Journal of New Results in Science 11(1) (2022) 62-76

Journal of New Results in Science

https://dergipark.org.tr/en/pub/jnrs

Research Article

Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.1095343

A curve theory on sliced almost contact manifolds

Mehmet Gümüş[1], **Çetin Camcı**[2]

Keywords
Sliced almost contact manifolds, Sliced contact metric manifolds, $\pi - Legendre$ *curve*

Abstract — We have realized a gap between almost contact metric manifolds and contact metric manifolds in our studies. The examples that were given as Sasaki manifolds don't satisfy the condition of being contact metric manifold. As a result of our work, the sliced almost contact manifolds were formed and defined in [1]. In this paper we applied the theory of sliced almost contact manifolds to curves as a curve theory in three dimensional space. We define the $\pi - regular$ and $\pi - Legendre$ curves, also we give basic theorems on $\pi - Legendre$ curves and an example to $\pi - Legendre$ curves.

Subject Classification (2020): 53A04, 53A05.

1. Introduction

Contact geometry, indeed it is not a new area in geometry, has become an important place in different parts of mathematics and mathematical physics. Especially it has useful applications in differential equations, optics, general relativity and etc. Christian Huygens [2], Barrow [3] and Isaac Newton [4] are the first mathematicians that we see the contact geometry in their works. Many mathematicians like Sophus Lie [5], Gray [6] used the contact structures in their works on differential equations. Gibbs [7] used contact geometry in his work on thermodynamics and the others. 20th century is an important period in differantial geometry because semi-Riemannian geometry took important place in the research in mathematics and physics. Because the importance comes from the non-empty intersection of the tangent bundle and the orthogonal bundle which is called radical space by O'neill [8]. During 1950s, Marcel Berger published the major developments of Riemannian geometry. In 1960s S. Sasaki defined Sasaki manifolds [9]. After these works in 1970s the research focused on Lorentzian geometry. Then, many mathematicians worked on contact manifolds, almost contact manifolds, almost contact metric manifolds, contact metric manifolds and etc. Up to now lots of different papers and books published on lightlike submanifolds of Sasakian manifolds, Keahler manifolds, Legendre curves and some of them can be found in [10–16]. In our works we saw that there is a problem between indefinite-Sasakian manifolds and the submanifolds theory. We investigated the prob-

[1]mehmetgumus@comu.edu.tr (Corresponding Author); [2]ccamci@comu.edu.tr
[1]Department of Accounting and Tax Treatment, Lapseki Vocational School, Çanakkale Onsekiz Mart University, Çanakkale, Turkey
[2]Department of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, Çanakkale, Turkey
Article History: Received: 29 Mar 2022 - Accepted: 25 Apr 2022 - Published: 30 Apr 2022

lem occurs on contact metric manifolds. Because, the examples of Sasakian manifolds that given are not contact metric manifolds. Because the equation $d\eta = \Phi$ is non-satisfied by the examples. After our works we constructed in [1] a new structure to get rid of this problem. We named these structures sliced almost contact manifolds. We used projection morphisms on the tangent bundle. We divided slices the tangent bundle by projection morphisms. So, for every slice, the conditions are satisfied for theory. As a result we get a wider class of contact metric manifolds and the others. Furthermore we calculated the Riemannian curvature tensor for sliced almost contact metric manifolds [17].

Curve theory is a milestone in the geometry especially understanding the world that we live in. It is well known that there are characteristic features of every curve. Up to now many curves were defined and studied by many mathematicians. With the use of differential geometry, the curves began to be studied in depth. We chose Legendre curves and applied our theory to them. On sliced almost contact metric manifolds we defined π-regular and π-Legendre curves as an application of our theory to the theory of curves.

2. Preliminaries

In contact geometry if M is a $(2n+1)-$dimensional differentiable manifold and η is a 1-form on M which satisfies

$$\eta\Lambda(d\eta)^n \neq 0 \tag{2.1}$$

everywhere on M, then M is called a contact manifold. Let M be a contact manifold. On M the contact distribution denoted by D_p and it is defined by

$$D_p = \{X \in T_pM \,|\, \eta(X) = 0\}$$

Blair defined the almost contact manifolds in 1976 as follows.

Definition 2.1. [10] Let M be a $(2n+1)-$dimensional manifold and the tensor fields ϕ, ξ, and η are of the type $(1,1), (1,0)$, and $(0,1)$, respectively, defined on M. If these tensor fields satisfy the equations below, then (ϕ,ξ,η) is called a contact structure on M and (M,ϕ,ξ,η) is called an almost contact manifold.

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi \\ \eta(\xi) &= 1 \end{aligned} \tag{2.2}$$

After this definition a new manifold was needed to construct different geometric structures in contact geometry. For this aim the necessity of a metric was occurred. After almost contact manifolds, almost contact metric manifolds are defined by Gray.

Definition 2.2. [10] Assume that (M,ϕ,ξ,η) is an almost contact manifold with dimension $2n+1$. If g is a Riemannian or Lorentzian metric and g satisfies the equation

$$g(\phi(X),\phi(Y)) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all $X, Y \in \chi(M)$, then (ϕ,ξ,η,g) is called an almost contact metric structure and (M,ϕ,ξ,η,g) is called an almost contact metric manifold.

Definition 2.3. [6] Assume that (M,ϕ,ξ,η,g) is a $(2n+1)-$dimensional almost contact metric manifold. If the equation

$$d\eta(X,Y) = g(X,\phi(Y)) \tag{2.4}$$

is satisfied on M, then (M,ϕ,ξ,η,g) is called a contact metric manifold.

Let F be a tensor field of type $(1,1)$ on manifold M. If we define the tensor field $N_F : \chi(M)\times\chi(M) \longrightarrow \chi(M)$ by

$$N_F(X,Y) = F^2[X,Y] + [F(X),F(Y)] - F[F(X),Y] - F[X,F(Y)] \tag{2.5}$$

then N_F is a tensor field of type (1,2) [18].

Definition 2.4. If J is an almost complex structure on manifold M and $N_J \equiv 0$, then J is called integrable on M.

Definition 2.5. Let J be an almost complex structure on $M\times\mathbb{R}$. If J is integrable, then (ϕ,ξ,η) is called a normal structure.

Sasaki and Hatakeyama defined Sasaki manifold with the following definition.

Definition 2.6. [9] If $(2n+1)-$dimensional manifold M has (ϕ,ξ,η,g) normal contact metric structure, then the manifold M is called a Sasakian manifold.

3. Sliced Almost Contact Manifolds

In this paper we carried on the sliced almost contact manifolds Gümüş defined in [1] and [17]. And Gümüş published in 2018 sliced almost contact manifolds by the following definition.

Definition 3.1. [17] Assume that M is a manifold and TM is a tangent bundle of the manifold M. Let's accept, H is a distribution on the tangent bundle TM and $\xi \in H$. If we choose the projection π, ω tensor field of type (0,1) and ϕ_π tensor field of type (1,1) by , $\pi,\phi_\pi : TM \to H$, $\omega : TM \to C^\infty(M,\mathbb{R})$ and these tensor fields satisfy the following conditions,

$$\begin{aligned} \phi_\pi^2 X &= -\pi(X) + \omega(X)\xi \\ \omega(\xi) &= 1 \end{aligned} \tag{3.1}$$

then $(M,\phi_\pi,\omega,\pi,\xi)$ is called a sliced almost contact manifold.

Theorem 3.2. If $(M,\phi_\pi,\omega,\pi,\xi)$ is a sliced almost contact manifold, then the following equations are hold.

i. $\omega\circ\phi_\pi = 0$

ii. $\phi_\pi(\xi) = 0$

iii. $Ker\phi_\pi = \pi^{-1}(Sp\{\xi\})$

Proof.
ii. If we put ξ instead of X, then we get the following equation

$$\phi_\pi^2(\xi) = -\pi(\xi) + \xi = 0$$

From this result we get

$$\phi_\pi^3(\xi) = \phi_\pi(\phi_\pi^2(\xi)) = \phi_\pi^2(\phi_\pi(\xi)) = 0$$

$$\phi_\pi^2(\phi_\pi(\xi)) = -\pi\phi_\pi(\xi) + \omega(\phi_\pi(\xi))\xi = 0$$

We know that the equality $\pi\phi_\pi = \phi_\pi$ is valid. If we use this fact, then we reach the equation

$$\phi_\pi(\xi) = \omega(\phi_\pi(\xi))\xi$$

If we assume that $\omega(\phi_\pi(\xi)) \neq 0$, then we get

$$\phi_\pi^2(\xi) = \omega(\phi_\pi(\xi))\phi_\pi(\xi) = 0$$

From here we conclude that $\phi_\pi(\xi) = 0$. If we assume that $\omega(\phi_\pi(\xi)) = 0$, then it is clear that $\phi_\pi(\xi) = 0$.

i. On $\chi(M)$ for all $X \in \chi(M)$ we have $\phi_\pi^3(X) = -\phi_\pi(X)$. Also

$$\phi_\pi^3(X) = \phi_\pi^2(\phi_\pi(X))$$

is true. From here we can write

$$-\phi_\pi(X) + \omega(\phi_\pi(X))\xi = -\phi_\pi(X)$$

From this equation we conclude that $\omega \circ \phi_\pi = 0$ because $\xi \neq 0$.

iii. If $X \in Ker\phi_\pi$, then we can say $\phi_\pi(X) = 0$. In this equation if we apply ϕ_π both sides we get $\phi_\pi^2(X) = -\pi(X) + \omega(X)\xi = 0$. Here, when we do the necessary calculations we reach the following equality.

$$Ker\phi_\pi = \pi^{-1}(Sp\{\xi\})$$

Definition 3.3. [1, 17] Assume that (M,ϕ,η,ξ) is an almost contact manifold and let H be a distribution on M. When $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ is a sliced almost contact manifold and the following equalities

i. $i)$ $\phi \circ \pi = \phi_\pi$

ii. $ii)$ $\eta \circ \pi = \omega_\pi$

are satisfied by$(M,\phi_\pi,\omega_\pi,\pi,\xi)$, then it is called that the manifold$(M,\phi_\pi,\omega_\pi,\pi,\xi)$is a compatible sliced almost contact manifold with(M,ϕ,η,ξ).

Definition 3.4. Assume that $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ is a sliced almost contact manifold. If there is a Riemaniann metric $g : TM \times TM \to C^\infty(M,\mathbb{R})$ defined on M which satisfies

$$g(\phi_\pi X, \phi_\pi Y) = g(\pi X, \pi Y) - \omega_\pi(X)\omega_\pi(Y) \tag{3.2}$$

then $(M,\phi_\pi,\omega_\pi,g,\xi)$ is called a sliced almost contact metric manifold.

Definition 3.5. [17] Let's accept $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ is a sliced almost contact manifold and compatible sliced almost contact manifold by (M,ϕ,η,ξ). If there exists a Riemannian metric g and (M,ϕ,η,g,ξ) is an almost contact metric manifold $\forall X,Y \in \chi(M)$ where the following equation

$$g(\phi_\pi X, \phi_\pi Y) = g(\pi X, \pi Y) - \omega_\pi(X)\omega_\pi(Y) \tag{3.3}$$

is satisfied, then $(M,\phi_\pi,\omega_\pi,\pi,g,\xi)$ is named compatible sliced almost contact metric manifold with

(M,ϕ,η,g,ξ). If we take $g|_H = \overline{g}$, then we reach

$$\overline{g}(\phi_\pi X,\phi_\pi Y) = \overline{g}(X,Y) - \omega_\pi(X)\omega_\pi(Y)$$

where $\omega_\pi(X) = g(\pi X,\xi)$.

Definition 3.6. [1, 17] Assume that $(M,\phi_\pi,\omega_\pi,\pi,g,\xi)$ is a sliced almost contact metric manifold. In that case Φ_π is called second fundamental form and it is described as the following:

$$\Phi_\pi(X,Y) = g(\pi X,\phi_\pi Y) \tag{3.4}$$

Definition 3.7. [1, 17] Let's accept $(M,\phi_\pi,\omega_\pi,\pi,g,\xi)$ is a sliced almost contact metric manifold. When $(M,\phi_\pi,\omega_\pi,\pi,g,\xi)$ satisfies the equation $\epsilon d\omega_\pi = \Phi_\pi$, then $(M,\phi_\pi,\omega_\pi,\pi,g,\xi,\epsilon)$ is named as $\epsilon-$sliced contact metric manifold.

4. Normal Sliced Almost Contact Metric and Sasaki Manifolds

Definition 4.1. Assume that $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ is a sliced almost contact manifold. If we define $\widetilde{\pi}$ as

$$\begin{array}{rcl} \widetilde{\pi} & : & \chi(M\times\mathbb{R}) \to \chi(H\times\mathbb{R}) \\ (X,f\dfrac{d}{dt}) & \to & \widetilde{\pi}(X,f\dfrac{d}{dt}) = (\pi X,f\dfrac{d}{dt}) \end{array}$$

then we see that $\widetilde{\pi}^2 = \widetilde{\pi}$. This means that $\widetilde{\pi}$ is a projection morphism on $M\times\mathbb{R}$.

It is known that if $[\ ,\ ]$ is a bracket operator, then $\widetilde{\pi}[\ ,\ ] = [\widetilde{\pi},\widetilde{\pi}]$. So, $[\ ,\ ]$ is a bracket operator on $M\times\mathbb{R}$ and it is defined as following

$$[\ ,\ ] : \chi(M\times\mathbb{R})\times\chi(M\times\mathbb{R}) \to \chi(M\times\mathbb{R})$$

$$((X,f\frac{d}{dt}),(Y,g\frac{d}{dt})) \to [(X,f\frac{d}{dt}),(Y,g\frac{d}{dt})]$$

$$[(X,f\frac{d}{dt}),(Y,g\frac{d}{dt})] = ([X,Y],(Xg-Yf)\frac{d}{dt}) \tag{4.1}$$

([10]). Here, it is easy to show that

$$\widetilde{\pi}[\ ,\ ] = [\widetilde{\pi},\widetilde{\pi}]$$

On the other hand if we define

$$\begin{array}{rcl} J_\pi & : & \chi(M\times\mathbb{R}) \to \chi(H\times\mathbb{R}) \\ (X,f\dfrac{d}{dt}) & \to & J_\pi(X,f\dfrac{d}{dt}) = (\phi_\pi X - f\xi,\omega_\pi(X)\dfrac{d}{dt}) \end{array}$$

then J_π satisfies the following properties:

i. J_π is linear

ii. $J_\pi^2 = -\widetilde{\pi}$

iii. $J_\pi(\chi(M\times\mathbb{R})) = H\times\mathbb{R}$

So, J_π is a sliced almost complex structure on $M\times\mathbb{R}$.

Definition 4.2. [1, 17] If $N_{J_\pi}\equiv 0$, then J_π sliced almost complex structure is called integrable.

Definition 4.3. [1, 17] If J_π sliced almost complex structure is integrable on $M\times\mathbb{R}$, then $(\phi_\pi,\omega_\pi,\pi,\xi)$ sliced almost contact structure is called a sliced normal structure.

Here, if we compute $N_{J_\pi}((X,0),(Y,0))$ and $N_{J_\pi}((X,0),(0,f\frac{d}{dt}))$ for N_{J_π}, then we get the components

$$\begin{aligned} N_{J_\pi}((X,0),(Y,0)) &= (N_\pi^1(X,Y),N_\pi^2(X,Y)) \\ N_{J_\pi}((X,0),(0,f\frac{d}{dt})) &= (N_\pi^3(X),N_\pi^4(X)) \end{aligned}$$

Let's start with $N_{J_\pi}((X,0),(Y,0))$.

$$\begin{aligned} N_{J_\pi}((X,0),(Y,0)) &= -[(X,0),(Y,0)]+[J_\pi(X,0),J_\pi(Y,0)] \\ &\quad -J_\pi[J_\pi(X,0),(Y,0)]-J_\pi[(X,0),J_\pi(Y,0)] \\ &= -([X,Y],0)+[(\phi_\pi X,\omega_\pi(X)\frac{d}{dt}),(\phi_\pi Y,\omega_\pi(Y)\frac{d}{dt})] \\ &\quad -J_\pi[(\phi_\pi X,\omega_\pi(X)\frac{d}{dt}),(Y,0)] \\ &\quad -J_\pi[(X,0),(\phi_\pi Y,\omega_\pi(Y)\frac{d}{dt})] \end{aligned}$$

If we do the necessary operations, then we can write

$$\begin{aligned} N_{J_\pi}((X,0),(Y,0)) &= (-[X,Y]+[\phi_\pi X,\phi_\pi Y]-\phi_\pi[\phi_\pi X,Y] \\ &\quad -\phi_\pi[X,\phi_\pi Y]+(-Y\omega_\pi(X)+ \\ &\quad X\omega_\pi(Y))\xi,(\phi_\pi(X)\omega_\pi(Y)-\phi_\pi(Y)\omega_\pi(X) \\ &\quad -\omega_\pi([\phi_\pi X,Y])-\omega_\pi([X,\phi_\pi Y]))\frac{d}{dt}) \end{aligned}$$

From the equality above we get the components N_π^1 and N_π^2.

$$\begin{aligned} N_\pi^1(X,Y) &= -[X,Y]+[\phi_\pi X,\phi_\pi Y]-\phi_\pi[\phi_\pi X,Y]-\phi_\pi[X,\phi_\pi Y] \\ &\quad +(X\omega_\pi(Y)-Y\omega_\pi(X))\xi-\omega_\pi[X,Y]\xi \end{aligned}$$

If we simplify this equation, then we get the following result.

$$\begin{aligned} N_\pi^1(X,Y) &= \phi_\pi^2[X,Y]+[\phi_\pi X,\phi_\pi Y]-\phi_\pi[\phi_\pi X,Y]-\phi_\pi[X,\phi_\pi Y] \\ &\quad +(X\omega_\pi(Y)-Y\omega_\pi(X)-\omega_\pi[X,Y])\xi \end{aligned}$$

We know that

$$\begin{aligned} N_{\phi_\pi}(X,Y) &= \phi_\pi^2[X,Y]+[\phi_\pi X,\phi_\pi Y]-\phi_\pi[\phi_\pi X,Y]-\phi_\pi[X,\phi_\pi Y] \\ 2d\omega_\pi(X,Y) &= X\omega_\pi(Y)-Y\omega_\pi(X)-\omega_\pi[X,Y] \end{aligned}$$

From here as a result we get

$$N_\pi^1(X,Y)=N_{\phi_\pi}(X,Y)+2d\omega_\pi(X,Y)\xi \tag{4.2}$$

And for $N_\pi^2(X,Y)$ we can write

$$\begin{aligned} N_\pi^2(X,Y) &= \phi_\pi(X)\omega_\pi(Y)-\phi_\pi(Y)\omega_\pi(X)-\omega_\pi([\phi_\pi X,\pi Y]) \\ &\quad -\omega_\pi([X,\phi_\pi Y]) \end{aligned}$$

Now we look for $N_{J_\pi}((X,0),(0,f\frac{d}{dt}))$ to get $N_\pi^3(X)$ and $N_\pi^4(X)$. By similar calculations we reach the components N_π^3 and N_π^4.

$$\begin{aligned} N_\pi^3(X) &= -[\phi_\pi(X),\xi]+\phi_\pi[X,\xi] \\ N_\pi^4(X) &= \xi\omega_\pi(X)+\omega_\pi[X,\xi] \end{aligned} \tag{4.3}$$

It is known that the following equations are valid.

$$\begin{aligned} (L_\xi\phi_\pi)X &= [\xi,\phi_\pi(X)]-\phi_\pi[\xi,X] \\ (L_\xi\omega_\pi)X &= \xi\omega_\pi(X)-\omega_\pi[\xi,X] \end{aligned} \tag{4.4}$$

If we use these equations, then we reach the following equations.

$$\begin{aligned} N_\pi^3(X) &= (L_\xi\phi_\pi)X \\ N_\pi^4(X) &= (L_\xi\omega_\pi)X \end{aligned} \tag{4.5}$$

Definition 4.4. The necessary and sufficient condition of the sliced almost contact manifold $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ to be normal is the tensors $N_\pi^1, N_\pi^2, N_\pi^3$ and N_π^4 are all equivalent to zero identically.

Theorem 4.5. Let $(M,\phi_\pi,\omega_\pi,\pi,\xi)$ be a sliced almost contact manifold. If $N_\pi^1\equiv 0$, then $N_\pi^2\equiv N_\pi^3\equiv N_\pi^4\equiv 0$.

Theorem 4.6. Let $(M,\phi_\pi,\omega_\pi,\pi,\xi,g)$ be a sliced almost contact metric manifold. On this manifold M, for all $X,Y\in\chi(M)$ we have the following equation.

$$\begin{aligned} 2g((\nabla_X\phi_\pi)Y,Z) &= 3d\Phi_{\omega_\pi}(\pi X,\phi_\pi Y,\phi_\pi Z)-3d\Phi_{\omega_\pi}(\pi X,\pi Y,\pi Z) \\ &\quad +g(N_\pi^2(Y,Z),\phi_\pi X)+\omega_\pi(X)N_\pi^2(Y,Z) \\ &\quad +2d\omega_\pi(\phi_\pi Y,\pi X)\omega_\pi(Z)-2d\omega_\pi(\phi_\pi Z,\pi X)\omega_\pi(Y) \end{aligned}$$

Here, we have used $\Phi_{\omega_\pi}(X,Y)=g(\pi X,\phi_\pi Y)$.

Definition 4.7. We define the morphism h_π ($h_\pi:\chi(M)\to H$, $X\to h_\pi(X)=\frac{1}{2}(L_\xi\phi_\pi)X$). Here, it is clear that

$\pi h_\pi = h_\pi \pi = h_\pi$.

Theorem 4.8. Let $(M, \phi_\pi, \omega_\pi, \pi, \xi, g)$ be a sliced almost contact metric manifold. $\forall X, Y \in \chi(M)$ we have

$$\pi(\nabla_X \xi) = -\phi_\pi X - \phi_\pi h_\pi X \tag{4.6}$$

Here, we have $\phi_\pi h_\pi = -h_\pi \phi_\pi$ and $tr h_\pi = 0$. Also, we can see that h_π is symmetric.

Proof.
On sliced almost contact metric manifolds we have

$$\nabla_\xi \phi_\pi = 0 \quad \text{and} \quad \nabla_\xi \xi = 0$$

So, we get

$$\begin{aligned} g((L_\xi \phi_\pi)X, Y) &= g(\nabla_\xi \phi_\pi X - \nabla_{\phi_\pi X}\xi - \phi_\pi(\nabla_\xi X) + \phi_\pi \nabla_X \xi, Y) \\ &= g(-\nabla_{\phi_\pi X}\xi + \phi_\pi \nabla_X \xi, Y) \end{aligned}$$

In this equation, if one of X or Y is equal to ξ, then it is equal to 0. If we take X and Y orthogonal to ξ, then we have $N_\pi^2 = 0$ and because of the following

$$\omega_\pi(X) = \omega_\pi(Y) = 0$$

we get the equation

$$\omega_\pi[\phi_\pi X, \pi Y] + \omega_\pi[\pi X, \phi_\pi Y] = 0$$

So, we have

$$g((L_\xi \phi_\pi)X, Y) = \omega_\pi(\nabla_{\phi_\pi X} Y) + \omega_\pi(\nabla_X \phi_\pi Y)$$

If we use the fact $g(Y, \xi) = 0$, then we get the following equation

$$g(\nabla_{\phi_\pi X} Y, \xi) + g(Y, \nabla_{\phi_\pi X}\xi) = 0 \tag{4.7}$$

From this equation we conclude that

$$g(-\nabla_{\phi_\pi X}\xi, Y) = \omega(\nabla_{\phi_\pi X} Y)$$

Similarly we have $g(\phi_\pi Y, \xi) = 0$. From this result we get the equation $g(\nabla_X \phi_\pi Y, \xi) + g(\phi_\pi Y, \nabla_X \xi) = 0$. This equation gives us the result

$$g(\phi_\pi(\nabla_X \xi), Y) = \omega_\pi(\nabla_X \phi_\pi Y)$$

From this result we get the equation

$$g((L_\xi \phi_\pi)X, Y) = \omega_\pi(\nabla_{\phi_\pi X} Y) + \omega_\pi(\nabla_X \phi_\pi Y) \tag{4.8}$$

On the other hand because of the following equality

$$\omega_\pi([\phi_\pi X, Y]) + \omega_\pi([X, \phi_\pi Y]) = 0 \tag{4.9}$$

we have

$$g(\nabla_{\phi_\pi X} Y - \nabla_Y \phi_\pi X, \xi) + g(\nabla_X \phi_\pi Y - \nabla_{\phi_\pi Y} X, \xi) = 0 \tag{4.10}$$

So at the end we reach the following equalities.

$$g(h_\pi X, Y) = \omega(\nabla_{\phi_\pi X} Y) + \omega_\pi(\nabla_X \phi_\pi Y) = \omega_\pi(\nabla_Y \phi_\pi X) + \omega_\pi(\nabla_{\phi_\pi Y} X)$$

$g(h_\pi X, Y) = g(X, h_\pi Y)$.

From here we conclude that h_π is symmetric. Since $\Phi_\pi = d\omega_\pi$ and $N_\pi^2 = 0$, we can write the following equation

$$2g((\nabla_X \phi_\pi)\xi, Z) = g(N_\pi^1(\xi, Z), \phi_\pi X) - 2d\omega_\pi(\phi_\pi Z, X)$$

On the other hand we know that

$$N_\pi^1(\xi, Z) = \phi_\pi^2[\xi, Z] - \phi_\pi[\xi, \phi_\pi Z] \quad \text{and} \quad (L_\xi \phi_\pi) Z = [\xi, \phi_\pi Z] - \phi_\pi[\xi, Z] \tag{4.11}$$

From these equalities we reach the equation below.

$$-\phi_\pi(L_\xi \phi_\pi) Z = \phi_\pi^2[\xi, Z] - \phi_\pi[\xi, \phi_\pi Z] \tag{4.12}$$

Since the equation

$$\begin{aligned} 2g((\nabla_X \phi_\pi)\xi, Z) &= g(-\phi_\pi(L_\xi \phi_\pi) Z, \phi_\pi X) - 2d\omega_\pi(\phi_\pi Z, X) \\ &= g(-(L_\xi \phi_\pi) Z, X) + 2\omega_\pi((L_\xi \phi_\pi) Z)\omega_\pi(X) - 2g(\phi_\pi Z, \phi_\pi X) \end{aligned}$$

is true, we get

$$\begin{aligned} g((\nabla_X \phi_\pi)\xi, Z) &= \frac{1}{2} g(-(L_\xi \phi_\pi) Z, X) - g(\pi Z, \pi X) + \omega_\pi(X)\omega_\pi(Z) \\ &= g(-\frac{1}{2}(L_\xi \phi_\pi) X, Z) - g(\pi X, Z) + g(\omega_\pi(X)\xi, Z) \end{aligned}$$

From this equation we get $(\nabla_X \phi_\pi)\xi = -h_\pi X - \pi X + \omega_\pi(X)\xi$. As a result we have the following equations.

$$-\phi_\pi(\nabla_X \xi) = -h_\pi X - \pi X + \omega_\pi(X)\xi \quad \text{and} \quad \pi(\nabla_X \xi) = -\phi_\pi X - \phi_\pi h_\pi X$$

Theorem 4.9. [1, 17] If $(M, \phi_\pi, \omega_\pi, \pi, \xi, g)$ is a sliced almost contact metric manifold, then it is a sliced Sasakian manifold if and only if the following equation is valid $\forall X, Y \in \chi(M)$;

$$(\nabla_X \phi_\pi) Y = g(\pi X, \pi Y)\xi - \omega_\pi(Y)\pi X \tag{4.13}$$

Proof.

Since $N^1_\pi \equiv 0$ we have

$$\begin{aligned} 2g((\nabla_X\phi_\pi)Y,\pi Z) &= 2d\omega_\pi(\phi_\pi Y,X)\omega_\pi(Z)-2d\omega_\pi(\phi_\pi Z,X)\omega_\pi(Y) \\ &= 2g(g(\pi Y,\pi X)\omega_\pi(Z)-\omega_\pi(X)\omega_\pi(Y)\omega_\pi(Z)) \\ &\quad -2g(g(\pi Z,\pi X)\omega_\pi(Y)-\omega_\pi(X)\omega_\pi(Y)\omega_\pi(Z)) \\ &= 2g(g(\pi Y,\pi X)\omega_\pi(Z)-g(\pi Z,\pi X)\omega_\pi(Y)) \\ &= 2g(g(\pi X,\pi Y)\xi-\omega_\pi(Y)\pi X,Z) \end{aligned}$$

So, we have

$$(\nabla_X\phi_\pi)Y = g(\pi X,\pi Y)\xi-\omega_\pi(Y)\pi X \tag{4.14}$$

Definition 4.10. If (M,ϕ,η,g,ξ) is an (ϵ)-contact metric manifold and (ϕ,η,ξ) is a normal structure on M, then (M,ϕ,η,ξ) is called Sasaki manifold.

Theorem 4.11. Let $(M,\phi_\pi,\pi,\omega_\pi,g,\xi)$ be a compatible sliced almost contact metric manifold with (M,ϕ,η,g,ξ). If (M,ϕ,η,g,ξ) is a Sasakian manifold, then $(M,\phi_\pi,\pi,\omega_\pi,g,\xi)$ is a sliced Sasakian manifold.

Proof.

We know that $\forall X,Y \in \chi(M)$ we have $\pi X,\pi Y \in H \subset \chi(M)$. Because of (M,ϕ,η,g,ξ) is a Sasakian manifold and $\phi\circ\pi=\phi_\omega$, $\eta\circ\pi=\omega$ we can write

$$\pi(\bigtriangledown_X\phi)Y = g(\pi X,\pi Y)\xi-\omega(Y)\pi(X) \tag{4.15}$$

In this case it is shown that $(M,\phi_\pi,\pi,\omega_\pi,g,\xi)$ is a sliced Sasakian manifold.

Theorem 4.12. If $(M,\phi,\eta,\xi,\epsilon)$ is an $(\epsilon)-$contact metric manifold, then N^2 and N^4 are identically equal to zero, where $\epsilon=\mp1$.

5. A Curve Theory in Sliced Almost Contact Metric Manifolds

In this section we constructed the frame vector fields of a curve in the 3-dimensional subdistribution H^3 of TM where $(M^{2n+1},\phi_\pi,\omega_\pi,\xi,g)$ is a sliced almost contact metric manifold. Also, we define $\pi-Legendre$ curve by using the classical definition of $Legendre$ curve.

Let $(M^{2n+1},\phi_\pi,\omega_\pi,\xi,g)$ be a sliced almost contact metric manifold and H^3 is a 3-dimensional distribution in TM. At the same time, we define a projection morphism π on TM as $\pi: TM\to H^3$, $X\to\pi(X)$ where $\forall X\in TM$. Let $(M^{2n+1},\phi_\pi,\omega_\pi,\xi,g)$ be a sliced almost contact metric manifold and H^3 is a 3-dimensional distribution in TM. If a γ curve which is not parameterized by arclength, defined as $\gamma: I\to M$, $t\to\gamma(t)$ for all $t\in I$, then it is clear that $\dot{\gamma}(t)\in T_{\gamma(t)}M$ and $\pi(\dot{\gamma}(t))\in H^3$ are true.

Definition 5.1. Let γ be a curve defined by the same arguments in the Definition 5.2. If the curve γ satisfies $g(\pi(\dot{\gamma}(t)),\pi(\dot{\gamma}(t)))\neq 0$ for all $t\in I$, then the γ curve is called as π-regular curve. As a consequence we define the velocity v_π as

$$v_\pi(t) = \left\|\pi(\dot{\gamma}(t))\right\| \tag{5.1}$$

If $v_\pi(t)=\left\|\pi(\gamma'(t))\right\|=1$, then the parameter t is called as the arclength parameter in the distribution H^3

and denoted by s_π.

Definition 5.2. Let $(M^{2n+1}, \phi_\pi, \omega_\pi, \xi, g)$ be a sliced almost contact metric manifold and H^3 is a 3-dimensional distribution in TM. In this case we define a new cross product $\wedge_\pi$ on the distribution H^3 as following

$$X \wedge_\pi Y = \pi(X) \wedge_\pi \pi(Y) = -g(X, \phi_\pi Y)\xi - \omega(Y)\phi_\pi X + \omega(X)\phi_\pi Y \tag{5.2}$$

for all $X, Y \in TM$.

Theorem 5.3. Let $(M^{2n+1}, \phi_\pi, \omega_\pi, \xi, g)$ be a sliced almost contact metric manifold and $\wedge_\pi$ a cross product on the distribution H^3. In this situation the cross product $\wedge_\pi$, $\forall X, Y, Z \in TM$ satisfies the following properties:

i. $\wedge_\pi$ is bilinear and antisymmetric $i.e. X \wedge_\pi Y = -Y \wedge_\pi X$.

ii. Vector field $X \wedge_\pi Y$ is perpendicular to both X and Y vector fields.

iii.

$$\begin{aligned} Y \wedge_\pi \phi_\pi X &= g(\pi X, Y)\xi - \omega(Y)\pi(X) \\ \phi_\pi X &= \xi \wedge_\pi X \end{aligned}$$

iv. If we define the triple cross product by $(X, Y, Z) = g(X \wedge_\pi Y, \pi Z)$, then the following equations are valid.

$$\begin{aligned} (X, Y, Z) &= -(g(X, \phi_\pi Y)\omega(Z) + g(Y, \phi_\pi Z)\omega(X) + g(Z, \phi_\pi X)\omega(Y)) \\ (X, Y, Z) &= (Y, Z, X) = (Z, X, Y) \end{aligned}$$

v.

$$g(X, \phi_\pi Y)\pi Z + g(Y, \phi_\pi Z)\pi X + g(Z, \phi_\pi X)\pi Y = -\det(\pi X, Y, Z)\xi$$

$$(X \wedge_\pi Y) \wedge_\pi Z = g(\pi X, Z)\pi Y - g(\pi Y, Z)\pi X$$

$$g(X \wedge_\pi Y, Z \wedge_\pi W) = g(\pi X, Z)g(\pi Y, W) - g(\pi Y, Z)g(\pi X, W)$$

$$\|X \wedge_\pi Y\|^2 = g(\pi X, \pi X)g(\pi Y, \pi Y) - g(\pi X, Y)^2$$

$$(X \wedge_\pi Y) \wedge_\pi Z + (Y \wedge_\pi Z) \wedge_\pi X + (Z \wedge_\pi X) \wedge_\pi Y = 0$$

Proof.

The properties can be proved by similar to the proof of the Theorem 2.1 in [19].

Theorem 5.4. Let $(M^{2n+1}, \phi_\pi, \omega_\pi, \xi, g)$ be a sliced almost contact metric manifold. In this manifold $\forall X, Y, Z \in TM$ we have the following equation where ∇ is a Levi-Civita connection on M^{2n+1}

$$\nabla_Z (X \wedge_\pi Y) = (\nabla_Z X) \wedge_\pi Y + X \wedge_\pi (\nabla_Z Y) \tag{5.3}$$

Proof.

The proof is similar to the Theorem 2.2 in [19].

Now, let γ be a $\pi - regular$ curve in $(M^{2n+1}, \phi_\pi, \omega_\pi, \xi, g)$ sliced almost contact metric manifold. If we define $h(t) = s_\pi = \int_0^t \|\pi\gamma(u)\|\, du$, then the curve $\beta(s_\pi)$ is going to be $\beta(s_\pi) = \pi\gamma(h^{-1}(s_\pi))$. Here, s_π is the arclength parameter of the curve γ. In order to construct the Serret-Frenet frame fields of the curve γ, we define the frame fields T_π, N_π and B_π by

$$\begin{aligned} T_\pi &= \pi\beta^{'}(s) \\ N_\pi &= \frac{\pi\beta''(s)}{\left\|\pi\beta''(s)\right\|} \\ B_\pi &= T_\pi \wedge_\pi N_\pi \end{aligned} \tag{5.4}$$

the above equalities. When we define the fields T_π, N_π and B_π above, then we can say that κ_π and τ_π can be defined as following.

$$\begin{aligned} \kappa_\pi &= \left\|\pi\beta''(s)\right\| \\ \tau_\pi &= g(N'_\pi, B_\pi) \end{aligned} \tag{5.5}$$

From these equations we can get the following results.

$$\begin{aligned} T'_\pi &= \kappa_\pi N_\pi \\ N'_\pi &= -\tau_\pi T_\pi + \kappa_\pi B_\pi \\ B'_\pi &= -\tau_\pi N_\pi \end{aligned} \tag{5.6}$$

In the classical theory of contact structures $Legendre\ curves$ play an important role. Because contact distribution carry the $Legendre\ curves$ to $Legendre\ curves$. With the new definition of sliced Sasaki manifolds we will define new type of curves.

Definition 5.5. Let $(M^{2n+1}, \phi_\pi, \omega_\pi, \xi, g)$ be a sliced almost contact metric manifold and H^3 is a 3-dimensional distribution in TM. Assume that γ curve is $\pi - regular$. If $\omega_\pi(\pi(\dot{\gamma}(t))) = 0$ is true $\forall t \in I$, then the curve γ is called $\pi - Legendre\ curve$.

If we consider the definition of $\pi - Legendre\ curve$ and (5.9) we get

$$\omega_\pi(T_\pi)T_\pi + \omega_\pi(N_\pi)N_\pi + \omega_\pi(B_\pi)B_\pi = \xi$$
$$\omega_\pi(T_\pi)^2 + \omega_\pi(N_\pi)^2 + \omega_\pi(B_\pi)^2 = 1$$
$$T_\pi \wedge_\pi N_\pi = B_\pi, N_\pi \wedge_\pi B_\pi = T_\pi, B_\pi \wedge_\pi T_\pi = N_\pi$$

Proposition 5.6. Let $(M^3, \phi_\pi, \omega_\pi, \xi, g)$ be a 3-dimensional sliced almost contact metric manifold and γ a π-regular curve in M^3 parameterized by arclength. Then, the following equations are valid.

$$\phi_\pi T_\pi = \omega_\pi(B_\pi)N_\pi - \omega_\pi(N_\pi)B_\pi$$
$$\phi_\pi N_\pi = \omega_\pi(T_\pi)B_\pi - \omega_\pi(B_\pi)T_\pi$$
$$\phi_\pi B_\pi = \omega_\pi(N_\pi)T_\pi - \omega_\pi(T_\pi)N_\pi$$

Proof.
The proof is similar to the Proposition 3.1 in [19].

Proposition 5.7. If $(M^3, \phi_\pi, \omega_\pi, \xi, g)$ is a 3-dimensional sliced contact metric manifold and γ is a π-regular curve in M^3 parametrized by arclength, then the following equations hold.

$$\begin{aligned} \sigma'_{t_\pi} &= \kappa_\pi \sigma_{n_\pi} - g(t_\pi, \phi_\pi h t_\pi) \\ \sigma'_{n_\pi} &= -\kappa_\pi \sigma_{t_\pi} + (\tau_\pi - 1)\sigma_{b_\pi} - g(n_\pi, \phi_\pi h t_\pi) \\ \sigma'_{b_\pi} &= -(\tau_\pi - 1)\sigma_{n_\pi} - g(b_\pi, \phi_\pi h t_\pi), \end{aligned}$$

where $\sigma_{t_\pi}(s) = \omega_\pi(t_\pi) = g(t_\pi, \xi)$, $\sigma_{n_\pi}(s) = \omega_\pi(n_\pi) = g(n_\pi, \xi)$, and $\sigma_{b_\pi}(s) = \omega_\pi(b_\pi) = g(b_\pi, \xi)$.

Proof.
The proof is similar to the Proposition 3.2 in [19].

Theorem 5.8. Let γ be a π-regular curve in a 3-dimensional sliced Sasaki manifold M^3 and parameterized by arclength. Then, the following equations hold.

$$\begin{aligned} \sigma'_{t_\pi} &= \kappa_\pi \sigma_{n_\pi} \\ \sigma'_{n_\pi} &= -\kappa_\pi \sigma_{t_\pi} + (\tau_\pi - 1)\sigma_{b_\pi} \\ \sigma'_{b_\pi} &= -(\tau_\pi - 1)\sigma_{n_\pi} \end{aligned}$$

Proof.
The proof is similar to the Theorem 4.1 in [19].

Proposition 5.9. If γ is a π-Legendre curve. in a 3-dimensional sliced Sasakian manifold, then the torsion of γ is equal to 1.

Proof.
The proof is similar to the Remark 4.1 in [19].

Theorem 5.10. Let γ be a π-regular curve in a 3-dimensional sliced Sasakian manifold and $\sigma = \omega_\pi(\dot{\gamma})$. If $\tau = 1$ and $\sigma = \dot{\sigma} = 0$ for at least one point, then γ is a π-Legendre curve.

Proof.
The proof is similar to the A new proof of Theorem 1.1 in [19].

Theorem 5.11. If the torsion of a π-Legendre curve. is equal to 1 on a 3-dimensional sliced contact metric manifold, then the manifold is sliced Sasakian.

Proof.
The proof is similar to the A new proof of Theorem 1.2 in [19].

Example 5.12. Let $(M^5, \phi_\pi, \omega_\pi, \xi, g)$ be a sliced almost contact metric manifold and $x_1, x_2. y_1, y_2, z$ coordinate functions. On this manifold we define the projection morphism π ($\pi : TM \to H^3$, $X \to \pi(X) = (X_1, 0, X_3, 0, X_5)$) where $X = (X_1, X_2, X_3, X_4, X_5) \in TM$ and $H^3 = \mathrm{Sp}\{\partial x_1, \partial y_1, \partial z\}$. While we defined π, we can define the tensor fields ϕ_π and ω_π as ($\phi_\pi : TM \to H^3$, $X \to \phi_\pi(X) = \xi \wedge_\pi \pi X$) and ($\omega_\pi : H^3 \to C^\infty(H^3, \mathbb{R})$, $X \to \omega_\pi(X) = \frac{1}{2}(dz - y_1 dx_1)$) for all $X = (X_1, 0, X_3, 0, X_5) \in H^3$ we can take the metric by

$$ds^2 = \frac{1}{4}\sum_{i=1}^{2}(dx_i^2 + dy_i^2) + \frac{1}{4}(dz - \sum_{i=1}^{2} y_i dx_i) \otimes (dz - \sum_{i=1}^{2} y_i dx_i)$$

Here, we choose the characteristic vector field $\xi = (0,0,0,0,2)$. So, it is clear that $\omega_\pi(\xi) = 1$. Under these assumptions and definitions the contact distribution D_π is defined by

$$D_\pi = \{X \in H^3 | \omega_\pi(X) = 0\}$$

Now we define the curve $\gamma : I \to M^5$ as

$$\gamma(t) = (2\cos(t+3)+5, t^3, 2\sin(t+3)-5, 2t^2, \sin 2(t+3) - 10\cos(t+3) - 2t + 4)$$

So, when we apply the projection morphism to the curve γ, then it will be

$$\gamma = (2\cos(t+3)+5, 0, 2\sin(t+3)-5, 0, \sin 2(t+3) - 10\cos(t+3) - 2t + 4)$$

When we differentiate the γ we get the following:

$$\pi(\dot{\gamma}(s)) = (-2\sin(t+3), 0, 2\cos(t+3), 0, 2\cos 2(t+3) + 10\sin(t+3) - 2)$$

If we calculate $\omega_\pi(\pi(\dot{\gamma}(t)))$, then we get $\omega_\pi(\pi(\dot{\gamma}(t))) = 0$ which means that γ is a $\pi - Legendre\ curve$. Here, the π-curvature of γ is equal to 1.

6. Conclusion

The authors showed that sliced almost contact manifolds include the almost contact manifolds. Hence sliced almost contact manifolds is a wider class of almost contact manifolds. In this paper they gived the fundamental properties of sliced almost contact manifolds and as an application of the theory they applied the sliced almost contact manifolds to the curve theory. For this aim they choosed Legendre curves as an example because the Legendre curves are important for contact manifolds. The authors defined $\pi - regular$, $\pi - curvature$ and $\pi - Legendre$ curves and gave some properties on $\pi - Legendre$ curves which are Legendre curve in special case. This idea can be applied to different curves or the theory of submanifolds.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] M. Gümüş, *A new construction of Sasaki manifolds in semi-Riemann space and applications*, PhD Dissertation, Çanakkale Onsekiz Mart University (2018) Çanakkale, Turkey (in Turkish).

[2] C. Huygens, The pendulum clock or geometrical demonstrations concerning the motion of pendula as applied to clocks, Iowa State University Press, Iowa, 1986.

[3] J. M. Child, The geometrical lectures of Isaac Barrow, The Open Court Publishing Company, 1916.

[4] S. I. Newton, The mathematical principles of natural philosophy: Philosophiae naturalis principia

mathematica, CreateSpace Independent Publishing Platform, 2016.

[5] S. Lie, *Verallgemeinerung und neue Verwertung der Jacobischen Multiplikatortheorie*, Forhandlinger Christiania, (1874) 255–274.

[6] J. Gray, *Some global properties of contact structures*, Annals of Mathematics, 69, (1959) 421–450.

[7] J. W. Gibbs, *A method of geometrical representation of the thermodynamic properties of substances by means of surfaces*, Transactions of the Connecticut Academy of Arts and Sciences, 2, (1873) 382–404.

[8] B. O'Neill, Semi-Riemannian geometry. with applications to relativity. Pure and Applied Mathematics, 103. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[9] S. Sasaki, Y. Hatakeyama, *On differentiable manifolds with contact metric structures*, Journal of the Mathematical Society of Japan, 14, (1962) 249–271.

[10] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Springer-Verlag, 509, 1976.

[11] C. Baikoussis, D. E. Blair, *On Legendre curves in contact 3-manifolds*, Geometriae Dedicata, 49, (1994) 135–142.

[12] K. L. Duggal, A. Bejancu, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic, 364, 1996.

[13] M. Belkhelfa, I. E. Hiric, R. Rosca, L. Verstraelen, *On Legendre curves in Riemannian and Lorentzian Sasaki spaces*, Soochow Journal of Mathematics, 28, (2002) 81–91.

[14] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progressin Mathematics 203. Birkhauser Boston, 2002.

[15] Ç. Camcı, *A curve theory in contact geometry*, PhD Dissertation, Ankara University (2007) Ankara, Turkey (in Turkish).

[16] K. L. Duggal, B. Şahin, *Lightlike submanifolds of indefinite Sasakian manifolds*, International Journal of Mathematics and Mathematical Sciences, 2007, (2007) Article ID: 057585.

[17] M. Gümüş, Ç. Camcı, *Riemannian curvature of a sliced contact metric manifold*, Çanakkale Onsekiz Mart University, Journal of Graduate School of Natural and Applied Sciences, 4(2), (2018) 1–14.

[18] K. Yano, M. Kon, Structures on manifolds, World Scientific, 1984.

[19] Ç. Camcı, *Extended cross product in 3-dimensional almost contact metric manifold with applications to curve theory*, Turkish Journal of Mathematics, 36, (2012) 305–318.