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A generalized integral problem for a system of hyperbolic equations and its applications

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Abstract

A nonlocal boundary value problem for a system of hyperbolic equations of second order with generalized integral condition is considered. By method of introduction of functional parameters the investigated problem is transformed to the inverse problem for the system of hyperbolic equations with unknown parameters and additional functional relations. Algorithms of finding solution to the inverse problem for the system of hyperbolic equations are constructed, and their convergence is proved. The conditions for existence of unique solution to the inverse problem for the system of hyperbolic equations are obtained in the terms of initial data. The coefficient conditions for unique solvability of nonlocal boundary value problem for the system of hyperbolic equations with generalized integral condition are established. The results are illustrated by numerical examples.

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1. Statement of the problem

Consider on $\Omega = [0,T] \times [0,\omega]$ the nonlocal boundary value problem for the system of hyperbolic equations with generalized integral condition

$$\frac{\partial^2 u}{\partial x \partial t} = A(t,x)\frac{\partial u}{\partial x} + B(t,x)\frac{\partial u}{\partial t} + C(t,x)u + f(t,x), \tag{1.1}$$

$$\sum_{i=0}^{m+1} L_i(x)u(t_i,x) + \sum_{j=1}^{m+1} \int\limits_{t_{j-1}}^{t_j} K_j(\tau,x)u(\tau,x)d\tau = \varphi(x), \quad x \in [0,\omega], \tag{1.2}$$

$$u(t,0) = \psi(t), \qquad t \in [0,T], \tag{1.3}$$

where $u = col(u_1, u_2, ..., u_n)$ is unknown function, the $(n \times n)$ matrices $A(t,x)$, $B(t,x)$, and $C(t,x)$, the n vector-function $f(t,x)$ are continuous on Ω, the $(n \times n)$ matrices $L_i(x)$, $i = \overline{0, m+1}$, the n vector-function $\varphi(x)$ are continuously differentiable on $[0,\omega]$, $t_0 = 0 < t_1 < t_2 < ... < t_m < t_{m+1} = T$, the $(n \times n)$ matrices $K_j(t,x)$ are continuous and continuously differentiable by x on $[t_{j-1}, t_j] \times [0,\omega]$, the n vector-function $\psi(t)$ is continuously differentiable on $[0,T]$, and the following compatibility condition for the initial data holds:

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$$L_0(0)\psi(0) + \sum_{i=1}^{m} L_i(0)\psi(t_i) + L_{m+1}(0)\psi(T) + \sum_{j=1}^{m+1} \int_{t_{j-1}}^{t_j} K_j(\tau,0)\psi(\tau)d\tau = \varphi(0).$$

A solution to problem (1.1)–(1.3) is a function $u(t,x)$, which is continuous on Ω and has continuous partial derivatives $\frac{\partial u(t,x)}{\partial x}, \frac{\partial u(t,x)}{\partial t}, \frac{\partial^2 u(t,x)}{\partial x \partial t}$ on Ω and satisfies the system of equations (1.1), boundary conditions (1.2) and (1.3).

In recent years, the nonlocal problems with multi-point and integral conditions for hyperbolic equations are of great interest to the specialists. Multi-point boundary value problems for a system of hyperbolic equations arise while studying the motion of adsorbed mixtures of substances, consisting of many components through an porous environment pre-saturated by one or several substances for small or large concentrations of adsorbed substances at a constant or variable speed filter [8,21,22,25,30]. Boundary value problems with integral condition for hyperbolic equations describe the mathematical models of processes on heat propagation, plasma physics, the metal treatment, and moisture transfer in capillary-porous environments [3,17,21,23,26,27,31]. The solvability of some classes of nonlocal boundary value problems with multi-point and integral conditions are studied, and the methods for their solution are suggested. In the papers [1,2], there considered a boundary value problem for system of hyperbolic equations (1.1) with nonlocal condition in the form of linear combination of desired solution values and its derivatives by t and x on the characteristics $t = 0$, $t = T$ and condition (1.3). By introducing the additional functional parameters, the considered problem was transformed to an equivalent problem, consisting of the Goursat problem for a system of hyperbolic equations with unknown parameters and functional relations. Note, that the resulting equivalent problem can be seen as the inverse nonlocal problem for the system of hyperbolic equations [7,10–15,18–20,28,32–35]. Sufficient conditions for existence of unique classical solution to the investigated problem were established in the terms of initial data, and the algorithms for finding solutions are constructed under assumption on the continuity of coefficients of system (1.1). Sufficient and necessary conditions of well-posedness of nonlocal problem with integral condition on the characteristics $t = 0$, $t = T$ for system of hyperbolic equations are obtained in [3].

In the present paper, this approach is developed for nonlocal boundary value problem with generalized integral condition (1.1)–(1.3). By introduction of additional functions the problem (1.1)–(1.3) is transformed to the inverse problem for system of hyperbolic equations with unknown parameters and integral equations. Algorithms for finding a solution to the resulting problem are constructed, and their convergence is proved. The conditions for existence of unique solution to the inverse problem for system of hyperbolic equations with unknown parameters are obtained in the terms of initial data. Sufficient conditions for the existence of unique classical solution to the nonlocal boundary value problem for the system of hyperbolic equations with generalized integral condition (1.1)–(1.3) are set in the terms of coefficients and boundary matrices.

First the idea on reduction of nonlocal problem to the inverse problem was proposed by Cannon for a heat equation in the work [9]. This approach has been also applied to a nonlocal problem for one class of differential equations of high order in [21]. In contrast to those works, the method proposed in the present paper uses a substitution of desired function and reduction of original problem to an equivalent problem. The domain is divided into parts by the variable t, then functional parameters are introduced as the values of the desired function on the partition lines. We introduce some new unknown functions, which are the differences between the desired function and functional parameter in the corresponding sub-domains. Examining further the obtained equivalent problem and setting the conditions for its unique solvability, we go back to the original problem and then formulate the solvability conditions for it.

Consider problem (1.1)–(1.3). In Section 2, a scheme of method of functional parameter introduction without division on domain is provided. Algorithms of finding solution to the inverse problem for the system of hyperbolic equations with single unknown parameter are constructed. Conditions for the convergence of proposed algorithms and existence of unique solution to the inverse problem for hyperbolic equations with parameter are established.

In Section 3, there provided a general scheme of the method of functional parameters introduction with division of domain. Algorithms of finding solution to the inverse problem for the system of hyperbolic equations with many unknown parameters are constructed. Conditions for the convergence of proposed algorithms and existence of unique solution to the inverse problem for the system of hyperbolic equations with parameters are established.

In Section 4, the statements on unique solvability of original nonlocal problem for the system of hyperbolic equations with generalized integral condition (1.1)–(1.3) are formulated. The main condition here is an invertibility of some matrix composed by initial data of the problem. Auxiliary lemmas on invertibility of this matrix and recurrence formulas of its block-level elements are provided. In Section 5, we offer a numerical implementation of the results based on Runge-Kutta method of 4-th order and Simpson's method.

2. Transformation to the inverse problem for the system of hyperbolic equations with parameter and the algorithms for finding of its solutions

Denote by $\mu(x)$ the value of function $u(t,x)$ at $t=0$, and make in the problem(1.1)–(1.3) the replacement $\widetilde{u}(t,x)=u(t,x)-\mu(x)$:

$$\frac{\partial^2\widetilde{u}}{\partial x\partial t}=A(t,x)\frac{\partial\widetilde{u}}{\partial x}+B(t,x)\frac{\partial\widetilde{u}}{\partial t}+C(t,x)\widetilde{u}+A(t,x)\dot{\mu}(x)+C(t,x)\mu(x)+f(t,x), \tag{2.1}$$

$$\widetilde{u}(0,x)=0, \qquad x\in[0,\omega], \tag{2.2}$$

$$\widetilde{u}(t,0)=\psi(t)-\psi(0), \qquad t\in[0,T], \tag{2.3}$$

$$\sum_{i=0}^{m+1}L_i(x)\mu(x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)d\tau\mu(x)+\sum_{i=1}^{m+1}L_i(x)\widetilde{u}(t_i,x)+$$
$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)\widetilde{u}(\tau,x)d\tau=\varphi(x), \qquad x\in[0,\omega]. \tag{2.4}$$

In the relation (2.3) the compatibility condition at the point $(0,0)$: $\psi(0)=\mu(0)$ is taken into account.

Solution to (2.1)–(2.4) is a pair of functions $(\widetilde{u}(t,x),\mu(x))$ satisfying the system of equations (2.1) and conditions (2.2)–(2.4), where the function $\widetilde{u}(t,x)$ has continuous partial derivatives $\dfrac{\partial\widetilde{u}(t,x)}{\partial x}$, $\dfrac{\partial\widetilde{u}(t,x)}{\partial t}$, $\dfrac{\partial^2\widetilde{u}(t,x)}{\partial x\partial t}$ on Ω, the function $\mu(x)$ is continuously differentiable on $[0,\omega]$.

The problem (2.1)–(2.4) is the inverse problem for the system of hyperbolic equations (2.1) with unknown functions $\mu(x)$, $\dot{\mu}(x)$, conditions on characteristics (2.2), (2.3), and additional condition (2.4) for the determination of parameters introduced. Inverse problems for equations and systems of hyperbolic type are investigated by many authors, the bibliography and references can be found in [12,18–20,24,29,32]. Methods for solving some classes of inverse problems for hyperbolic equations with mixed derivatives are proposed in [11–15].

For fixed $\mu(x)$ and $\dot{\mu}(x)$ the function $\widetilde{u}(t,x)$ is a solution of the Goursat problem (2.1)–(2.3). For fixed $\widetilde{u}(t,x)$ the functional parameter is to be determined from the relation (2.4).

Introduce the notation $\widetilde{v}(t,x)=\dfrac{\partial\widetilde{u}(t,x)}{\partial x}$, $\widetilde{w}(t,x)=\dfrac{\partial\widetilde{u}(t,x)}{\partial t}$. From (2.2), (2.3) we obtain $\widetilde{v}(0,x)=0$ and $\widetilde{w}(t,0)=\dot{\psi}(t)$. The Goursat problem is equivalent to the system of integral equations

$$\widetilde{w}(t,x)=\dot{\psi}(t)+\int\limits_0^x\Big[A(t,\xi)\widetilde{v}(t,\xi)+B(t,\xi)\widetilde{w}(t,\xi)+C(t,\xi)\widetilde{u}(t,\xi)+f(t,\xi)\Big]d\xi+$$

$$+\int\limits_0^x\Big[A(t,\xi)\dot{\mu}(\xi)+C(t,\xi)\mu(\xi)\Big]d\xi, \tag{2.5}$$

$$\widetilde{v}(t,x)=\int\limits_0^t\Big[A(s,x)\widetilde{v}(s,x)+B(s,x)\widetilde{w}(\tau,x)+C(s,x)\widetilde{u}(s,x)+f(s,x)\Big]ds+$$

$$+\int\limits_0^t\Big[A(s,x)\dot{\mu}(x)+C(s,x)\mu(x)\Big]ds, \tag{2.6}$$

$$\widetilde{u}(t,x)=\psi(t)-\psi(0)+\int\limits_0^t\int\limits_0^x\Big[A(s,\xi)\widetilde{v}(s,\xi)+B(s,\xi)\widetilde{w}(s,\xi)++C(s,\xi)\widetilde{u}(s,\xi)+f(s,\xi)\Big]d\xi ds+$$

$$+\int\limits_0^t\int\limits_0^x\Big[A(s,\xi)\dot{\mu}(\xi)+C(s,\xi)\mu(\xi)\Big]d\xi ds. \tag{2.7}$$

Substitute the corresponding right-hand side of (2.6) instead of $\widetilde{v}(s,x)$ and repeating this process l $(l=1,2,...)$ times, we get

$$\widetilde{v}(t,x)=G_l(t,x,\widetilde{v})+H_l(t,x,\widetilde{u},\widetilde{w})+D_l(t,x)\dot{\mu}(x)+E_l(t,x)\mu(x)+F_l(t,x), \tag{2.8}$$

where $(t,x)\in\Omega$,

$$D_l(t,x)=\int\limits_0^t A(s_1,x)ds_1+...+\int\limits_0^t A(s_1,x)\int\limits_0^{s_1}A(s_2,x)...\int\limits_0^{s_{l-1}}A(s_l,x)ds_l...ds_2ds_1,$$

$$H_l(t,x,\widetilde{u},\widetilde{w})=\int\limits_0^t\Big[B(s_1,x)\widetilde{w}(s_1,x)+C(s_1,x)\widetilde{u}(s_1,x)\Big]ds_1+...+$$

$$+\int\limits_0^t A(s_1,x)...\int\limits_0^{s_{l-2}}A(s_{l-1},x)\int\limits_0^{s_{l-1}}\Big[B(s_l,x)\widetilde{w}(s_l,x)+C(s_l,x)\widetilde{u}(s_l,x)\Big]ds_l...ds_1,$$

$$G_l(t,x,\widetilde{v})=\int\limits_0^t A(s_1,x)\int\limits_0^{s_1}A(s_2,x)...\int\limits_0^{s_{l-1}}A(s_l,x)\widetilde{v}(s_l,x)ds_l...ds_2ds_1,$$

$$E_l(t,x)=\int\limits_0^t C(s_1,x)ds_1+...+\int\limits_0^t A(s_1,x)...\int\limits_0^{s_{l-2}}A(s_{l-1},x)\int\limits_0^{s_{l-1}}C(s_l,x)ds_l...ds_1,$$

$$F_l(t,x)=\int\limits_0^t f(s_1,x)ds_1+...+\int\limits_0^t A(s_1,x)...\int\limits_0^{s_{l-2}}A(s_{l-1},x)\int\limits_0^{s_{l-1}}f(s_l,x)ds_l...ds_1.$$

Assumptions on initial data allow us to differentiate the relation (2.4) by the variable x:

$$\Big\{\sum_{i=0}^{m+1}\dot{L}_i(x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\frac{\partial K_j(\tau,x)}{\partial x}d\tau\Big\}\mu(x)+\Big\{\sum_{i=0}^{m+1}L_i(x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)d\tau\Big\}\dot{\mu}(x)+$$

$$+\sum_{i=1}^{m+1}\Big[\dot{L}_i(x)\widetilde{u}(t_i,x)+L_i(x)\widetilde{v}(t_i,x)\Big]+$$

$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\Big[\frac{\partial K_j(\tau,x)}{\partial x}\widetilde{u}(\tau,x)+K_j(\tau,x)\widetilde{v}(\tau,x)\Big]d\tau=\varphi'(x),\qquad x\in[0,\omega]. \tag{2.9}$$

Relation (2.9) together with compatibility condition are equivalent to condition (2.4).

From the representation (2.8) we find the values of function $\widetilde{v}(t,x)$ at $t=t_i$, $i=\overline{0,m+1}$, and $t=\tau$. Substituting these values into (2.9), for unknown vector-function $\mu(x)$, we obtain a system of n ordinary differential equations of first order, unsolved with respect to derivatives:

$$M_l(x)\dot{\mu}(x)=-\widetilde{E}_l(x)\mu(x)-\widetilde{H}_l(x,\widetilde{u},\widetilde{w})-\widetilde{G}_l(x,\widetilde{v})-\widetilde{F}_l(x),\qquad x\in[0,\omega], \tag{2.10}$$

where

$$M_l(x)=L_0(x)+\sum_{i=1}^{m+1}L_i(x)[I+D_l(t_i,x)]+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)[I+D_l(\tau,x)]d\tau,$$

$$\widetilde{E}_l(x)=\sum_{i=0}^{m+1}\dot{L}_i(x)+\sum_{i=1}^{m+1}L_i(x)E_l(t_i,x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\frac{\partial K_j(\tau,x)}{\partial x}d\tau+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)E_l(\tau,x)d\tau,$$

$$\widetilde{H}_l(x,\widetilde{u},\widetilde{w})=\sum_{i=1}^{m+1}\dot{L}_i(x)\widetilde{u}(t_i,x)+\sum_{i=1}^{m+1}L_i(x)H_l(t_i,x,\widetilde{u},\widetilde{w})+$$

$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\frac{\partial K_j(\tau,x)}{\partial x}\widetilde{u}(\tau,x)d\tau+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)H_l(\tau,x,\widetilde{u},\widetilde{w})d\tau,$$

$$\widetilde{G}_l(x,\widetilde{v})=\sum_{i=1}^{m+1}L_i(x)G_l(t_i,x,\widetilde{v})+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)G_l(\tau,x,\widetilde{v})d\tau,$$

$$\widetilde{F}_l(x)=\sum_{i=1}^{m+1}L_i(x)F_l(t_i,x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)F_l(\tau,x)d\tau-\varphi'(x).$$

Taking into account the compatibility condition

$$\mu(0)=\psi(0), \tag{2.11}$$

we obtain the Cauchy problem for system (2.10).

Suppose that the $(n\times n)$ matrix $M_l(x)$ is invertible for all $x\in[0,\omega]$.

Introduction of additional functional parameter allows us to divide the process of finding the unknown functions into next two stages:

1) finding the function $\mu(x)$ ($\dot{\mu}(x)$) from system (2.10) under the condition (2.11).

2) finding the unknown functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, and $\widetilde{u}(t,x)$ from the system of integral equations (2.5)–(2.7).

If the functions $\dot{\mu}(x)$, and $\mu(x)$, are known, then solving the system of integral equations (2.5)–(2.7), we find the functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, $\widetilde{u}(t,x)$. Function $\mu(x)+\widetilde{u}(t,x)$ is the solution to the initial problem. If the functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, and $\widetilde{u}(t,x)$ are known, then solving the equation (2.10) under condition (2.11), we find $\dot{\mu}(x)$, and $\mu(x)$. Determining again the sum of functions $\mu(x)+\widetilde{u}(t,x)$, we define the solution to problem (1.1)–(1.3).

Here the functions $\dot{\mu}(x)$ and $\mu(x)$ are unknown, as well as the functions $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, and $\widetilde{u}(t,x)$. Therefore, the inverse problem for the system of hyperbolic equations (2.1)–(2.4) is to be solved using the iterative method. Solution to system (2.5)–(2.7), (2.10) with

condition (2.11) is to be found as the limits of sequences $\{\dot{\mu}^{(k)}(x)\}$, $\{\mu^{(k)}(x)\}$, $\{\widetilde{v}^{(k)}(t,x)\}$, $\{\widetilde{w}^{(k)}(t,x)\}$, $\{\widetilde{u}^{(k)}(t,x)\}$, $k = 0,1,2,...$, defined by the following algorithm:
0th step. Supposing $\mu(x) = \psi(0)$, $\widetilde{v}(t,x) = 0$, $\widetilde{w}(t,x) = \dot{\psi}(t)$ in the right-hand part of (2.10) $\widetilde{u}(t,x) = \psi(t) - \psi(0)$, and taking into account the invertibility of matrix $M_l(x)$ for all $x \in [0,\omega]$, from equation (2.10) we define $\dot{\mu}^{(0)}(x)$. Using the condition (2.11), we find the function $\mu^{(0)}(x)$:

$$\mu^{(0)}(x) = \psi(0) + \int_0^x \dot{\mu}^{(0)}(\xi)d\xi.$$

From the system of integral equations (2.5)–(2.7), where $\mu(x) = \mu^{(0)}(x)$, $\dot{\mu}(x) = \dot{\mu}^{(0)}(x)$, we determine the functions $\widetilde{v}^{(0)}(t,x)$, $\widetilde{w}^{(0)}(t,x)$, $\widetilde{u}^{(0)}(t,x)$.
1st step. From equation (2.10), where in the right-hand part $\mu(x) = \mu^{(0)}(x)$, $\widetilde{v}(t,x) = \widetilde{v}^{(0)}(t,x)$, $\widetilde{w}(t,x) = \widetilde{w}^{(0)}(t,x)$, and $\widetilde{u}(t,x) = \widetilde{u}^{(0)}(t,x)$ due to the invertibility of $M_l(x)$ for $x \in [0,\omega]$ we determine $\dot{\mu}^{(1)}(x)$. Using again condition (2.11), we find $\mu^{(1)}(x)$:

$$\mu^{(1)}(x) = \psi(0) + \int_0^x \dot{\mu}^{(1)}(\xi)d\xi,$$

and from the system of integral equations (2.5)–(2.7), where $\mu(x) = \mu^{(1)}(x)$ and $\dot{\mu}(x) = \dot{\mu}^{(1)}(x)$, we determine the functions $\widetilde{w}^{(1)}(t,x)$, $\widetilde{v}^{(1)}(t,x)$, $\widetilde{u}^{(1)}(t,x)$. And so on.

Introduce the notation

$$\alpha(x) = \max_{t\in[0,T]} ||A(t,x)||, \quad \beta(x) = \max_{t\in[0,T]} ||B(t,x)||, \quad \sigma(x) = \max_{t\in[0,T]} ||C(t,x)||,$$

$$\theta_i(x) = \sup_{t\in[t_{i-1},t_i)} ||K_i(t,x)||, \quad \vartheta_i(x) = \sup_{t\in[t_{i-1},t_i)} ||\tfrac{\partial K_i(t,x)}{\partial x}||, \quad i = \overline{1,m+1},$$

$$f_0(x) = \max_{t\in[0,T]} ||f(t,x)||.$$

Conditions of feasibility and convergence of the proposed algorithm provide the next assertion

Theorem 2.1. *Let for some l, $l = 1,2,...$, the $(n \times n)$ matrix $M_l(x)$ is invertible for all $x \in [0,\omega]$, and the following inequalities are valid:*

a) $||[M_l(x)]^{-1}|| \le \gamma_l(x)$, where $\gamma_l(x)$ is a function, positive and continuous on $x \in [0,\omega]$;

b) $q_l(x) = \gamma_l(x)\cdot\Big\{\sum_{i=1}^{m+1} ||L_i(x)|| + \theta_i(x)(t_i - t_{i-1})\Big\}\Big[e^{\alpha(x)t_i} - \sum_{j=0}^{l} \frac{[\alpha(x)t_i]^j}{j!}\Big] \le \chi < 1$, χ-const.

Then there exists a unique solution to inverse problem for the system of hyperbolic equations (2.1)–(2.4).

Proof. In virtue of the continuity of $L_i(t,x)$, $i = \overline{0,m+1}$, $A(t,x)$, $K_j(t,x)$, $j = \overline{1,m+1}$, the matrix $M_l(x)$ is continuous by $x \in [0,\omega]$. Then in view of theorem conditions and the inequality

$$||[M_l(x)]^{-1} - [M_l(\bar{x})]^{-1}|| \le ||[M_l(x)]^{-1}|| \cdot ||M_l(x) - M_l(\bar{x})|| \cdot ||[M_l(\bar{x})]^{-1}||,$$

where $x, \bar{x} \in [0,\omega]$, the matrix $[M_l(x)]^{-1}$ is also continuous for all $x \in [0,\omega]$.

The next inequalities are valid:

$$||\widetilde{E}_l(x)|| \le \sum_{i=0}^{m+1} ||\dot{L}_i(x)|| + \sum_{j=1}^{m+1} \vartheta_j(x)(t_j - t_{j-1}) +$$

$$+\sigma(x) \sum_{i=1}^{m+1} \Big(||L_i(x)||t_i + \theta_i(x)\frac{(t_i^2 - t_{i-1}^2)}{2}\Big) \sum_{j=0}^{l-1} \frac{[\alpha(x)t_i]^j}{j!} = a_0(x),$$

$$||\widetilde{F}_l(x)|| \le$$

$$\leq \Big\{1+\sum_{i=1}^{m+1}\Big(||L_i(x)||t_i+\theta_i(x)\frac{(t_i^2-t_{i-1}^2)}{2}\Big)\sum_{j=0}^{l-1}\frac{[\alpha(x)t_i]^j}{j!}\Big\}\max\Big(||\varphi'(x)||, f_0(x)\Big)=\phi(x),$$

$$||\widetilde{H}_l(x,\widetilde{u},\widetilde{w})|| \leq \Big\{\sum_{i=1}^{m+1}||\dot{L}_i(x)||+\sum_{j=1}^{m+1}\vartheta_j(x)(t_j-t_{j-1})+\sum_{i=1}^{m+1}\Big(||L_i(x)||t_i+$$

$$+\theta_i(x)\frac{(t_i^2-t_{i-1}^2)}{2}\Big)\sum_{j=0}^{l-1}\frac{[\alpha(x)t_i]^j}{j!}\max(\beta(x),\sigma(x))\Big\}\max_{t\in[0,T]}\Big[||\widetilde{w}(t,x)||+||\widetilde{u}(t,x)||\Big]=$$

$$=a_1(x)\max_{t\in[0,T]}\Big[||\widetilde{w}(t,x)||+||\widetilde{u}(t,x)||\Big]. \tag{2.12}$$

In view of conditions a) at fixed $\mu(x)$, $\widetilde{v}(t,x)$, $\widetilde{w}(t,x)$, and $\widetilde{u}(t,x)$, the function $\dot{\mu}(x)$ is defined uniquely from equation (2.10) and

$$\dot{\mu}(x)=-[M_l(x)]^{-1}\Big\{\widetilde{E}_l(x)\mu(x)+\widetilde{F}_l(x)+\widetilde{H}_l(x,\widetilde{u},\widetilde{w})+\widetilde{G}_l(x,\widetilde{v})\Big\},\qquad x\in[0,\omega].$$

Taking into account the condition (2.11), we define $\mu(x)$. At fixed $\mu(x)\in C([0,\omega],R^n)$ and $\dot{\mu}(x)\in C([0,\omega],R^n)$ the system of integral equations (2.5)–(2.7) has a unique solution $\{\widetilde{v}(t,x),\ \widetilde{w}(t,x),\ \widetilde{u}(t,x)\}$, where $\widetilde{v}$, $\widetilde{w}$, $\widetilde{u}$ belong to $C(\bar{\Omega},R^n)$, and the following estimations hold:

$$||\widetilde{v}(t,x)|| \leq \Big(e^{\alpha(x)t}-1\Big)||\dot{\mu}(x)||+$$

$$+Te^{\alpha(x)t}\Big\{\sigma(x)\cdot||\mu(x)||+f_0(x)+\max\{\beta(x),\sigma(x)\}\max_{t\in[0,T]}\Big[||\widetilde{u}(t,x)||+||\widetilde{w}(t,x)||\Big]\Big\}, \tag{2.13}$$

$$\max_{t\in[0,T]}\Big[||\widetilde{u}(t,x)||+||\widetilde{w}(t,x)||\Big]\leq\Big\{\max_{t\in[0,T]}||\psi(t)-\psi(0)||+\max_{t\in[0,T]}||\dot{\psi}(t)||+$$

$$+(1+T)\int_0^x\alpha(\xi)Te^{\alpha(\xi)T}||\dot{\mu}(\xi)||d\xi+(1+T)\int_0^x\Big[1+\alpha(\xi)Te^{\alpha(\xi)T}\Big]\Big(f_0(\xi)+\sigma(\xi)||\mu(\xi)||\Big)d\xi\Big\}\times$$

$$\times\exp\Big\{\int_0^x\Big[1+\alpha(\xi)Te^{\alpha(\xi)T}\Big]\max\{\beta(\xi),\sigma(\xi)\}d\xi\Big\}. \tag{2.14}$$

From the integral equation (2.5) using the inequality Bellman - Gronwall for the difference of successive approximations $\widetilde{v}^{(k)}(t,x)-\widetilde{v}^{(k-1)}(t,x)$, we obtain

$$||\widetilde{v}^{(k)}(t,x)-\widetilde{v}^{(k-1)}(t,x)|| \leq$$

$$\leq(e^{\alpha(x)t}-1)\cdot||\dot{\mu}^{(k)}(x)-\dot{\mu}^{(k-1)}(x)||+Te^{\alpha(x)t}\Big(\sigma(x)||\mu^{(k)}(x)-\mu^{(k-1)}(x)||+$$

$$+\max\{\beta(x),\sigma(x)\}\max_{t\in[0,T]}\Big[||\widetilde{w}^{(k)}(t,x)-\widetilde{w}^{(k-1)}(t,x)||+||\widetilde{u}^{(k)}(t,x)-\widetilde{u}^{(k-1)}(t,x)||\Big]\Big). \tag{2.15}$$

For the differences of successive approximations
$\mu^{(k)}(x)-\mu^{(k-1)}(x)$, $\widetilde{w}^{(k)}(t,x)-\widetilde{w}^{(k-1)}(t,x)$, and $\widetilde{u}^{(k)}(t,x)-\widetilde{u}^{(k-1)}(t,x)$, $k=1,2,...$, in virtue of inequalities (2.13)–(2.15) the next estimates are valid:

$$||\mu^{(k)}(x)-\mu^{(k-1)}(x)|| \leq \int_0^x||\dot{\mu}^{(k)}(\xi)-\dot{\mu}^{(k-1)}(\xi)||d\xi, \tag{2.16}$$

$$\max_{t\in[0,T]}\Big[||\widetilde{w}^{(k)}(t,x)-\widetilde{w}^{(k-1)}(t,x)||+||\widetilde{u}^{(k)}(t,x)-\widetilde{u}^{(k-1)}(t,x)||\Big]\leq$$

$$\leq\int_0^x a_2(\xi,x)||\dot{\mu}^{(k)}(\xi)-\dot{\mu}^{(k-1)}(\xi)||d\xi, \tag{2.17}$$

where $a_2(\xi,x)=e^{a_3(x)}(1+T)[\alpha(\xi)Te^{\alpha(\xi)T}+a_4(x)]$,

$$a_3(x) = \int_0^x [1 + \alpha(\xi) T e^{\alpha(\xi)T}] \max\{\beta(\xi), \sigma(\xi)\} d\xi, \qquad a_4(x) = \int_0^x [1 + \alpha(\xi) T e^{\alpha(\xi)T}] \sigma(\xi) d\xi.$$

Then for difference $\dot{\mu}^{(k+1)}(x) - \dot{\mu}^{(k)}(x)$, taking into account the inequality (2.10), we have the estimate

$$||\dot{\mu}^{(k+1)}(x) - \dot{\mu}^{(k)}(x)|| \le \gamma_l(x) ||\widetilde{E}_l(x)|| \cdot ||\mu^{(k)}(x) - \mu^{(k-1)}(x)|| +$$

$$+ \gamma_l(x) ||\widetilde{H}_l(x, \widetilde{u}^{(k)} - \widetilde{u}^{(k-1)}, \widetilde{w}^{(k)} - \widetilde{w}^{(k-1)})|| + \gamma_l(x) ||\widetilde{G}_l(x, \widetilde{v}^{(k)} - \widetilde{v}^{(k-1)})||.$$

In the last term, using estimates (2.12)–(2.17) and evaluating the repeated integrals, we get

$$||\dot{\mu}^{(k+1)}(x) - \dot{\mu}^{(k)}(x)|| \le \chi ||\dot{\mu}^{(k)}(x) - \dot{\mu}^{(k-1)}(x)|| + \int_0^x a_5(\xi, x) ||\dot{\mu}^{(k)}(\xi) - \dot{\mu}^{(k-1)}(\xi)|| d\xi, \quad (2.18)$$

where $a_5(\xi, x) = \gamma_l(x) \Big[a_0(x) + a_1(x) a_2(\xi, x) + [\sigma(x) + \max\{(\beta(x), \sigma(x)\} a_2(\xi, x)] a_6(x) \Big]$,

$$a_6(x) = T \sum_{i=1}^{m+1} \Big(||L_i(x)|| + \theta_i(x)(t_i - t_{i-1}) \Big) \Big[e^{\alpha(x) t_i} - \sum_{s=0}^{l-1} \frac{[\alpha(x) t_i]^s}{s!} \Big].$$

From the zero and first steps of algorithm there follow the estimates:

$$||\dot{\mu}^{(0)}(x)|| \le \gamma_l(x) \Big(a_0(x) ||\psi(0)|| + \phi(x) + a_1(x) \max_{t \in [0,T]} \{ ||\dot{\psi}(t)|| + ||\psi(t) - \psi(0)|| \} \Big) = d_1(x),$$

$$||\mu^{(0)}(x) - \psi(0)|| \le \int_0^x ||\dot{\mu}^{(0)}(\xi)|| d\xi \le \int_0^x d_1(\xi) d\xi = d_2(x),$$

$$||\dot{\mu}^{(1)}(x) - \dot{\mu}^{(0)}(x)|| \le \gamma_l(x) a_0(x) d_2(x) +$$

$$+ q_l(x) d_1(x) + \int_0^x [\gamma_l(x) a_1(x) a_2(\xi, x) + a_6(x) f_0(x) + a_5(\xi, x)] d_1(\xi) d\xi = d(x). \quad (2.19)$$

Based on (2.18), (2.19) we set the inequality

$$\Delta\mu_k(x) = ||\dot{\mu}^{(k+1)}(x) - \dot{\mu}^{(k)}(x)|| \le$$

$$\le \sum_{j=0}^{k} \frac{k! \, \chi^{k-j}}{(k-j)! \cdot j!} \frac{1}{j!} \Big\{ \int_0^x a_5(\xi, x) d\xi \Big\}^j \max_{x \in [0,\omega]} d(x) \le \chi^k \sum_{j=0}^{k} \frac{k!}{(k-j)! j!} \frac{1}{j!} \Big\{ \frac{\tilde{a}}{\chi} \Big\}^j \tilde{d}, \quad (2.20)$$

where

$$\tilde{a} = \max_{x \in [0,\omega]} \int_0^x a_5(\xi, x) d\xi, \quad \tilde{d} = \max_{x \in [0,\omega]} d(x).$$

From the theory of limits, using the consequence of the Töplitz theorem, it is easy to establish that the sequence $\Delta\mu_k(x)$ is majorizing by geometric progression. Hence there follows the uniform convergence of series $\sum_{k=1}^{\infty} \Delta\mu_k(x)$ for $x \in [0, \omega]$, which guarantees the uniform convergence of sequence $\dot{\mu}^{(k)}(x)$ to a function $\dot{\mu}^*(x)$, continuous by $x \in [0, \omega]$. From inequality (2.16) there follows the uniform convergence of sequence $\mu^{(k)}(x)$ to the function $\mu^*(x) \in C([0, \omega], R^n)$. Based on the estimates (2.15) and (2.17), there follows the convergence of sequences $\widetilde{v}^{(k)}(t, x)$, $\widetilde{w}^{(k)}(t, x)$, and $\widetilde{u}^{(k)}(t, x)$, uniform with respect to $(t, x) \in \bar{\Omega}$, to the functions $\widetilde{v}^*(t, x)$, $\widetilde{w}^*(t, x)$, and $\widetilde{u}^*(t, x)$ belonging $C(\Omega, R^n)$, respectively. It is obvious that the pair $(\mu^*(x), \widetilde{u}^*(t, x))$ is a solution to problem (2.1)–(2.4). The uniqueness of solution to problem (2.1)–(2.4) is proved by contradiction. Theorem 2.1 is proved. □

Thus, in case of introducing one parameter, the main condition for the solvability of problem (2.1)–(2.4) is the invertibility of matrix $M_l(x)$ composed by the matrices of boundary condition $L_i(x)$, $K_j(t,x)$ and the sums of repeated integrals of matrix $A(t,x)$ of dimension coinciding with the dimension of system (2.1). Choosing the number l, the amount of repeated integrals of matrix $A(t,x)$, we can check the invertibility of matrix $M_l(x)$, for all $x \in [0,\omega]$.

3. General scheme of the method of reduction to the inverse problem for the system of hyperbolic equations

In the case of introduction of single parameter, to check the solvability conditions for the investigated problem, as a tool to operate we have only l, the number of repeated integrals of matrix $A(t,x)$. However, for large values of $t = t_i$, $i = \overline{1, m+1}$, we have to choose the number l large enough, what leads to the difficulties with evaluating the repeated integrals and their sums. In this regard, in the section we propose a general scheme of the method of functional parameters, where the domain is divided into parts by lines $t = t_i$, $i = \overline{0, m}$, and on these partition lines, the parameters are introduced as the values of the desired function. This allows to operate with two quantities while solving the problem (2.1)–(2.4): the distances $t_i - t_{i-1}$ and l, the number of repeated integrals.

Make a partition on domain Ω: $\Omega = \bigcup\limits_{r=1}^{m+1} \Omega_r$, $\Omega_r = \bigcup\limits_{r=1}^{m+1} [t_{r-1}, t_r) \times [0,\omega]$. Denote by $u_r(t,x)$ the restriction of function $u(t,x)$ to Ω_r, $r = 1, 2, ..., m+1$. Then pass from problem (1.1)–(1.3) to the boundary value problem

$$\frac{\partial^2 u_r}{\partial x \partial t} = A(t,x)\frac{\partial u_r}{\partial x} + B(t,x)\frac{\partial u_r}{\partial t} + C(t,x)u_r + f(t,x), \qquad (t,x) \in \Omega_r, \tag{3.1}$$

$$\sum_{i=0}^{m} L_i(x) u_{i+1}(t_i, x) + L_{m+1}(x) \lim_{t \to T-0} u_{m+1}(t,x) + $$
$$+ \sum_{j=1}^{m+1} \int\limits_{t_{j-1}}^{t_j} K_j(\tau, x) u_j(\tau, x) d\tau = \varphi(x), \qquad x \in [0,\omega], \tag{3.2}$$

$$u_r(t,0) = \psi(t), \qquad t \in [t_{r-1}, t_r), \quad r = \overline{1, m+1}, \tag{3.3}$$

$$\lim_{t \to t_p - 0} \frac{\partial u_p(t,x)}{\partial x} = \frac{\partial u_{p+1}(t_p, x)}{\partial x}, \qquad x \in [0,\omega], \qquad p = \overline{1, m}. \tag{3.4}$$

Here the interrelations (3.4) are the conditions of gluing (continuity) the derivatives of solutions by x in the internal lines partitioning the domain Ω: $t = t_p$, $p = \overline{1, m}$.

Solution to problem (3.1)–(3.4) is a system of functions $u([t], x) = (u_1(t,x), u_2(t,x), ..., u_{m+1}(t,x))'$, where each function $u_r(t,x)$ with continuous partial derivatives of first order, mixed partial derivatives of second order, which is bounded on its definition domain Ω_r together with derivatives and satisfies the system of equations (3.1), boundary conditions (3.2), (3.3), and continuity condition (3.4). $[t]$ in the notation $u([t], x)$ means that partition is made by variable t.

Continuity and boundedness of function $u_r(t,x)$ together with derivatives on Ω_r, $r = \overline{1, m+1}$, leads to the existence of left-hand side limits $\lim\limits_{t \to t_r - 0} u_r(t,x)$, $\lim\limits_{t \to t_r - 0} \dfrac{\partial u_r(t,x)}{\partial x}$, and $\lim\limits_{t \to t_r - 0} \dfrac{\partial u_r(t,x)}{\partial t}$. The values $u_1(0,x)$, $u_{i+1}(t_i, x)$, $i = \overline{1, m}$, and $\lim\limits_{t \to T-0} u_{m+1}(t,x)$ satisfy the relation (3.3), the values $\lim_{t\to t_p-0}\dfrac{\partial u_p(t,x)}{\partial x}$, and $\dfrac{\partial u_{p+1}(t_p,x)}{\partial x}$, $p=\overline{1,m}$, satisfy the relations (3.4).

Denote by $\mu_r(x)$ the values of function $u_r(t,x)$ at $t=(r-1)h$. Making the replacement $\widetilde{u}_r(t,x)=u_r(t,x)-\mu_r(x)$, $r=\overline{1,N}$ in problem (3.1) – (3.4), we obtain the equivalent boundary problem with unknown functions $\mu_r(x)$:

$$\frac{\partial^2\widetilde{u}_r}{\partial x\partial t}=A(t,x)\frac{\partial\widetilde{u}_r}{\partial x}+B(t,x)\frac{\partial\widetilde{u}_r}{\partial t}+C(t,x)\widetilde{u}_r+f(t,x)+$$
$$+A(t,x)\dot{\mu}_r(x)+C(t,x)\mu_r(x),\qquad (t,x)\in\Omega_r,\quad r=\overline{1,m+1},\tag{3.5}$$
$$\widetilde{u}_r(t_{r-1},x)=0,\qquad x\in[0,\omega],\qquad r=\overline{1,m+1},\tag{3.6}$$
$$\widetilde{u}_r(t,0)=\psi(t)-\psi(t_{r-1}),\qquad t\in[t_{r-1},t_r),\qquad r=\overline{1,m+1},\tag{3.7}$$
$$\sum_{i=0}^{m}L_i(x)\mu_{i+1}(x)+L_{m+1}(x)\mu_{m+1}(x)+L_{m+1}(x)\lim_{t\to T-0}\widetilde{u}_{m+1}(t,x)+$$
$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)d\tau\mu_j(x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)\widetilde{u}_j(\tau,x)d\tau=\varphi(x),\qquad x\in[0,\omega],\tag{3.8}$$
$$\dot{\mu}_p(x)+\lim_{t\to t_p-0}\frac{\partial\widetilde{u}_p(t,x)}{\partial x}=\dot{\mu}_{p+1}(x),\qquad x\in[0,\omega],\qquad p=\overline{1,m}.\tag{3.9}$$

In condition (3.7), the compatibility condition at the points $(t_{r-1},0)$ is taken into account:

$$\mu_r(0)=\psi(t_{r-1}),\qquad r=\overline{1,m+1}.\tag{3.10}$$

Problems (3.1)–(3.4) is equivalent to problem (3.5)–(3.9) in the following sense: if the system of functions $u([t],x)=(u_1(t,x),u_2(t,x),...,u_{m+1}(t,x))'$ is a solution to (3.1)–(3.4), the system of pairs $(\mu(x),\ \widetilde{u}([t],x))$, where $\mu(x)=(\mu_1(x),\mu_2(x),...,\mu_{m+1}(x))'$, $\widetilde{u}([t],x)=(\widetilde{u}_1(t,x),\widetilde{u}_2(t,x),...,\ \widetilde{u}_{m+1}(t,x))'$, $\mu_r(x)=u_r(t_{r-1},x)$, $\widetilde{u}_r(t,x)=u_r(t,x)-u_r(t_{r-1},x)$, $r=\overline{1,m+1}$, is a solution to (3.5)–(3.9), and vice versa, if $(\mu_r(x),\ \widetilde{u}_r(t,x))$, $r=\overline{1,m+1}$, is a solution to (3.5)–(3.9), then $(\mu_r(x)+\widetilde{u}_r(t,x))$, $r=\overline{1,m+1}$, is a solution to (3.1)–(3.4).

Problem (3.5)–(3.9) is the inverse problem for the system of hyperbolic equations with functional parameters, where relations (3.8) and (3.9) allow us to determine the unknown functions $\mu_r(x)$, $r=\overline{1,m+1}$.

Let $C(\Omega,\Omega_r,R^{n(m+1)})$ be a space of function systems $u([t],x)=(u_1(t,x),u_2(t,x),...,$ $u_{m+1}(t,x))'$, where the function $u_r:\Omega_r\longrightarrow R^n$ is continuous and has a finite left-hand side limit $\lim_{t\to t_r-0}u_r(t,x)$, $r=\overline{1,m+1}$, uniformly with respect to $x\in[0,\omega]$, with the norm

$$||u||_1=\max_{r=\overline{1,m+1}}\ \sup_{(t,x)\in\Omega_r}||u_r(t,x)||.$$

At fixed $\mu_r(x),\dot{\mu}_r(x)$, $r=\overline{1,m+1}$, the functions $\widetilde{u}_r(t,x)$, $r=\overline{1,m+1}$, are the solutions to Goursat problem on Ω_r with conditions (3.6), (3.7).

Introducing notation $\widetilde{v}_r(t,x)=\dfrac{\partial\widetilde{u}_r(t,x)}{\partial x}$, $\widetilde{w}_r(t,x)=\dfrac{\partial\widetilde{u}_r(t,x)}{\partial t}$, from (3.6) and (3.7) we obtain $\widetilde{v}_r(t_{r-1},x)=0$, $\widetilde{w}_r(t,0)=\dot{\psi}(t)$, and reduce the Goursat problem to the system of three integral equations

$$\widetilde{w}_r(t,x)=\dot{\psi}(t)+\int\limits_0^x[A(t,\xi)\widetilde{v}_r(t,\xi)+B(t,\xi)\widetilde{w}_r(t,\xi)+C(t,\xi)\widetilde{u}_r(t,\xi)+$$
$$+f(t,\xi)+A(t,\xi)\dot{\mu}_r(\xi)+C(t,\xi)\mu_r(\xi)]d\xi,\tag{3.11}$$
$$\widetilde{v}_r(t,x)=\int\limits_{(r-1)h}^{t}[A(s,x)\widetilde{v}_r(s,x)+B(s,x)\widetilde{w}_r(s,x)+C(s,x)\widetilde{u}_r(s,x)+$$

$$+f(s,x)+A(s,x)\dot{\mu}_r(x)+C(s,x)\mu_r(x)\big]ds, \tag{3.12}$$

$$\widetilde{u}_r(t,x)=\psi(t)-\psi(t_{r-1})+\int\limits_{t_{r-1}}^{t}\int\limits_{0}^{x}\big[A(s,\xi)\widetilde{v}_r(s,\xi)+B(s,\xi)\widetilde{w}_r(s,\xi)+C(s,\xi)\widetilde{u}_r(s,\xi)+$$
$$+f(s,\xi)+A(s,\xi)\dot{\mu}_r(\xi)+C(s,\xi)\mu_r(\xi)\big]d\xi ds. \tag{3.13}$$

Substituting the corresponding right-hand side of (3.12) instead of $\widetilde{v}_r(s,x)$ and repeating the process l $(l=1,2,...)$ times, we get

$$\widetilde{v}_r(t,x)=D_{l,r}(t,x)\dot{\mu}_r(x)+E_{l,r}(t,x)\mu_r(x)+H_{l,r}(t,x,\widetilde{u}_r,\widetilde{w}_r)+$$
$$+G_{l,r}(t,x,\widetilde{v}_r)+F_{l,r}(t,x),\quad (t,x)\in\Omega_r,\ r=\overline{1,m+1}, \tag{3.14}$$

$$D_{l,r}(t,x)=\int\limits_{t_{r-1}}^{t}A(s_1,x)ds_1+...+\int\limits_{t_{r-1}}^{t}A(s_1,x)\int\limits_{t_{r-1}}^{s_1}A(s_2,x)...\int\limits_{t_{r-1}}^{s_{l-1}}A(s_l,x)ds_l...ds_2ds_1,$$

$$E_{l,r}(t,x)=\int\limits_{t_{r-1}}^{t}C(s_1,x)ds_1+...+\int\limits_{t_{r-1}}^{t}A(s_1,x)...\int\limits_{t_{r-1}}^{s_{l-2}}A(s_{l-1},x)\int\limits_{t_{r-1}}^{s_{l-1}}C(s_l,x)ds_l...ds_1,$$

$$H_{l,r}(t,x,\widetilde{u}_r,\widetilde{w}_r)=\int\limits_{t_{r-1}}^{t}\Big[B(s_1,x)\widetilde{w}_r(s_1,x)+C(s_1,x)\widetilde{u}_r(s_1,x)\Big]ds_1+...+$$
$$+\int\limits_{t_{r-1}}^{t}A(s_1,x)...\int\limits_{t_{r-1}}^{s_{l-2}}A(s_{l-1},x)\int\limits_{t_{r-1}}^{s_{l-1}}\Big[B(s_l,x)\widetilde{w}_r(s_l,x)+C(s_l,x)\widetilde{u}_r(s_l,x)\Big]ds_l...ds_1,$$

$$G_{l,r}(t,x,\widetilde{v}_r)=\int\limits_{t_{r-1}}^{t}A(s_1,x)...\int\limits_{t_{r-1}}^{s_{l-2}}A(s_{l-1},x)\int\limits_{t_{r-1}}^{s_{l-1}}A(s_l,x)\widetilde{v}_r(s_l,x)ds_l...ds_1,$$

$$F_{l,r}(t,x)=\int\limits_{t_{r-1}}^{t}f(s_1,x)ds_1+...+\int\limits_{t_{r-1}}^{t}A(s_1,x)...\int\limits_{t_{r-1}}^{s_{l-2}}A(s_{l-1},x)\int\limits_{t_{r-1}}^{s_{l-1}}f(s_l,x)ds_l...ds_1.$$

Differentiating the relation (3.8) by x

$$\sum_{i=0}^{m}\dot{L}_i(x)\mu_{i+1}(x)+\sum_{i=0}^{m}L_i(x)\dot{\mu}_{i+1}(x)+\dot{L}_{m+1}(x)\mu_{m+1}(x)+L_{m+1}(x)\dot{\mu}_{m+1}(x)+$$
$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\frac{\partial K_j(\tau,x)}{\partial x}d\tau\mu_j(x)+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)d\tau\dot{\mu}_j(x)+$$
$$+\dot{L}_{m+1}(x)\lim_{t\to T-0}\widetilde{u}_{m+1}(t,x)+L_{m+1}(x)\lim_{t\to T-0}\widetilde{v}_{m+1}(t,x)+$$
$$+\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\Big\{\frac{\partial K_j(\tau,x)}{\partial x}\widetilde{u}_j(\tau,x)+K_j(\tau,x)\widetilde{v}_j(\tau,x)\Big\}d\tau=\varphi'(x),\qquad x\in[0,\omega], \tag{3.15}$$

then passing to the limit at $t\to t_r-0$ in right-hand part of (3.14), we find $\lim\limits_{t\to t_r-0}\widetilde{v}_r(t,x)$, $r=\overline{1,m+1}$, $\widetilde{v}_r(\tau,x)$, $x\in[0,\omega]$. Substituting them into (3.9) and (3.15), multiplying both parts of (3.15) by $h=t_{m+1}-t_m>0$, for the unknown vector-functions $\mu_r(x)$, $r=\overline{1,m+1}$, we obtain the system of $m+1$ ordinary differential equations of first order unsolved with respect to derivatives:

$$Q_l(m,x)\dot{\mu}(x)=-\widetilde{E}_l(m,x)\mu(x)-$$

$$-\widetilde{H}_l(m,x,\widetilde{u},\widetilde{w})-\widetilde{G}_l(m,x,\widetilde{v})-\widetilde{F}_l(m,x), \qquad x\in[0,\omega], \tag{3.16}$$

where

$$Q_l(m,x)=$$

$$\begin{vmatrix} h\widetilde{L}_1(x) & h\widetilde{L}_2(x) & h\widetilde{L}_3(x) & .. & h\widetilde{L}_m(x) & h(\widetilde{L}_{m+1}(x)+L_{m+1}(x)[I+D_{l,m+1}(T,x)]) \\ I+D_{l,1}(t_l,x) & -I & 0 & .. & 0 & 0 \\ 0 & I+D_{l,2}(t_2,x) & -I & .. & 0 & 0 \\ \dots & \dots & \dots & .. & \dots & \dots \\ 0 & 0 & 0 & .. & I+D_{l,m}(t_m,x) & -I \end{vmatrix}$$

$$\widetilde{L}_j(x)=L_{j-1}(x)+\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)[I+D_{l,j}(\tau,x)]d\tau,$$

I is identity matrix on dimension $n\times n$,

$$\widetilde{E}_l(m,x)=$$

$$\begin{vmatrix} h\widetilde{K}_1(x) & h\widetilde{K}_2(x) & h\widetilde{K}_3(x) & .. & h\widetilde{K}_m(x) & h(\widetilde{K}_{m+1}(x)+\dot{L}_{m+1}(x)+L_{m+1}(x)E_{l,m+1}(T,x)) \\ E_{l,1}(t_l,x) & 0 & 0 & .. & 0 & 0 \\ 0 & E_{l,2}(t_2,x) & 0 & .. & 0 & 0 \\ \dots & \dots & \dots & .. & \dots & \dots \\ 0 & 0 & 0 & .. & E_{l,m}(t_m,x) & 0 \end{vmatrix},$$

$$\widetilde{K}_j(x)=\dot{L}_{j-1}(x)+\int\limits_{t_{j-1}}^{t_j}\frac{\partial K_j(\tau,x)}{\partial x}d\tau+\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)E_{l,j}(\tau,x)d\tau, \quad j=\overline{1,m+1},$$

$$\widetilde{H}_l(m,x,\widetilde{u},\widetilde{w})=\Big(h\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}\Big\{\frac{\partial K_j(\tau,x)}{\partial x}\widetilde{u}_j(\tau,x)+K_j(\tau,x)H_{l,j}(\tau,x,\widetilde{u}_j,\widetilde{w}_j)\Big\}d\tau+$$

$$+h\dot{L}_{m+1}(x)\lim_{t\to T-0}\widetilde{u}_{m+1}(t,x),H_{l,1}(m,x,\widetilde{w}_1,\widetilde{u}_1),...,H_{l,m}(m,x,\widetilde{w}_m,\widetilde{u}_m)\Big)',$$

$$\widetilde{G}_l(m,x,\widetilde{v})=\Big(h\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)G_{l,j}(\tau,x,\widetilde{v}_j)d\tau+hL_{m+1}(x)G_{l,m+1}(T,x,\widetilde{v}_{m+1}),$$

$$G_{l,1}(m,x,\widetilde{v}_1),\dots,G_{l,m}(m,x,\widetilde{v}_m)\Big)',$$

$$\widetilde{F}_l(m,x)=\Big(-h\varphi'(x)+h\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}K_j(\tau,x)F_{l,j}(\tau,x)d\tau+hL_{m+1}(x)F_{l,m+1}(T,x),$$

$$F_{l,1}(t_1,x),\dots,F_{l,m}(t_m,x)\Big)'.$$

The matrix $Q_l(m,x)$ has a special structure, so for any $x\in[0,\omega]$ it translates the elements of $R^{n(m+1)}$ into $R^{n(m+1)}$, and

$$||Q_l(m,x)||\le$$

$$\le 1+h\sum_{i=0}^{m}||L_i(x)||+\max\Big(h||L_{m+1}(x)||+h\sum_{j=1}^{m+1}\int\limits_{t_{j-1}}^{t_j}||K_j(\tau,x)||d\tau,1\Big)\sum_{s=0}^{l}\frac{[\alpha(x)\tilde{h}]^s}{s!},$$

where $\tilde{h}=\max\limits_{i=\overline{1,m+1}}(t_i-t_{i-1})$. Continuity of matrices $A(t,x)$ on Ω, $K_j(t,x)$ on Ω_j, $j=\overline{1,m+1}$, and $L_i(x)$, $i=\overline{0,m+1}$, on $[0,\omega]$, respectively, leads to its continuity by $x\in[0,\omega]$ for any $l\in\mathbb{N}$.

If the functions $\dot{\mu}(x)\in C([0,\omega],R^{n(m+1)})$, and $\mu(x)\in C([0,\omega],R^{n(m+1)})$ with components

$\dot{\mu}_r(x)$, $\mu_r(x)$, $r=\overline{1,m+1}$, are known, then solving the system of integral equations (3.11)–(3.13), we find the functions $\widetilde{w}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$, $\widetilde{v}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$, and $\widetilde{u}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$ with components $\widetilde{w}_r(t,x)$, $\widetilde{v}_r(t,x)$, and $\widetilde{u}_r(t,x)$. Function $u(t,x)$ defined by the equalities

$$u(t,x)=\mu_r(x)+\widetilde{u}_r(t,x), \qquad (t,x)\in\Omega_r, \qquad r=\overline{1,m+1},$$

$$u(T,x)=\mu_{m+1}(x)+\lim_{t\to T-0}\widetilde{u}_{m+1}(t,x), \qquad x\in[0,\omega],$$

is a solution to initial problem (1.1)–(1.3).

If the functions $\widetilde{w}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$, $\widetilde{v}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$, and $\widetilde{u}([t],x)\in C(\Omega,\Omega_r,R^{n(m+1)})$ with components $\widetilde{w}_r(t,x)$, $\widetilde{v}_r(t,x)$, and $\widetilde{u}_r(t,x)$, are known, then solving the equation (3.16) under condition (3.10), we find $\dot{\mu}(x)\in C([0,\omega],R^{n(m+1)})$, and $\mu(x)\in C([0,\omega],R^{n(m+1)})$ with components $\dot{\mu}_r(x)$, $\mu_r(x)$, $r=\overline{1,m+1}$, and defining again the function $u(t,x)$ by equalities

$u(t,x)=\mu_r(x)+\widetilde{u}_r(t,x)$, $\quad (t,x)\in\Omega_r$, $r=\overline{1,m+1}$,

$u(T,x)=\mu_{m+1}(x)+\lim_{t\to T-0}\widetilde{u}_{m+1}(t,x)$, $\quad x\in[0,\omega]$,

we find a solution to problem (1.1)–(1.3).

Here the unknowns are the functions $\dot{\mu}(x)$, $\mu(x)$, as well as the functions $\widetilde{w}([t],x)$, $\widetilde{v}([t],x)$, and $\widetilde{u}([t],x)$. Therefore, we apply an iterative method and find the solution of functional relations (3.11)–(3.13), (3.16) with condition (3.10) as the limits of sequences $\{\dot{\mu}^{(k)}(x)\}$, $\{\mu^{(k)}(x)\}$, $\{\widetilde{w}^{(k)}([t],x)\}$, $\{\widetilde{v}^{(k)}([t],x)\}$, $\{\widetilde{u}^{(k)}([t],x)\}$, $k=0,1,2,...$, defining by the following algorithm:

0th step. In the right-hand side of (3.16), assuming $\mu_r(x)=\psi(t_{r-1})$, $\widetilde{v}_r(t,x)=0$, $\widetilde{w}_r(t,x)=\dot{\psi}(t)$, and $\widetilde{u}_r(t,x)=\psi(t)-\psi(t_{r-1})$, $r=\overline{1,m+1}$, and taking into account that under selected $\in\mathbb{N}$, the matrix $Q_l(m,x)$ is invertible for all $x\in[0,\omega]$, from the system of equations (3.16) we determine $\dot{\mu}^{(0)}(x)\in C([0,\omega],R^{n(m+1)})$. Using the condition (3.10), we find the function $\mu_r^{(0)}(x)$:

$$\mu_r^{(0)}(x)=\psi(t_{r-1})+\int\limits_0^x\dot{\mu}_r^{(0)}(\xi)d\xi, \qquad r=\overline{1,m+1}.$$

From the system of integral equations (3.11)–(3.13), where $\mu_r(x)=\mu_r^{(0)}(x)$, $\dot{\mu}_r(x)=\dot{\mu}_r^{(0)}(x)$, $r=\overline{1,m+1}$, we find the functions $\widetilde{w}_r^{(0)}(t,x)$, $\widetilde{v}_r^{(0)}(t,x)$, $\widetilde{u}_r^{(0)}(t,x)$, $(t,x)\in\Omega_r$, $r=\overline{1,m+1}$.

1st step. From the system of equations (3.16), where we have $\mu(x)=\mu^{(0)}(x)$, $\widetilde{v}(t,x)=\widetilde{v}^{(0)}(t,x)$, $\widetilde{w}(t,x)=\widetilde{w}^{(0)}(t,x)$, and $\widetilde{u}(t,x)=\widetilde{u}^{(0)}(t,x)$ in the right-hand side, in virtue of invertibility of $Q_l(m,x)$ for $x\in[0,\omega]$, we determine $\dot{\mu}^{(1)}(x)\in C([0,\omega],R^{n(m+1)})$. Using again the conditions (3.10), we find $\mu_r^{(1)}(x)$:

$$\mu_r^{(1)}(x)=\psi(t_{r-1})+\int\limits_0^x\dot{\mu}_r^{(1)}(\xi)d\xi, \qquad r=\overline{1,m+1}.$$

From the system of integral equations (3.11)–(3.13), where $\mu_r(x)=\mu_r^{(1)}(x)$ and $\dot{\mu}_r(x)=\dot{\mu}_r^{(1)}(x)$, we determine the functions $\widetilde{w}_r^{(1)}(t,x)$, $\widetilde{v}_r^{(1)}(t,x)$, and $\widetilde{u}_r^{(1)}(t,x)$, $(t,x)\in\Omega_r$, $r=\overline{1,m+1}$. And so on.

The process of finding unknown functions is also divided into two parts:

1) finding the introduced parameters $\dot{\mu}_r(x)$, $\mu_r(x)$ from the system of functional equations (3.16) with condition (3.10).

2) finding the unknown functions $\widetilde{w}_r(t,x)$, $\widetilde{v}_r(t,x)$, and $\widetilde{u}_r(t,x)$, from the Goursat problems for the system of hyperbolic equations (3.5)–(3.7).

Sufficient conditions for the feasibility and convergence of the algorithm proposed above, which provide the unique solvability for inverse problem (3.5)–(3.9), are given by the following theorem

Theorem 3.1. *For some $l, l \in \mathbb{N}$, let the matrix $Q_l(m,x) : R^{n(m+1)} \to R^{n(m+1)}$ be invertible for all $x \in [0,\omega]$ and the following inequalities be valid:*

a) $||[Q_l(m,x)]^{-1}|| \leq \gamma_l(m,x)$;

$\gamma_l(m,x)$ *is a function, positive and continuous by* $x \in [0,\omega]$;

b) $$q_l(m,x) = \gamma_l(m,x) \cdot \Big\{\max\Big(h||L_{m+1}(x)||, 1\Big) + h \sum_{i=1}^{m+1} \theta_i(x)(t_i - t_{i-1})\Big\} \times$$
$$\times \Big[e^{\alpha(x)\tilde{h}} - \sum_{j=0}^{l} \frac{[\alpha(x)\tilde{h}]^j}{j!}\Big] \leq \chi < 1, \qquad \chi\text{- const.}$$

Then there exists a unique solution to problem (3.5)–(3.9).

Proof of Theorem 3.1 is analogous to the proof of Theorem 2.1 and is based on the algorithm provided above.

4. Criteria of unique solvability of problem (1.1)–(1.3)

The proof of Theorem 2.1 yields that the function $u^*(t,x)$, received as the sum $\mu^*(x) + \widetilde{u}^*(t,x)$, belongs to $C(\Omega, R^n)$ and is a classical solution to problem (1.1)–(1.3).

Then the equivalence of problem (1.1)–(1.3) to problem (2.1)–(2.4) leads to the following assertion

Theorem 4.1. *For some l, $l = 1, 2, ...$, let the $(n \times n)$ matrix $M_l(x)$ be invertible for all $x \in [0,\omega]$ and the inequalities a), b) of Theorem 3.1 be hold.*

Then there exists a unique solution to problem (1.1)–(1.3).

If the system of pairs $(\mu^*(x), \widetilde{u}^*([t], x))$ is a solution to problem (3.5)–(3.9), found by the algorithm given in Section 3, then the function $u^*(t,x)$ defined by the equalities

$$u^*(t,x) = \mu_r^*(x) + \widetilde{u}_r^*(t,x), \qquad (t,x) \in \Omega_r, \qquad r = \overline{1, m+1},$$

$$u^*(T,x) = \mu_{m+1}^*(x) + \lim_{t \to T-0} \widetilde{u}_{m+1}^*(t,x), \qquad x \in [0,\omega],$$

where $\mu_r^*(x)$, $\widetilde{u}_r^*(t,x)$ are the components of $\mu^*(x)$, $\widetilde{u}^*([t],x)$, $r = \overline{1, m+1}$, respectively, is a solution to problem (1.1)–(1.3).

Again from the equivalence of problem (1.1)–(1.3) to problem (3.5)–(3.9) we get

Theorem 4.2. *For some $l, l \in \mathbb{N}$, let the matrix $Q_l(m,x) : R^{n(m+1)} \to R^{n(m+1)}$ be invertible for all $x \in [0,\omega]$ and the inequalities a), b) of Theorem 3.1 be hold.*

Then there exists a unique solution to problem (1.1)–(1.3).

Existence of number $l \in \mathbb{N}$, for which the $(n \times n)$ matrix $M_l(x)$ is invertible at all $x \in [0,\omega]$, the $(n(m+1) \times n(m+1))$ matrix $Q_l(m,x)$ is invertible at all $x \in [0,\omega]$, is the main condition for the unique solvability of investigated problems (2.1)–(2.4) and (3.5)–(3.9), respectively. Dimension of matrix $M_l(x)$ coincides with dimension of initial system (1.1). Dimension of matrix $Q_l(m,x)$ depends on the numbers of points t_i, $i = \overline{0, m+1}$. Since the $(n(m+1) \times n(m+1))$ matrix $Q_l(m,x)$ has a special structure, then the next lemmas are valid

Lemma 4.3. *The $(n(m+1) \times n(m+1))$ matrix $Q_l(m,x)$ is invertible for all $x \in [0,\omega]$ if only if the $(n \times n)$ matrix*

$$\widetilde{M}_l(m,x) = L_0(x) + \sum_{i=1}^{m+1} L_i(x) \prod_{s=i}^{1} [I + D_{l,s}(t_s, x)] +$$

$$+\int\limits_{t_0}^{t_1} K_1(\tau,x)[I+D_{l,1}(\tau,x)]d\tau+\sum_{i=2}^{m+1}\int\limits_{t_{i-1}}^{t_i} K_i(\tau,x)[I+D_{l,i}(\tau,x)]d\tau\prod_{s=i-1}^{1}[I+D_{l,s}(t_s,x)]$$

is invertible.

Lemma 4.4. *If the matrix* $\widetilde{M}_l(m,x)$ *is invertible, then*

$$[Q_l(m,x)]^{-1}=\{\nu_{r,j}(x)\},\qquad r,j=\overline{1,m+1},$$

where

$$\nu_{1,1}(x)=h^{-1}\widetilde{M}_l^{-1}(m,x);$$

$$\nu_{1,j}(x)=\widetilde{M}_l^{-1}(m,x)\Big\{L_{j-1}(x)+\sum_{i=j}^{m+1}L_i(x)\prod_{s=i}^{j}[I+D_{l,s}(t_s,x)]+$$

$$+\int\limits_{t_{j-1}}^{t_j} K_j(\tau,x)[I+D_{l,j}(\tau,x)]d\tau+$$

$$+\sum_{r=j+1}^{m+1}\int\limits_{t_{r-1}}^{t_r} K_r(\tau,x)[I+D_{l,r}(\tau,x)]d\tau\prod_{s=r-1}^{j}[I+D_{l,s}(t_s,x)]\Big\},\quad 1<j\le m+1;$$

$$\nu_{r,r}(x)=[I+D_{l,r-1}(t_{r-1},x)]\nu_{r-1,r}(x)-I,\qquad r=2,3,...,m+1,$$

$$\nu_{r,j}(x)=[I+D_{l,r-1}(t_{r-1},x)]\nu_{r-1,j}(x),\qquad j\neq r.$$

Lemma 4.3 demonstrates that the invertibility of matrix $Q_l(m,x)$ of dimension $(n(m+1)\times n(m+1))$ is equivalent to the invertibility of matrix $\widetilde{M}_l(m,x)$ of dimension, coinciding with the dimension of initial system (1.1). Lemma 4.4 allows us identify block by block the elements of the inverse matrix $[Q_l(m,x)]^{-1}$.

So, we investigated the nonlocal boundary value problem for the system of hyperbolic equations of the second order with generalized integral condition. By method of introduction of functional parameters the considered problem is reduced to the inverse problem for the system of hyperbolic equations with unknown parameters and additional functional relations. Algorithms of finding solution to the inverse problem for the system of hyperbolic equations are constructed, and their convergence is proved. The conditions for existence of unique solution to the inverse problem for the system of hyperbolic equations are obtained in the terms of initial data. The coefficient conditions for unique solvability of nonlocal boundary value problem for the system of hyperbolic equations with generalized integral condition are established. Further, we propose an one numerical approach for solve this nonlocal problem. Numerical method based on algorithms of parametrization method [16]. This numerical approach is illustrated by examples.

5. Numerical examples

Consider the following problem with integral condition for system of hyperbolic equations

$$\frac{\partial^2 u}{\partial x\partial t}=A(t)\frac{\partial u}{\partial x}+f(t),\tag{5.1}$$

$$\sum_{i=0}^{3}L_iu(t_i,x)+\sum_{j=1}^{3}\int\limits_{t_{j-1}}^{t_j}K_j(\tau)u(\tau,x)d\tau=\varphi\cdot x,\qquad x\in[0,\omega],\tag{5.2}$$

$$u(t,0)=\psi(t),\qquad t\in[0,T].\tag{5.3}$$

Here $u = (u_1, u_2)$, $\quad t_0 = 0, \quad t_1 = \frac{1}{4}, \quad t_2 = \frac{1}{2}, \quad t_3 = T = 1, \quad \omega = 1,$

$$A(t) = \begin{pmatrix} \cos t & t^2 \\ 7 & t+9 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} e^t - e^t\cos t + 6\cos t - t^5 - 9t^3 + 4t^2 \\ -7e^t - t^4 - 9t^3 - 6t^2 - 77t + 87 \end{pmatrix},$$

$$L_0 = \begin{pmatrix} 1 & 3 \\ -5 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 4 & -6 \\ 8 & 3 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -1 & 7 \\ 0 & -1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 6 & 8 \\ -5 & -1 \end{pmatrix},$$

$$K_j(t) = K(t) = \begin{pmatrix} t & -3 \\ 0 & 12 \end{pmatrix}, \ j = \overline{1,4}, \quad \varphi = \begin{pmatrix} 6e - e^{\frac{1}{2}} + 4e^{\frac{1}{4}} - \frac{399}{32} \\ -5e + 8e^{\frac{1}{4}} + \frac{11}{64} \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}.$$

Conditions of Theorem 2.1 are fulfilled for $l = 2$.

We introduce a new function $v(t,x) = \frac{\partial u(t,x)}{\partial x}$ and reduce to an equivalent problem

$$\frac{\partial v}{\partial t} = A(t)v + f(t), \tag{5.4}$$

$$L_0 v(t_0,x) + L_1 v(t_1,x) + L_2 v(t_2,x) + L_3 v(t_3,x) + \int_0^T K(\tau)v(\tau,x)d\tau = \varphi, \quad x \in [0,1], \tag{5.5}$$

$$u(t,x) = \psi(t) + \int_0^x v(t,\xi)d\xi. \tag{5.6}$$

Here integral relation (5.6) allow us to determine of function $u(t,x)$ for all $(t,x) \in \Omega = [0,1] \times [0,1]$.

From (5.4) and (5.5) it follows that $v(t,x) = v(t)$, i.e. the function v not depends of x. Then, for for finding of approximate solution to problem (5.4)–(5.5) we we use the numerical implementation of algorithm of the parametrization method.

Let $\Phi(t)$ is fundamental matrix of differential equation $\frac{\partial v}{\partial t} = A(t)v$.

Consider the system of equations [[16], p. 347]

$$L_0\mu_1 + L_1\mu_2 + L_3\mu_3 + L_4\mu_4 + \sum_{k=1}^{4} \int_{t_{k-1}}^{t_k} K(t)\Phi(t) \int_{t_{k-1}}^{t} \Phi^{-1}(\tau)A(\tau)d\tau\mu_k dt =$$

$$= \varphi - \sum_{k=1}^{4} \int_{t_{k-1}}^{t_k} K(t)\Phi(t) \int_{t_{k-1}}^{t} \Phi^{-1}(\tau)f(\tau)d\tau dt, \tag{5.7}$$

$$\mu_p + \Phi(t_p) \int_{t_{p-1}}^{t_p} \Phi^{-1}(\tau)A(\tau)d\tau\mu_p - \mu_{p+1} = -\Phi(t_p) \int_{t_{p-1}}^{t_p} \Phi^{-1}(\tau)f(\tau)d\tau, \qquad p = 1,2,3. \tag{5.8}$$

We provide the results of the numerical implementation of algorithm in [16] by partitioning the subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 1]$ with step $h_1 = h_2 = h_3 = 0.025$.

Solving the system of equations (5.7), (5.8) we obtain the numerical values of the parameters

$$\mu_1^{\widetilde{h}} = \begin{pmatrix} -4.99999905 \\ -4.00000042 \end{pmatrix}, \quad \mu_2^{\widetilde{h}} = \begin{pmatrix} -4.71597337 \\ -1.73437503 \end{pmatrix}, \quad \mu_3^{\widetilde{h}} = \begin{pmatrix} -4.35127721 \\ 0.62500044 \end{pmatrix},$$

We find the numerical solutions at the other points of the subintervals using the Runge-Kutta method of the 4th order to the following Cauchy problems

$$\frac{d\widetilde{v}_r}{dt} = A(t)\widetilde{v}_r + f(t), \qquad t \in [t_{r-1}, t_r), \qquad r = \overline{1,3},$$

$$\widetilde{v}_r(t_{r-1}) = \mu_r^{\widetilde{h}}, \qquad r = \overline{1,3},$$

where $\widetilde{v}_r(t) = v(t)$, $\quad t \in [t_{r-1}, t_r)$, $r = 1,2,3$.

Exact solution of the problem (5.4), (5.5) is

$$v^*(t) = \begin{pmatrix} e^t - 6 \\ t^3 + 9t - 4 \end{pmatrix}.$$

Therefore, from integral relation (5.6) we find of exact solution of problem (5.1)–(5.3):

$$u^*(t,x) = \psi(t) + \int\limits_0^x v^*(t,\xi)d\xi = \begin{pmatrix} t^2 + (e^t - 6)x \\ t^3 + (t^3 + 9t - 4)x \end{pmatrix}.$$

The results of calculations of numerical and exact solutions of problem (5.4), (5.5) at discrete points are presented in the table 1.

For the difference of the corresponding values of the exact and constructed solutions of problem (5.4), (5.5) the following estimate is true

$$\max_{j=\overline{0,40}} \|v^*(t_j) - \widetilde{v}(t_j)\| < 0.000002.$$

Using integral relation (5.6) we find of the difference of the corresponding values of the exact and constructed solutions of problem (5.1)–(5.3) of an example the following estimate is true

$$\max_{x\in[0,1]} \max_{j=\overline{0,40}} \|u^*(t_j,x) - \widetilde{u}(t_j,x)\| < 0.000002.$$

6. Conclusion

In this paper, we propose a constructive method for solving the generalized integral problem for the system of hyperbolic equations (1.1)–(1.3). Conditions for the unique solvability of problem (1.1)–(1.3) are established in terms of solvability of the system of functional equations and in terms of input data. We develop the algorithms for finding approximate solutions to problem (1.1)–(1.3) and prove their convergence to the exact solution. Based on the approach proposed, we also construct numerical method for solving problem (1.1)–(1.3). Further, these results will be developed to problem for partial differential equations of Sobolev-type [4], nonlocal problems for pseudo-parabolic equations [5] singular problem for hyperbolic equations [6].

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Table 1. Numerical analysis.

t	$\widetilde{v}_1(t)$	$v_1^*(t)$	$\widetilde{v}_2(t)$	$v_2^*(t)$
0	-4.99999905	-5	-4.00000042	-4
0.025	-4.9746839	-4.97468488	-3.77498476	-3.77498437
0.05	-4.9487279	-4.9487289	-3.54987535	-3.549875
0.075	-4.92211482	-4.92211585	-3.32457844	-3.32457812
0.1	-4.89482803	-4.89482908	-3.09900027	-3.099
0.125	-4.86685047	-4.86685155	-2.87304711	-2.87304688
0.15	-4.83816465	-4.83816576	-2.6466252	-2.646625
0.175	-4.80875265	-4.80875378	-2.41964078	-2.41964062
0.2	-4.77859608	-4.77859724	-2.19200012	-2.192
0.225	-4.7476761	-4.74767728	-1.96360945	-1.96360937
0.25	-4.71597337	-4.71597458	-1.73437503	-1.734375
0.275	-4.68346808	-4.68346933	-1.50420312	-1.50420313
0.3	-4.65013992	-4.65014119	-1.27299995	-1.273
0.325	-4.61596805	-4.61596935	-1.04067178	-1.04067187
0.35	-4.58093112	-4.58093245	-0.80712486	-0.807125
0.375	-4.54500722	-4.54500859	-0.57226544	-0.57226563
0.4	-4.50817391	-4.5081753	-0.33599976	-0.336
0.425	-4.47040815	-4.47040958	-0.09823409	-0.09823437
0.45	-4.43168636	-4.43168781	0.14112534	0.141125
0.475	-4.39198432	-4.3919858	0.38217226	0.38217187
0.5	-4.35127721	-4.35127873	0.62500044	0.625
0.525	-4.3095396	-4.30954115	0.86970361	0.86970313
0.55	-4.2667454	-4.26674698	1.11637554	1.116375
0.575	-4.22286787	-4.22286947	1.36510997	1.36510937
0.6	-4.17787956	-4.1778812	1.61600064	1.616
0.625	-4.13175238	-4.13175404	1.86914131	1.86914063
0.65	-4.08445748	-4.08445917	2.12462573	2.124625
0.675	-4.03596531	-4.03596702	2.38254764	2.38254688
0.7	-3.98624555	-3.98624729	2.6430008	2.643
0.725	-3.93526713	-3.9352689	2.90607895	2.90607812
0.75	-3.8829982	-3.88299998	3.17187583	3.171875
0.775	-3.82940607	-3.82940787	3.4404852	3.44048438
0.8	-3.77445725	-3.77445907	3.71200079	3.712
0.825	-3.7181174	-3.71811923	3.98651635	3.98651563
0.85	-3.6603513	-3.66035315	4.26412562	4.264125
0.875	-3.60112286	-3.60112471	4.54492233	4.54492188
0.9	-3.54039505	-3.54039689	4.82900021	4.829
0.925	-3.47812992	-3.47813174	5.116453	5.11645313
0.95	-3.41428855	-3.41429034	5.40737442	5.407375
0.975	-3.34883105	-3.34883279	5.70185816	5.70185938
1	-3.2817165	-3.28171817	5.99999793	6

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