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A Note On E-Injective Modules

Abuzer GÜNDÜZ [*1], Osama NAJİ [1]

Abstract

Let R be a commutative ring with identity, M an R-module and E a torsion-free R-module. A submodule N of M is said to be essential (large) in M if the intersection of N with each nonzero submodule of M is nonzero, that is, $N \cap Rm \neq 0$ for any nonzero element $m \in M$ and we write $N \leq_e M$. It is clear that the class of $e-exact$ sequences is larger than the class of $exact$ sequences. In this study we present the concept of e-injective modules as a generalization of injective modules. The main goal is to give a characterization of e-injective modules in terms of contravariant functor $Hom(-,E)$.

Keywords: E-injective modules, e-exact sequences, contravariant functor

1. INTRODUCTION

Let R be a commutative ring with identity and M an R-module. A submodule N of M is said to be essential (large) in M if the intersection of N with each nonzero submodule of M is nonzero, that is, $N \cap Rm \neq 0$ for any nonzero element $m \in M$ and we write $N \leq_e M$. A sequence of R-modules and R-module homomorphisms f_i

$$\ldots \rightarrow M_{i-1} \overset{f_{i-1}}{\longrightarrow} M_i \overset{f_i}{\rightarrow} M_{i+1} \overset{f_{i+1}}{\longrightarrow} \ldots$$

is called $exact$ at M_i if $Im(f_{i-1}) = Ker(f_i)$. Akray and Zebari in [1] introduced the $e-exact$ sequences as a generalization of exact sequences. The above sequence is called $e-exact$ at M_i if $Im(f_{i-1}) \leq_e Ker(f_i)$ and it is called $e-exact$ if it is $e-exact$ at each M_i. Expectedly, they defined the sequence

$$0 \rightarrow A_1 \overset{f_1}{\rightarrow} A_2 \overset{f_2}{\rightarrow} A_3 \rightarrow 0$$

to be short $e-exact$ if $Ker(f_1) = 0$, $Im(f_1) \leq_e Ker(f_2)$ and $Im(f_2) \leq_e A_3$, where $f_i: A_i \rightarrow A_{i+1}$ is an R-module homomorphism for $i = 1,2$. Recall from [1] that an R-morphism $f: A_1 \rightarrow A_2$ is called $epic$ if $Im(f_1) \leq_e A_2$ and essential $monic$ if $Ker f_1 = 0$. It is clear that the class of $e-exact$ sequences is larger than the class of $exact$ sequences. For example consider the short $e-exact$ sequence

$$0 \rightarrow 8\mathbb{Z} \overset{f_1}{\rightarrow} \mathbb{Z} \overset{f_2}{\rightarrow} \mathbb{Z} / 8\mathbb{Z} \rightarrow 0$$

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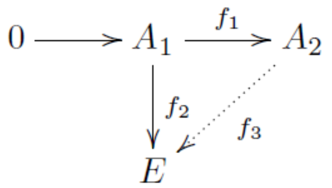
where $f_1(8n) = 4n$ and $f_2(n) = 2n + 8\mathbb{Z}$. Since f_1 and f_2 are epic, the sequence is $e-exact$. Note that f_2 is not an $epimorphism$, so the sequence is not $exact$.

In the sake of completeness, we recall from [2] some basic definitions. An element m of M is said to be torsion of M if there exists a regular element $r \in R$ such that $rm = 0$. The set of all torsion elements $T(M)$ is a submodule of M. Also, an R-module M is called $torsion$ if $T(M) = M$, and called $torsion - free$ when $T(M) = \{0\}$.

Let E be an $\mathrm{R}-$ module. E is said to be injective module if the following condition is satisfied: For any monic map $\mathrm{f_1}: \mathrm{A_1} \to \mathrm{A_2}$ and any map $\mathrm{f_2}: \mathrm{A_1} \to \mathrm{E}$, there exist $\mathrm{f_3}: \mathrm{A_2} \to \mathrm{E}$ such that $\mathrm{f_3 f_1} = \mathrm{f_2}$.



Moreover, if E is injective module, then the contravariant functor $\mathrm{Hom}(-, \mathrm{E})$ is an exact sequence [3].

A group D is called $divisible$ if for every positive integer n and every $d \in D$, there exists $0 \neq x \in D$ such that $nx = d$. It is known that a group D is $divisible$ if and only if it is $injective$ [3].

Throughout this note, all modules are assumed to be torsion-free. In section 2, we introduce the definition of $e - injective\ E$. It is shown that a module E is $e - injective$ if and only if the contravariant functor $Hom(-, E)$ is an $e - exact$ sequence.

2. CHARACTERIZATION OF E-INJECTIVE MODULE

In this part, we investigate some results about $\mathrm{e-}$injective modules such as when the contravariant functor $\mathrm{Hom}(-, \mathrm{M})$ is an $\mathrm{e-}$exact sequence, $\mathrm{E} = \prod_{\mathrm{i}\in\Delta} \mathrm{E_i}$ is $\mathrm{e-}$injective for each $\mathrm{E_i}$ be an $\mathrm{R_i}$-module for every $\mathrm{i} \in \Delta$ and short $\mathrm{e-}$exact sequence is $\mathrm{e-}$split.

The following theorem shows that the contravariant functor $Hom(-, M)$ is a left $e - exact$ functor when M is a $torsion - free$ R-module.

Theorem 1 *[1] Suppose that the following sequence of R-modules and R-morphism*

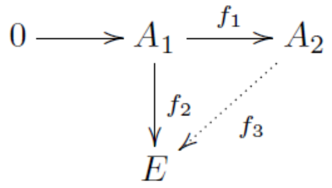
$$M_1 \overset{f_1}{\to} M_2 \overset{f_2}{\to} M_3 \to 0$$

is $e - exact$. Then for all $torsion - free$ R-module M, the sequence

$$0 \to Hom(M_3, M) \overset{f_2^*}{\to} Hom(M_2, M) \overset{f_1^*}{\to} Hom(M_1, M)$$

is $e - exact$. The converse is true if $M_3/Im(f_2)$ and $M_2/Im(f_1)$ are $torsion - free$ R-modules.

Definition 1 *Let R be a ring and E an $R-$ module. E is said to be $e - injective$ if the following condition is satisfied: For any monic map $f_1: A_1 \to A_2$ and any map $f_2: A_1 \to E$, there exist $0 \neq r \in R$ and $f_3: A_2 \to E$ such that $f_3 f_1 = r \cdot f_2$.*



Theorem 2 *Let R be a ring and E an R-module. Then the following statements are equivalent:*

(i) E is an $e - injective$ R-module.

(ii) $Hom(-, E)$ is an $e - exact$ sequence.

Proof. (i)⇒(ii): Suppose that E is an $e - injective$ R-module. Then by Theorem 1, $Hom(-, E)$ is left $e - exact$ functor. It remains to show that $Hom(-,E)$ is right $e-exact$ functor. Assume that

$$0 \to A_1 \overset{f_1}{\to} A_2$$

is an $e-exact$ sequence and we want to show that

$$Hom(A_2,E) \overset{f_1^*}{\to} Hom(A_1,E) \to 0$$

is $e-exact$. Since the contravariant functor $Hom(-,E)$ is left $e-exact$, it is enough to prove that $Im(f_1^*) \leq_e Hom(A_1,E)$. Since we have that f_1 is monic and let pick $f_2 \in Hom(A_1,E)$, by the definition of $e-injective$, we have $f_3 f_1 = r \cdot f_2$ for some $0 \neq r \in R$ and $f_3 : A_2 \to E$. This implies that $f_1^*(f_3) = r \cdot f_2$. Thus, $Im(f_1^*) \cap Rf_2 \neq 0$ and we obtian that $Im(f_1^*) \leq_e Hom(A_1,E)$.

(ii)⇒(i): Assume that $Hom(-,E)$ is an $e-exact$ functor. Let $f_1 : A_1 \to A_2$ be a monic map and $f_2 : A_1 \to E$ any map. Since the sequence

$$0 \to A_1 \overset{f_1}{\to} A_2$$

is $e-exact$, then by assumption, the sequence

$$Hom(A_2,E) \overset{f_1^*}{\to} Hom(A_1,E) \to 0$$

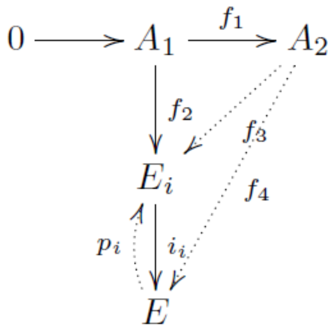
is also $e-exact$. Then we have $Im(f_1^*) \leq_e Hom(A_1,E)$. As $f_2 \in Hom(A_1,E)$, there exist $0 \neq r \in R$ and $f_3 \in Hom(A_2,E)$ such that $f_1^*(f_3) = r \cdot f_2$. This implies that $f_3 f_1 = r f_2$. Hence E is $e-injective$.

Theorem 3 *Let E_i be an R_i-module for each $i \in \Delta$, where Δ is an index set. Assume that $R = \prod_{i\in\Delta} R_i$ and $E = \prod_{i\in\Delta} E_i$. Then the following statements hold:*

(i) If E is an e-injective R-module, then E_i is an e-injective R_i-module for some $i \in \Delta$.

(ii) If E_i is an e-injective R_i-module for each $i \in \Delta$, then E is an e-injective R-module.

Proof. (i): Suppose that $f_1 : A_1 \to A_2$ is a monic map and $f_2 : A_1 \to E_i$. Consider the following diagram

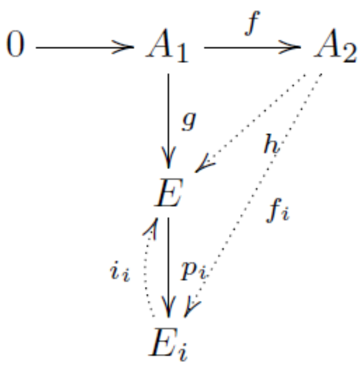


where $i_i : E_i \to E$ is the injective map and $p_i : E \to E_i$ is the projective map. Since $i_i f_2 : A_1 \to E$ and E is an $e-injective$ R-module, there exist $0 \neq r = (r_i)_{i\in\Delta} \in R$ and $f_4 : A_2 \to E$ such that $f_4 f_1 = r \cdot (i_i f_2)$. Assume that $r_k \neq 0$ for some $k \in \Delta$ and define $f_3 : A_2 \to E_k$ by $f_3 = p_k f_4$. Since $p_k \circ i_k = 1_{E_k}$, we obtain

$$f_3 f_1 = p_k f_4 f_1 = p_k(r \cdot i_k f_2) = r_k p_k i_k f_2 = r_k f_2$$

Therefore, E_k is an $e-injective$ R_k-module.

(ii): Assume that $f : A_1 \to A_2$ is a monic map and $g : A_1 \to E$. Consider the following diagram



Since E_i is $e-injective$ and $p_i g : A_1 \to E_i$ for each $i \in \Delta$, there exist $0 \neq r_i \in R_i$ and $f_i : A_2 \to E_i$ such that $f_i f = r_i(p_i g)$. Then there exists $h: A_2 \to E$ such that $f_i = p_i h$. Let $0 \neq r = (r_i) \in R$ and note that $p_i hf = f_i f = r_i p_i g = p_i(rg)$. Hence, we get $hf = rg$. Therefore, E is an e-injective R-module.

Theorem 4 *Let R be a ring and E be an $e-injective$ $R-$module. For any monic map*

$$0 \to E \overset{f}{\to} L$$

there exists an R-homomorphism $\alpha: L \to E$ such that $\alpha f = r \cdot 1_E$ for some $0 \neq r \in R$.

Proof. It is clear.

Definition 2 *[1] Let*

$$0 \to E \overset{f_1}{\to} A_1 \overset{f_2}{\to} A_2 \to 0$$

be a short $e-exact$ sequence. If for any map $f_1: E \to A_1$ there exist $g: A_1 \to E$ and $r \in R$ such that $gf_1 = r \cdot 1_E$. Then the above short $e-exact$ sequence is called $e-split$.

Theorem 5 *An $e-exact$ sequence*

$$0 \to E \overset{f_1}{\to} A_1 \overset{f_2}{\to} A_2 \to 0$$

is $e-split$ if E is an $e-injective$ module.

Proof. Suppose that

$$0 \to E \overset{f_1}{\to} A_1 \overset{f_2}{\to} A_2 \to 0$$

is an $e-exact$ sequence and E is an $e-injective$ R-module. Then by Theorem 2, the sequence

$$0 \to Hom(A_2, E) \overset{f_2^*}{\to} Hom(A_1, E) \overset{f_1^*}{\to} Hom(E, E) \to 0$$

is $e-exact$. Since f_1^* is epic, $Im(f_1^*) \leq_e Hom(E, E)$. Note that $1_E \in Hom(E, E)$, so there exist a map $g: A_1 \to E$ and $r \in R$ such that $f_1^*(g) = r \cdot 1_E$ and hence $gf_1 = r \cdot 1_E$. Therefore the sequence is $e-split$.

3. CONCLUSION

As a result, we get the definition of the e − injective R-module and some results. We hope that the results give rise to new results in Homological Algebra with regard to e − exact theory such as e − flat module and e − homology, e − functor.

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Authors' Contribution
The authors contributed equally to the study.

The Declaration of Ethics Committee Approval
The study does not require ethics committee permission or any special permission.

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