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Characterization of Tzitzéica Curves Using Positional Adapted Frame

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Abstract

In this study, Tzitzéica curves are taken into consideration in the Euclidean 3-space by using the Positional Adapted Frame (PAF). Such curves are characterized according to PAF elements. Also, some results are obtained on spherical Tzitzéica curves. The results obtained in this study are new contributions to the field. It is expected that these results will be useful in various application areas of differential geometry and applied mathematics in the future.

Keywords: *Differential equations; Kinematics of a particle; Positional Adapted Frame; Tzitzéica curves.*
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1. Introduction and Preliminaries

The theory of curves is a substantial concept of differential geometry, and several researchers have studied this concept from past to present. The curve theory attracts by researchers due to the various applications of it on different disciplines such as differential geometry, robotics, computer graphics and etc. Throughout this study, we consider the curves in the Euclidean 3-space.

Let the Euclidean 3-space E^3 be taken into account with the standard scalar product $\langle \rho, \upsilon \rangle = \rho_1\upsilon_1 + \rho_2\upsilon_2 + \rho_3\upsilon_3$ where $\rho = (\rho_1\,,\,\rho_2\,,\,\rho_3)$, $\upsilon = (\upsilon_1\,,\,\upsilon_2\,,\,\upsilon_3)$ are any vectors in space. The norm of ρ is given as $\|\rho\| = \sqrt{\langle \rho\,,\,\rho \rangle}$. If a differentiable curve $\alpha = \alpha(s) : I \subset \mathbb{R} \to E^3$ satisfies $\left\| \frac{d\alpha}{ds} \right\| = 1$ for every $s \in I$, it is called a unit speed curve. In this case, s is said to be arc-length parameter of α. A differentiable curve is called as regular curve if its derivative is never zero along the curve. All regular curves can be reparameterized by the arc-length of itself [12]. Throughout this work, we will show the differentiation with respect to the arc-length parameter s with a prime.

Suppose that a point particle of constant mass moves on a unit speed curve $\alpha = \alpha(s)$ in E^3. Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ show the Serret-Frenet frame of $\alpha = \alpha(s)$. $\mathbf{T}(s) = \alpha'(s)$, $\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|}$ and $\mathbf{B}(s) = \mathbf{T}(s) \wedge \mathbf{N}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. Also, the Serret-Frenet formulas are given as follows:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion function [12]. In the rest of the study, we assume everywhere $\kappa \neq 0$.

Until now, many researchers have developed new moving frames which have a common base vector with the Serret-Frenet frame (see [4, 13, 17] for some examples). One of the newest of them was developed in the study [10] by Özen and Tosun. They introduced the Positional Adapted Frame (PAF) for the trajectories with non-vanishing angular momentum in E^3. The same authors discussed the special trajectories generated by Smarandache curves according to PAF in Euclidean 3-space in [11]. Also, Gürbüz obtained PAF in Minkowski 3-space using the method given in [10] and investigated the evolution of an electric field with respect to this frame in $\mathbb{R}^3_1$ [8]. Then, in [14], Solouma investigated the characterization of some special Smarandache trajectory curves of moving point particles endowed with PAF.

The angular momentum vector of the aforementioned moving point particle about the origin is determined by vector product of the position vector and linear momentum vector of the moving point particle and given by $\mathbf{H}^O = m\langle \alpha(s), \mathbf{B}(s) \rangle \left(\frac{ds}{dt}\right) \mathbf{N}(s) - m\langle \alpha(s), \mathbf{N}(s) \rangle \left(\frac{ds}{dt}\right) \mathbf{B}(s)$

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where m and t indicate the mass and time, respectively. It is well known that when there is no net external forces, linear momentum is conserved. Similarly to this case when the net torque is zero, angular momentum is constant or conserved. Thus, in a closed system, the angular momentum is conserved always. Conservation of angular momentum is one of the most important conservation laws in physics [7]. Assume that the aforementioned angular momentum vector $\mathbf{H}^O$ does not equal to zero vector along the trajectory $\alpha = \alpha(s)$. This assumption warrants that the functions $\langle \alpha(s), \mathbf{N}(s) \rangle$ and $\langle \alpha(s), \mathbf{B}(s) \rangle$ do not equal to zero at the same time during the motion of the moving point particle. Thus, we can say that the tangent line of $\alpha = \alpha(s)$ never passes through the origin. In that case, there exists PAF denoted by $\{\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)\}$ along $\alpha = \alpha(s)$. Let us take into consideration the vector whose initial point is the foot of the perpendicular (from origin to instantaneous rectifying plane) and endpoint is the foot of the perpendicular (from origin to instantaneous osculating plane). The equivalent of this vector at the point $\alpha(s)$ helps us to determine the vector $\mathbf{Y}(s)$. Hence, $\mathbf{Y}(s)$ is calculated as (see [10] for more details):

$$\mathbf{Y}(s) = \frac{\langle -\alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{B}(s).$$

The second base vector of PAF is obtained by vector product $\mathbf{Y}(s) \wedge \mathbf{T}(s)$ as in the following:

$$\mathbf{M}(s) = \frac{\langle \alpha(s), \mathbf{B}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{N}(s) + \frac{\langle \alpha(s), \mathbf{N}(s) \rangle}{\sqrt{\langle \alpha(s), \mathbf{N}(s) \rangle^2 + \langle \alpha(s), \mathbf{B}(s) \rangle^2}} \mathbf{B}(s).$$

There is a relation between the Serret-Frenet frame and PAF as follows:

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\Omega(s) & -\sin\Omega(s) \\ 0 & \sin\Omega(s) & \cos\Omega(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{B}(s)$ and $\mathbf{Y}(s)$ which is positively oriented from $\mathbf{B}(s)$ to $\mathbf{Y}(s)$ (see Figure 1.1).

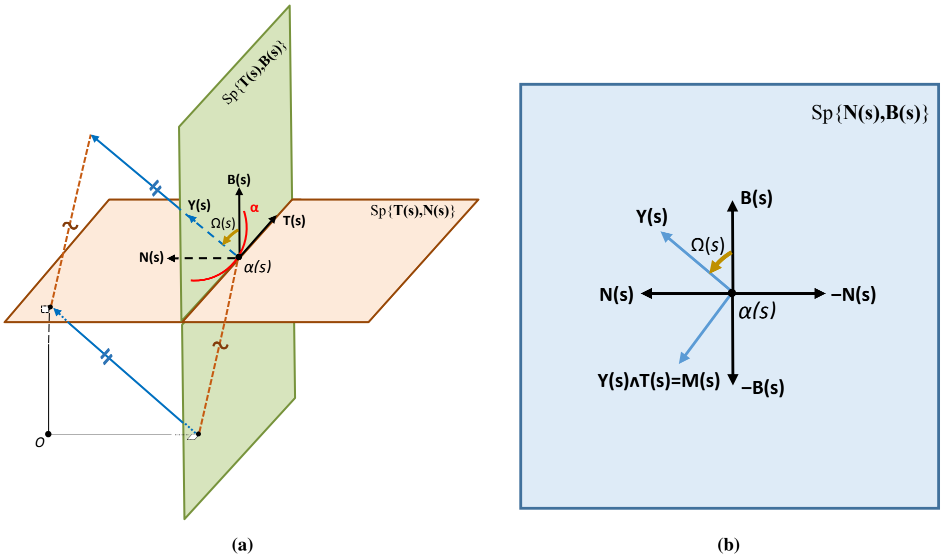

Figure 1.1: An illustration for the Positional Adapted Frame [10]

On the other hand, the derivative formulas of PAF are expressed as [10]:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{M}'(s) \\ \mathbf{Y}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & k_3(s) \\ -k_2(s) & -k_3(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{M}(s) \\ \mathbf{Y}(s) \end{pmatrix}$$

where

$$k_3(s) = \tau(s) - \Omega'(s)$$
$$k_1(s) = \kappa(s)\cos\Omega(s)$$
$$k_2(s) = \kappa(s)\sin\Omega(s).$$

The last two equations yield the followings:

$$\frac{k_2(s)}{k_1(s)} = \tan\Omega(s)$$
$$k_1(s) = \sqrt{k_1^2(s) + k_2^2(s)}\cos\Omega(s)$$
$$k_2(s) = \sqrt{k_1^2(s) + k_2^2(s)}\sin\Omega(s).$$

The aforementioned angle $\Omega(s)$ is calculated as in the following:

$$\Omega(s)=\begin{cases}\arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right) \; if \; \langle\alpha(s),\mathbf{B}(s)\rangle>0\\ \arctan\left(-\frac{\langle\alpha(s),\mathbf{N}(s)\rangle}{\langle\alpha(s),\mathbf{B}(s)\rangle}\right)+\pi \; if \; \langle\alpha(s),\mathbf{B}(s)\rangle<0\\ -\frac{\pi}{2} \; if \; \langle\alpha(s),\mathbf{B}(s)\rangle=0\,, \; \langle\alpha(s),\mathbf{N}(s)\rangle>0\\ \frac{\pi}{2} \; if \; \langle\alpha(s),\mathbf{B}(s)\rangle=0\,, \; \langle\alpha(s),\mathbf{N}(s)\rangle<0.\end{cases}$$

Any element of the set $\{\mathbf{T}(s),\mathbf{M}(s),\mathbf{Y}(s),k_1(s),k_2(s),k_3(s)\}$ is called PAF apparatus of $\alpha=\alpha(s)$ [10].

A class of surfaces and a class of curves which are nowadays called as Tzitzéica surfaces and Tzitzéica curves were introduced by Georges Tzitzéica in the early 1900s [15] and [16], respectively. A Tzitzéica curve in Euclidean 3-space is a spatial curve (with $\kappa>0$ and $\tau\neq 0$) for which the ratio of τ and the square of the distance from the origin to the osculating plane at an arbitrary point of this curve is a non-zero constant. Hence, the trajectory $\alpha=\alpha(s)$ of the aforesaid point particle is a Tzitzéica curve if

$$\frac{\tau}{\langle\alpha,\mathbf{B}\rangle^2}$$

is a constant function other than zero function. In the literature, Tzitzéica curves were studied widely by several researchers. In [5], depending upon the solution of the harmonic equation, Crasmareanu stated the elliptic and the hyperbolic cylindrical curves which satisfy the conditions of being the Tzitzéica curve. Moreover, Agnew et al. [1] studied on the Tzitzéica surfaces and curves. Furthermore, Bayram et al. [3] characterized the Tzitzéica curves considering the curvatures of these curves in E^3. These authors showed that the curves with constant curvatures are not Tzitzéica curves, and then the conditions of being Tzitzéica curves were examined with respect to the Salkowski and anti-Salkowski curves, as well [3]. In addition to this, Eren and Ersoy introduced the characterizations of these curves by using the Bishop frame in E^3 [6]. For more detailed information with respect to the Tzitzéica curves, we can refer to the studies [1, 2, 3, 5, 6, 9, 16] which are some of the studies on this topic.

In the rest of this paper, we continue to consider any moving point particle satisfying the aforementioned assumption (concerned with the angular momentum) and show the unit speed parameterization of the trajectory with $\alpha=\alpha(s)$. In the section 2, we characterize the Tzitzéica curves in terms of PAF apparatus and obtain some results on the spherical Tzitzéica curves. Also, we provide two examples for this characterization.

2. Tzitzéica Curves According to PAF

In this section, we give a characterization for the trajectory $\alpha=\alpha(s)$ to be a Tzitzéica curve. Also, we obtain some results on spherical Tzitzéica curves.

Note that if the trajectory $\alpha=\alpha(s)$ is a Tzitzéica curve, then $\langle\alpha(s),\mathbf{B}(s)\rangle\neq 0$ for all the values of the parameter s. So, PAF is well defined during the motion of this trajectory.

Theorem 2.1. *The trajectory $\alpha=\alpha(s)$ (with $\sqrt{k_1^2+k_2^2}>0$ and $k_3+\Omega'\neq 0$) is a Tzitzéica curve if and only if*

$$\frac{k_3+\Omega'}{\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2}$$

is a constant function other than zero function.

Proof. Using the well-known equalities $\tau=\frac{\langle\alpha'\wedge\alpha'',\alpha'''\rangle}{\|\alpha'\wedge\alpha''\|^2}$ and $\mathbf{B}=\frac{\alpha'\wedge\alpha''}{\|\alpha'\wedge\alpha''\|}$, one can immediately calculate the following:

$$\frac{\tau}{\langle\alpha,\mathbf{B}\rangle^2}=\frac{\langle\alpha'\wedge\alpha'',\alpha'''\rangle}{\langle\alpha,\alpha'\wedge\alpha''\rangle^2}. \tag{2.1}$$

On the other hand, we can easily write:

$$\begin{aligned}\alpha'&=\mathbf{T},\\ \alpha''&=\mathbf{T}'\\ &=k_1\mathbf{M}+k_2\mathbf{Y},\\ \alpha'''&=(k_1\mathbf{M}+k_2\mathbf{Y})'\\ &=k_1'\mathbf{M}+k_1(-k_1\mathbf{T}+k_3\mathbf{Y})+k_2'\mathbf{Y}+k_2(-k_2\mathbf{T}-k_3\mathbf{M})\\ &=-\left(k_1^2+k_2^2\right)\mathbf{T}+\left(k_1'-k_2k_3\right)\mathbf{M}+\left(k_2'+k_1k_3\right)\mathbf{Y}.\end{aligned}$$

These equations give us the followings:

$$\begin{aligned}
\alpha'\wedge\alpha'' &= \begin{vmatrix} \mathbf{T} & \mathbf{M} & \mathbf{Y} \\ 1 & 0 & 0 \\ 0 & k_1 & k_2 \end{vmatrix} \\
&= -k_2\mathbf{M}+k_1\mathbf{Y}, \\
\left\langle \alpha,\, \alpha'\wedge\alpha''\right\rangle^2 &= \langle \alpha,\, -k_2\mathbf{M}+k_1\mathbf{Y}\rangle^2 \\
&= \left\langle \alpha, -\sqrt{k_1^2+k_2^2}\sin\Omega\mathbf{M}+\sqrt{k_1^2+k_2^2}\cos\Omega\mathbf{Y}\right\rangle^2 \\
&= \left(\sqrt{k_1^2+k_2^2}\,\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle\right)^2 \\
&= \left(k_1^2+k_2^2\right)\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2, \\
\left\langle \alpha'\wedge\alpha'',\, \alpha'''\right\rangle &= \left\langle -k_2\mathbf{M}+k_1\mathbf{Y}, -\left(k_1^2+k_2^2\right)\mathbf{T}+\left(k_1'-k_2k_3\right)\mathbf{M}+\left(k_2'+k_1k_3\right)\mathbf{Y}\right\rangle \\
&= \left(k_1k_2'-k_2k_1'\right)+k_3\left(k_1^2+k_2^2\right).
\end{aligned}$$

In the light of the equations derived above, we get:

$$\begin{aligned}
\frac{\langle \alpha'\wedge\alpha'',\, \alpha'''\rangle}{\langle \alpha,\, \alpha'\wedge\alpha''\rangle^2} &= \frac{\left(k_1k_2'-k_2k_1'\right)+k_3\left(k_1^2+k_2^2\right)}{\left(k_1^2+k_2^2\right)\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2} \\
&= \frac{k_3+\frac{k_2'k_1-k_1'k_2}{k_1^2+k_2^2}}{\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2} \\
&= \frac{k_3+\left(\arctan\left(\frac{k_2}{k_1}\right)\right)'}{\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2} \\
&= \frac{k_3+\Omega'}{\langle \alpha, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2}.
\end{aligned}$$

Then, we can conclude

$$\frac{\tau}{\langle \alpha, \mathbf{B}\rangle^2} = \frac{k_3+\Omega'}{\langle \alpha,\, -\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2}$$

from the equation (2.1). Using the condition of being a Tzitzéica curve in this last equation completes the proof. □

Remark 2.2. *The following derivative formula can be given:*

$$\begin{aligned}
(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y})' &= -\Omega'\sin\Omega\mathbf{M}+\cos\Omega\left(-k_1\mathbf{T}+k_3\mathbf{Y}\right)+\Omega'\cos\Omega\mathbf{Y}+\sin\Omega\left(-k_2\mathbf{T}-k_3\mathbf{M}\right) \\
&= -\left(k_1\cos\Omega+k_2\sin\Omega\right)\mathbf{T}+\left(k_3+\Omega'\right)\left(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\right) \\
&= -\left(\sqrt{k_1^2+k_2^2}\cos\Omega\cos\Omega+\sqrt{k_1^2+k_2^2}\sin\Omega\sin\Omega\right)\mathbf{T}+\left(k_3+\Omega'\right)\left(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\right) \\
&= -\sqrt{k_1^2+k_2^2}\left(\cos^2\Omega+\sin^2\Omega\right)\mathbf{T}+\left(k_3+\Omega'\right)\left(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\right) \\
&= -\sqrt{k_1^2+k_2^2}\,\mathbf{T}+\left(k_3+\Omega'\right)\left(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\right)
\end{aligned}$$

for a trajectory equipped with PAF. Also, one can easily see

$$(-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y})' = -\left(k_3+\Omega'\right)\left(\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y}\right).$$

Note that this remark will play an important role in the proofs of the following two theorems.

Theorem 2.3. *If the trajectory $\alpha=\alpha(s)$ is a unit speed spherical curve on $S^2_{O,r}$ (2-sphere of radius r centered at the origin),*

$$\frac{k_3+\Omega'}{\sqrt{k_1^2+k_2^2}} = \left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{\left(k_3+\Omega'\right)\left(k_1^2+k_2^2\right)}\right)'$$

holds.

Proof. Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical curve on $S^2_{O,r}$. Then, we have

$$\langle \alpha,\, \alpha\rangle = r^2$$

where r is the radius of sphere. By differentiating this equation with respect to arc-length parameter s, we obtain

$$\langle \alpha, \mathbf{T} \rangle = 0. \tag{2.2}$$

Since the tangent vectors of the spherical curve $\alpha = \alpha(s)$ on $S^2_{O,r}$ never pass through the origin, PAF is well defined along $\alpha = \alpha(s)$. In that case, the equation (2.2) gives us the following

$$\begin{aligned}
0 &= \langle \alpha', \mathbf{T} \rangle + \langle \alpha, \mathbf{T}' \rangle \\
&= 1 + \langle \alpha, k_1\mathbf{M} + k_2\mathbf{Y} \rangle \\
&= 1 + \left\langle \alpha, \sqrt{k_1^2+k_2^2}\cos\Omega\mathbf{M} + \sqrt{k_1^2+k_2^2}\sin\Omega\mathbf{Y} \right\rangle \\
&= 1 + \left\langle \alpha, \sqrt{k_1^2+k_2^2}\left(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}\right) \right\rangle.
\end{aligned}$$

Hence we find

$$\left\langle \alpha, \sqrt{k_1^2+k_2^2}\left(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}\right) \right\rangle = -1. \tag{2.3}$$

Differentiating (2.3) yields

$$\begin{aligned}
0 &= \left\langle \alpha, \left(\sqrt{k_1^2+k_2^2}\right)'(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) + \sqrt{k_1^2+k_2^2}(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y})' \right\rangle \\
&= \left\langle \alpha, \left(\sqrt{k_1^2+k_2^2}\right)'(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) \right\rangle + \left\langle \alpha, \sqrt{k_1^2+k_2^2}\left(-\sqrt{k_1^2+k_2^2}\mathbf{T} + \left(k_3+\Omega'\right)(-\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y})\right) \right\rangle \\
&= -\left(k_1^2+k_2^2\right)\langle \alpha, \mathbf{T} \rangle + \left(\sqrt{k_1^2+k_2^2}\right)'\langle \alpha, (\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) \rangle + \sqrt{k_1^2+k_2^2}\left(k_3+\Omega'\right)\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle \\
&= \left(\sqrt{k_1^2+k_2^2}\right)'\langle \alpha, (\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) \rangle + \sqrt{k_1^2+k_2^2}\left(k_3+\Omega'\right)\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle.
\end{aligned}$$

If we derive the equation

$$\langle \alpha, \cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y} \rangle = -\frac{1}{\sqrt{k_1^2+k_2^2}} \tag{2.4}$$

from (2.3) and substitute this into the equation (2.4), we find

$$\frac{-\left(\sqrt{k_1^2+k_2^2}\right)'}{\sqrt{k_1^2+k_2^2}} + \sqrt{k_1^2+k_2^2}\left(k_3+\Omega'\right)\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle = 0$$

and so

$$\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{k_1^2+k_2^2} = \left(k_3+\Omega'\right)\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle. \tag{2.5}$$

The last equation gives us the following:

$$\begin{aligned}
\left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{k_1^2+k_2^2}\right)' &= \left(k_3+\Omega'\right)'\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle + \left(k_3+\Omega'\right)\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle' \\
&= \left(k_3+\Omega'\right)'\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle + \left(k_3+\Omega'\right)\left[\langle \mathbf{T}, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle + \left\langle \alpha, (-\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y})' \right\rangle\right] \\
&= \left(k_3+\Omega'\right)'\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle + \left(k_3+\Omega'\right)\left\langle \alpha, -\left(k_3+\Omega'\right)(\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) \right\rangle.
\end{aligned}$$

Hence we obtain

$$\left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{k_1^2+k_2^2}\right)' = \left(k_3+\Omega'\right)'\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle - \left(k_3+\Omega'\right)^2\langle \alpha, (\cos\Omega\mathbf{M} + \sin\Omega\mathbf{Y}) \rangle. \tag{2.6}$$

From (2.5),

$$\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y} \rangle = \frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{(k_1^2+k_2^2)\left(k_3+\Omega'\right)} \tag{2.7}$$

can be immediately written. By substituting (2.4) and (2.7) into (2.6), we get

$$\left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{k_1^2+k_2^2}\right)' = \frac{(k_3+\Omega')'\left(\sqrt{k_1^2+k_2^2}\right)'}{(k_1^2+k_2^2)(k_3+\Omega')} + \frac{(k_3+\Omega')^2}{\sqrt{k_1^2+k_2^2}}.$$

This equation yields

$$\frac{(k_3+\Omega')^2}{\sqrt{k_1^2+k_2^2}} = \left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{k_1^2+k_2^2}\right)' - \frac{(k_3+\Omega')'\left(\sqrt{k_1^2+k_2^2}\right)'}{(k_1^2+k_2^2)(k_3+\Omega')}.$$

In that case, we can write

$$\frac{(k_3+\Omega')^2}{\sqrt{k_1^2+k_2^2}} = \frac{\left(\sqrt{k_1^2+k_2^2}\right)''(k_3+\Omega')\left(k_1^2+k_2^2\right) - 2\left(k_3+\Omega'\right)\sqrt{k_1^2+k_2^2}\left(\left(\sqrt{k_1^2+k_2^2}\right)'\right)^2 - \left(k_1^2+k_2^2\right)\left(\sqrt{k_1^2+k_2^2}\right)'(k_3+\Omega')'}{(k_3+\Omega')\left(k_1^2+k_2^2\right)^2}.$$

Eventually, we find

$$\frac{k_3+\Omega'}{\sqrt{k_1^2+k_2^2}} = \frac{\left(\sqrt{k_1^2+k_2^2}\right)''(k_3+\Omega')\left(k_1^2+k_2^2\right) - 2\left(k_3+\Omega'\right)\sqrt{k_1^2+k_2^2}\left(\left(\sqrt{k_1^2+k_2^2}\right)'\right)^2 - \left(k_1^2+k_2^2\right)\left(\sqrt{k_1^2+k_2^2}\right)'(k_3+\Omega')'}{(k_3+\Omega')^2\left(k_1^2+k_2^2\right)^2} \tag{2.8}$$

and so

$$\frac{k_3+\Omega'}{\sqrt{k_1^2+k_2^2}} = \left(\frac{\left(\sqrt{k_1^2+k_2^2}\right)'}{(k_3+\Omega')\left(k_1^2+k_2^2\right)}\right)'.$$

This completes the proof. □

Theorem 2.4. *Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical curve on $S^2_{O,r}$. If it is a Tzitzéica curve,*

$$\frac{(k_3+\Omega')'}{2(k_3+\Omega')^3} = \frac{\sqrt{k_1^2+k_2^2}}{\left(\sqrt{k_1^2+k_2^2}\right)'} \tag{2.9}$$

holds.

Proof. Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical curve on $S^2_{O,r}$. Assume that $\alpha=\alpha(s)$ is a Tzitzéica curve. In that case

$$\frac{k_3+\Omega'}{\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle^2} = c \tag{2.10}$$

can be easily written where c is a non-zero constant. Differentiating the equation (2.10) yields

$$\left(k_3+\Omega'\right)'\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle + 2\left(k_3+\Omega'\right)^2\langle\alpha,\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y}\rangle = 0$$

and so

$$\left(k_3+\Omega'\right)'\langle\alpha,-\sin\Omega\mathbf{M}+\cos\Omega\mathbf{Y}\rangle = -2\left(k_3+\Omega'\right)^2\langle\alpha,\cos\Omega\mathbf{M}+\sin\Omega\mathbf{Y}\rangle. \tag{2.11}$$

Using (2.4) and (2.7) in the equation (2.11) gives us the desired result. □

Corollary 2.5. *Let the trajectory $\alpha=\alpha(s)$ be a unit speed spherical Tzitzéica curve on $S^2_{O,r}$. In this case, the equation*

$$k_3+\Omega' = \sqrt{\frac{\left(\sqrt{k_1^2+k_2^2}\right)''\sqrt{k_1^2+k_2^2} - 2\left(\left(\sqrt{k_1^2+k_2^2}\right)'\right)^2}{3\left(k_1^2+k_2^2\right)}}$$

is satisfied.

Proof. Suppose that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical Tzitzéica curve on $S^2_{O,r}$. We have the equation (2.8) given in the proof of Theorem 2.3. On the other hand, we can write

$$\left(k_3+\Omega'\right)' = 2\left(k_3+\Omega'\right)^3 \frac{\sqrt{k_1^2+k_2^2}}{\left(\sqrt{k_1^2+k_2^2}\right)'} \tag{2.12}$$

thanks to (2.9). Then, the desired result is obtained by substituting (2.12) into (2.8). □

Corollary 2.6. *Suppose that the trajectory $\alpha = \alpha(s)$ is a unit speed spherical Tzitzéica curve on $S^2_{O,r}$. Let the functions $k_3+\Omega'$ and $\sqrt{k_1^2+k_2^2}$ be constant function and non-constant function, respectively. Then the equation*

$$\sqrt{k_1^2+k_2^2} = c_2 \sec\left(\sqrt{3}\left(k_3+\Omega'\right)\left(c_1+s\right)\right)$$

is satisfied where c_1 and c_2 are any real constants.

Proof. Let the trajectory $\alpha = \alpha(s)$ be a unit speed spherical Tzitzéica curve on $S^2_{O,r}$. Assume that the function $k_3+\Omega'$ is a constant function and the function $\sqrt{k_1^2+k_2^2}$ is not a constant function. From Corollary 2.5, we have the following equation

$$k_3+\Omega' = \sqrt{\frac{\left(\sqrt{k_1^2+k_2^2}\right)''\sqrt{k_1^2+k_2^2} - 2\left(\left(\sqrt{k_1^2+k_2^2}\right)'\right)^2}{3\left(k_1^2+k_2^2\right)}}$$

which gives us the differential equation

$$\sqrt{k_1^2+k_2^2}\left(\sqrt{k_1^2+k_2^2}\right)'' - 2\left(\left(\sqrt{k_1^2+k_2^2}\right)'\right)^2 - 3\left(k_1^2+k_2^2\right)\left(k_3+\Omega'\right)^2 = 0. \tag{2.13}$$

The solution of (2.13) corresponds to the desired result. □

Now, we provide two original examples for the characterization given above. Note that the figures of this section are drawn in the Wolfram Mathematica (Wolfram Cloud).

Example 2.7. *In the Euclidean 3-space, assume that a point particle P of constant mass moves on the trajectory*

$$\begin{aligned} \alpha : \left(0, 15\sqrt{82}\right) &\to E^3 \\ s \mapsto \alpha(s) &= \left(9\cos\frac{s}{\sqrt{82}}, 9\sin\frac{s}{\sqrt{82}}, \frac{s}{\sqrt{82}}\right) \end{aligned} \tag{2.14}$$

which is a unit speed curve. In the following Figure 2.1, the trajectory $\alpha = \alpha(s)$ can be seen.

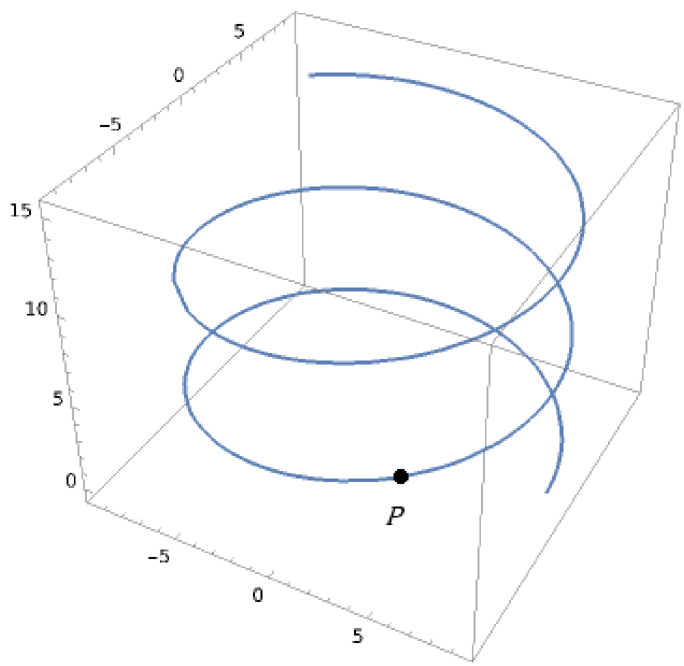

Figure 2.1: The trajectory of the moving point particle P given in (2.14)

In the light of the information given in the Section 1, we can obtain the rotation angle $\Omega(s)$ *as*

$$\Omega(s) = \arctan\left(\frac{82}{s}\right),$$

the PAF basis vectors $\mathbf{T}(s), \mathbf{M}(s), \mathbf{Y}(s)$ *as*

$$\begin{cases} \mathbf{T}(s) = \left(-\frac{9}{\sqrt{82}}\sin\frac{s}{\sqrt{82}}, \frac{9}{\sqrt{82}}\cos\frac{s}{\sqrt{82}}, \frac{1}{\sqrt{82}}\right), \\ \mathbf{M}(s) = \begin{pmatrix} -\cos\left(\arctan\left(\frac{82}{s}\right)\right)\cos\frac{s}{\sqrt{82}} - \frac{1}{\sqrt{82}}\sin\left(\arctan\left(\frac{82}{s}\right)\right)\sin\frac{s}{\sqrt{82}}, \\ -\cos\left(\arctan\left(\frac{82}{s}\right)\right)\sin\frac{s}{\sqrt{82}} + \frac{1}{\sqrt{82}}\sin\left(\arctan\left(\frac{82}{s}\right)\right)\cos\frac{s}{\sqrt{82}}, \\ -\frac{9}{\sqrt{82}}\sin\left(\arctan\left(\frac{82}{s}\right)\right) \end{pmatrix}, \\ \mathbf{Y}(s) = \begin{pmatrix} -\sin\left(\arctan\left(\frac{82}{s}\right)\right)\cos\frac{s}{\sqrt{82}} + \frac{1}{\sqrt{82}}\cos\left(\arctan\left(\frac{82}{s}\right)\right)\sin\frac{s}{\sqrt{82}}, \\ -\sin\left(\arctan\left(\frac{82}{s}\right)\right)\sin\frac{s}{\sqrt{82}} - \frac{1}{\sqrt{82}}\cos\left(\arctan\left(\frac{82}{s}\right)\right)\cos\frac{s}{\sqrt{82}}, \\ \frac{9}{\sqrt{82}}\cos\left(\arctan\left(\frac{82}{s}\right)\right) \end{pmatrix}, \end{cases}$$

and PAF curvatures $k_1(s), k_2(s), k_3(s)$ *as*

$$\begin{cases} k_1(s) = \frac{9}{82}\cos\left(\arctan\left(\frac{82}{s}\right)\right), \\ k_2(s) = \frac{9}{82}\sin\left(\arctan\left(\frac{82}{s}\right)\right), \\ k_3(s) = \frac{82}{s^2+6724} + \frac{1}{82}. \end{cases}$$

Using the above derivation, we find

$$\frac{k_3 + \Omega'}{\langle \alpha, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y}\rangle^2} = \frac{82}{81s^2}.$$

Then, by means of Theorem 2.1, we can say that the given trajectory is not a Tzitzéica curve.

Now, we will take into consideration a trajectory, which is a Tzitzéica curve, by using PAF in the following example. Note that this curve was studied according to Bishop frame in [6].

Example 2.8. *In the Euclidean 3-space, assume that a point particle P of constant mass moves on the trajectory*

$$\begin{aligned} &\alpha : (0, ln2) \to E^3 \\ &\qquad s \mapsto \alpha(s) = \left(e^{-s}\cos s,\, e^{-s}\sin s,\, e^{2s}\right). \end{aligned} \tag{2.15}$$

In the Figure 2.2, the trajectory $\alpha = \alpha(s)$ *can be seen. We can easily calculate the binormal vector of* α *as*

$$\mathbf{B}(s) = \frac{1}{\sqrt{20e^{2s} + e^{-4s}}}\left(4e^s\cos s - 2e^s\sin s,\, 4e^s\sin s + 2e^s\cos s,\, e^{-2s}\right).$$

Then, we find

$$\langle \alpha(s), \mathbf{B}(s)\rangle = \frac{5}{\sqrt{20e^{2s} + e^{-4s}}}.$$

Since $\langle \alpha(s), \mathbf{B}(s)\rangle = \frac{5}{\sqrt{20e^{2s}+e^{-4s}}}$ *is always non-zero, PAF is well defined during the motion of the particle P. Let the unit speed parameterization of the trajectory be denoted by* β *and PAF apparatus of* β *be denoted by* $\{\mathbf{T}, \mathbf{M}, \mathbf{Y}, k_1, k_2, k_3\}$*. One can easily find these and* Ω *and see the equality*

$$\frac{\langle \beta' \wedge \beta'', \beta'''\rangle}{\langle \beta, \beta' \wedge \beta''\rangle^2} = \frac{k_3 + \Omega'}{\langle \beta, -\sin\Omega\mathbf{M} + \cos\Omega\mathbf{Y}\rangle} = \frac{2}{5}.$$

Consequently, the given trajectory is a Tzitzéica curve due to the Theorem 2.1.

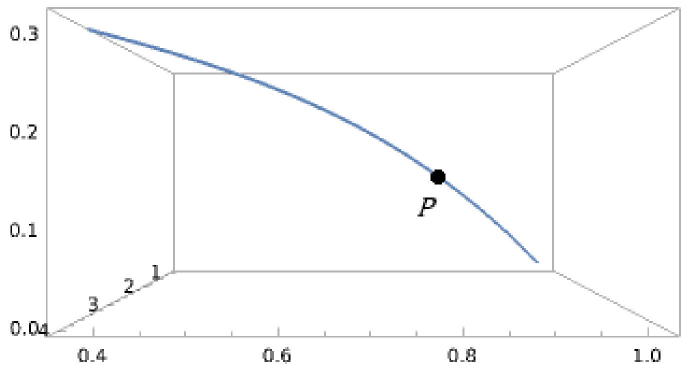

Figure 2.2: The trajectory of the moving point particle P given in (2.15)

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