M \to C_0 \to C_1 \to \cdots \to C_{n-1} \to C_n \to 0$ where $C_0, \ldots, C_n$ are S-cotorsion.*

Proof. The implications (3) ⇒ (2) ⇒ (1) are trivial.
(1) ⇒ (4). Set S-cd$(M) = m$. Hence, by Lemma 3.3 for any u-S-flat R-module F we have $0 = \mathrm{Ext}_R^{m+1}(F, M) = \mathrm{Ext}_R^1(F, K_m)$ with $K_m = \mathrm{Coker}(C_{m-2} \to C_{m-1})$. Hence, K_m is an S-cotorsion R-module. On the other hand, following Proposition 2.7, the class of all S-cotorsion modules is injectively resolving. Thus, using the exact sequence

$$0 \to K_m \to C_m \to \cdots \to C_{n-1} \to K_n \to 0$$

we deduce that K_n is an S-cotorsion R-module.
(4) ⇒ (5). Consider an exact sequence

$$0 \to M \to I_0 \to \cdots \to I_{n-1} \to C \to 0$$

where $I_0, \ldots, I_{n-1}$ are injective, and so S-cotorsion. Then, C is an S-cotorsion R-module, and so we have the desired S-cotorsion resolution of M.
(5) ⇒ (3). By Proposition 2.5 and Lemma 3.3, for any u-S-flat R-module F and any integer $j \geq 1$, we have $\mathrm{Ext}_R^{n+j}(F, M) = \mathrm{Ext}_R^j(F, C_n) = 0$. □

The proof of the next proposition is standard homological algebra. Thus we omit its proof.

Proposition 3.5. *Let R be a ring and S a multiplicative subset of R, $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If two of S-cd$_R(A)$, S-cd$_R(B)$, and S-cd$_R(C)$ are finite, so is the third. Moreover*

(1) *S-cd$_R(B) \leqslant \sup\{S\text{-cd}_R(A), S\text{-cd}_R(C)\}$.*
(2) *S-cd$_R(A) \leqslant \sup\{S\text{-cd}_R(B), S\text{-cd}_R(C) + 1\}$.*
(3) *S-cd$_R(C) \leqslant \sup\{S\text{-cd}_R(B), S\text{-cd}_R(A) - 1\}$.*

Corollary 3.6. *Let R be a ring and S a multiplicative subset of R, $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If B is S-cotorsion and S-cd$_R(A) > 0$, then S-cd$_R(A) = S$-cd$_R(C) + 1$.*

In the next result, we use the global S-cotorsion dimension to give an upper bound on the global dimension of rings.

Theorem 3.7. *Let R be a ring and S a multiplicative subset of R. We have*

$$\begin{aligned} S\text{-cot.D}(R) &= \sup\{\mathrm{pd}_R(F) \mid F \textit{ is } S\textit{-flat}\} \\ &\leqslant \mathrm{gl.dim}(R) \\ &\leqslant S\text{-cot.D}(R) + u\text{-}S\text{-w.gl.dim}(R) \end{aligned}$$

In particular:

- *If* S-cot.D$(R) = 0$ *(i.e* R *is strongly* S*-perfect), then* gl.dim$(R) = u$-S-w.gl.dim(R).
- *If* u-S-w.gl.dim$(R) = 0$ *(i.e* R *is* u-S*-von Neumann regular), then* gl.dim$(R) = S$-cot.D(R).

Proof. Assume that $\sup\{\mathrm{pd}_R(F) \mid F \text{ is } S\text{-flat}\} = n < \infty$ and let M be an arbitrary R-module. For any u-S-flat R-module F, we have $\mathrm{Ext}_R^{n+1}(F, M) = 0$. Hence, S-cd$(M) \leqslant n$. So, S-cot.D$(R) \leqslant n$. Now, assume that S-cot.D$(R) = n < \infty$ and let F be a u-S-flat R-module. For any R-module M, we have $\mathrm{Ext}_R^{n+1}(F, M) = 0$ since S-cd$(M) \leqslant n$. Thus, $\mathrm{pd}_R(F) \leqslant n$. Consequently, $\sup\{\mathrm{pd}_R(F) \mid F \text{ is } S\text{-flat}\} \leqslant n$. Thus, S-cot.D$(R) = \sup\{\mathrm{pd}_R(F) \mid F \text{ is } S\text{-flat}\}$.
The inequality $\sup\{\mathrm{pd}_R(F) \mid F \text{ is } S\text{-flat}\} \leqslant \mathrm{gl.dim}(R)$ is trivial.
To prove the last inequality, we can assume that S-cot.D$(R) = n$ and u-S-w.gl.dim$(R) = m$ are finite. Let M be an arbitrary R-module. For an R-module N, consider an exact sequence

$$0 \to F \to P_{m-1} \to \cdots \to P_0 \to N \to 0$$

where P_i are projective, and then F is u-S-flat (by [[11], Proposition 2.3]) since u-S-fd$(N) \leqslant m$). We have $\mathrm{Ext}_R^{n+m+1}(N, M) = \mathrm{Ext}_R^{n+1}(F, M) = 0$ since S-cd$_R(M) \leqslant n$. Consequently, $\mathrm{id}_R(M) \leqslant n + m$. Hence, we have the desired inequality. □

Next, we characterize the rings with global S-cotorsion dimension less than or equal to one.

Proposition 3.8. *Let* R *be a ring and* S *a multiplicative subset of* R*. The following conditions are equivalent:*

1. S-cot.D$(R) \leq 1$
2. *Every quotient of an* S*-cotorsion* R*-module is* S*-cotorsion.*
3. *Every quotient of an injective* R*-module is* S*-cotorsion.*
4. *Every* u-S*-flat* R*-module is of projective dimension* ≤ 1.
5. *For any* u-S*-pure (*u-S*)-exact sequence* $0 \to B \to A \to C \to 0$ *with* A *projective* R*-module,* B *is projective.*

Proof. (2) ⇒ (5). Let $0 \to B \to A \to C \to 0$ be an u-S-pure (u-S)-exact sequence with A projective R-module. Then, C is u-S-flat by [[12], Proposition 2.5]. For any R-module M, there exists an exact sequence $0 \to M \to E \to N \to 0$ with E injective. Note that N is S-cotorsion by (1), and hence $\mathrm{Ext}_R^2(C, M) \cong \mathrm{Ext}_R^1(C, N) = 0$. Thus, $\mathrm{pd}_R(C) \leq 1$, so B is projective.
(5) ⇒ (4). Let A be any u-S-flat R-module. There exists an exact sequence $0 \to B \to P \to A \to 0$ with P projective. Since $\mathrm{Tor}_1^R(A, M) \to B \otimes M \to P \otimes M \to A \otimes M \to 0$ is exact for any R-module M, this sequence is u-S-pure since $\mathrm{Tor}_1^R(A, M)$ is u-S-torsion. Thus, B is projective by (3). It follows that $\mathrm{pd}_R(A) \leq 1$.
(4) ⇒ (2). Let A be any S-cotorsion R-module and C a submodule of A. For any u-S-flat R-module B, the exactness of the sequence $0 \to C \to A \to A/C \to 0$ induces the exact sequence $0 = \mathrm{Ext}_R^1(B, A) \to \mathrm{Ext}_R^1(B, A/C) \to \mathrm{Ext}_R^2(B, C)$. By (2), $\mathrm{Ext}_R^2(B, C) = 0$, so $\mathrm{Ext}_R^1(B, A/C) = 0$. Hence, A/C is an S-cotorsion module.
(2) ⇒ (3). trivial.
(3) ⇒ (1). Let M be any R-module. Using the exact sequence $0 \to M \hookrightarrow E(M) \to E(M)/M \to 0$ where $E(M)$ is the injective envelope of M, it is clear that S-cd$_R(M) \leqslant 1$. Hence, S-cot.D$(R) \leq 1$.
(1) ⇔ (4). Follows from Theorem 3.7. □

Proposition 3.9. *Let* R_1 *and* R_2 *be two rings and* M_1 *and* M_2 *be* R_1*-module and* R_2*-module, respectively. Set* $S = S_1 \times S_2$ *Then,*

$$S\text{-}cd_{R_1 \times R_2}(M_1 \times M_2) = \sup\{S_1\text{-}cd_{R_1}(M_1), S_2\text{-}cd_{R_2}(M_2)\}$$

Proof. Let n be a positive integer.
Suppose $S\text{-cd}_{R_1\times R_2}(M_1\times M_2)\leq n$ and consider a u-S-flat R_1-module F. Then, by [[7], Theorem 10.75],

$$\mathrm{Ext}^{n+1}_{R_1}(F,M_1)\cong \mathrm{Ext}^{n+1}_{R_1}(F,\mathrm{Hom}_{R_1\times R_2}(R_1,M_1\times M_2))\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(F,M_1\times M_2)$$

Then, $\mathrm{Ext}^{n+1}_{R_1\times R_2}(F,M_1\times M_2)=0$ since $S\text{-cd}_{R_1\times R_2}(M_1\times M_2)\leq n$, then $\mathrm{Ext}^{n+1}_{R_1}(F,M_1)=0$. Hence, $S_1\text{-cd}_{R_1}(M_1)\leq n$ by Proposition 3.4. Similarly, $S_2\text{-cd}_{R_2}(M_2)\leq n$. Consequently, $\sup\{S\text{-cd}_{R_1}(M_1),S\text{-cd}_{R_2}(M_2)\}\leq S\text{-cd}_{R_1\times R_2}(M_1\times M_2)$.
Now, Suppose $\sup\{S_1\text{-cd}_{R_1}(M_1),S_2\text{-cd}_{R_2}(M_2)\}\leq n$. Let F be a u-S-flat $R_1\times R_2$-module, and $F_i=F\otimes R_i$ for $i=1,2$. It is clear that $F\cong F_1\times F_2$ and F_1, F_2 are u-S_1-flat R_1-module and u-S_2-flat R_2-module, respectively . On the other hand, by [[7], Theorem 10.75]

$$\begin{aligned}
&\mathrm{Ext}^{n+1}_{R_1}(F_1,M_1)\times \mathrm{Ext}^{n+1}_{R_2}(F_2,M_2)\\
&\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(F_1,M_1\times M_2)\times \mathrm{Ext}^{n+1}_{R_1\times R_2}(F_2,M_1\times M_2)\\
&\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(F_1\times 0,M_1\times M_2)\times \mathrm{Ext}^{n+1}_{R_1\times R_2}(0\times F_2,M_1\times M_2)\\
&\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(F_1\times F_1,M_1\times M_2)\\
&\cong \mathrm{Ext}^{n+1}_{R_1\times R_2}(F,M_1\times M_2).
\end{aligned}$$

On the hand, $\mathrm{Ext}^{n+1}_{R_1}(F_1,M_1)=0=\mathrm{Ext}^{n+1}_{R_2}(F_2,M_2)$, then $\mathrm{Ext}^{n+1}_{R_1\times R_2}(F,M_1\times M_2)=0$. Thus, $S\text{-cd}_{R_1\times R_2}(M_1\times M_2)\leq n$ by Proposition 3.4. Consequently,

$$S\text{-cd}_{R_1\times R_2}(M_1\times M_2)\leq \sup\{S_1\text{-cd}_{R_1}(M_1),S_2\text{-cd}_{R_2}(M_2)\}.$$

Finally, we have the desired equality. □

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