CHAOS
Theory and Applications
in Applied Sciences and Engineering

e-ISSN: 2687-4539
RESEARCH ARTICLE
Vol.4 / No.2 / 2022 / pp.71-81
https://doi.org/10.51537/chaos.1096278

Stability and Hopf Bifurcation Analysis of a Fractional-order Leslie-Gower Prey-predator-parasite System with Delay

Xiaoting Yang [*,1], **Liguo Yuan** [α,*,2] **and Zhouchao Wei** [γ,3]

[*]Department of Mathematics, College of Mathematics and Informatics, South China Agricultural University, Guangzhou, 510642, P. R. China, [α]School of Mathematical Sciences, Anhui University, Hefei, 230601, P. R. China, [γ]School of Mathematics and Physics, China University of Geosciences, Wuhan, 430074, P. R. China.

ABSTRACT A fractional-order Leslie-Gower prey-predator-parasite system with delay is proposed in this article. The existence and uniqueness of the solutions, as well as their non-negativity and boundedness, are studied. Based on the characteristic equations and the conditions of stability and Hopf bifurcation, the local asymptotic stability of each equilibrium point and Hopf bifurcation of interior equilibrium point are investigated. Moreover, a Lyapunov function is constructed to prove the global asymptotic stability of the infection-free equilibrium point. Lastly, numerical examples are studied to verify the validity of the obtained newly results.

KEYWORDS
Fractional derivative
Hopf bifurcation
Stability
Leslie-Gower prey-predator-parasite system

INTRODUCTION

Ecosystem is an extremely complex dynamics system. Mathematicians have great interest in dynamical characteristics of ecosystem. Especially, ecology and epidemiology attract more and more mathematicians' attention. Although ecology and epidemiology are two different fields, they get closer and closer for years(Anderson and May 1980; Zhou *et al.* 2010; Mbava *et al.* 2017; Shaikh *et al.* 2018; Adak *et al.* 2020). In 1980, Anderson and May first to study the eco-epidemiological model with disease in the prey(Anderson and May 1980). Recently, Zhou et al considered a predator-prey model with modified Leslie-Gower functional response and studied the Hopf bifurcation of this model(Zhou *et al.* 2010). They found that when the rate of infection exceeds a critical value, the strictly positive interior equilibrium experiences Hopf bifurcation. The eco-epidemic predator-prey model exhibits interesting dynamics with infected predators. So, Shaikh et al considered the stability of a Holling type III response mechanism for predation(Shaikh *et al.* 2018). The predator faced enormous competition from super-predators and even faced extinction. The disease was regarded as a biological control that allowed predator populations to recover from low numbers. Hence, Mbava et al considered a predator-prey model with disease in super-predator and studied its dynamic properties(Mbava *et al.* 2017). In addition, Adak et al analyzed the chaos and Hopf bifurcation of the delay-induced Leslie-Gower predator-prey-parasite model(Adak *et al.* 2020). It can be seen that research on eco-epidemiological models is a hot topic.

Fractional calculus is an extension of classical calculus. In recent years, fractional calculus has developed rapidly, which has gradually penetrated into scientific and engineering application fields. Furthermore, it also has become an important tool in many fields(Kilbas *et al.* 2006; Rajagopal *et al.* 2020; Li and Chen 2004). Compared with integer-order derivative, the fractional derivative has better memory. It can excellently describe long-range temporal memory(Rihan and Rajivganthi 2020). Since most biological models have long-range temporal memory, it is significant to consider the fractional derivative into account. Currently, research on this area has some outstanding results(Yousef *et al.* 2021; Li *et al.* 2017a; Boukhouima *et al.* 2017; Moustafa *et al.* 2020). Yousef et al analyzed the influence of fear and fractional-order derivative on system dynamics(Yousef *et al.* 2021). Li et al investigated the stability of a fractional-order predator-prey model, which incorporates a prey refuge(Li *et al.* 2017a). Boukhouima et al studied a fractional-order model to describe the dynamics of human immunodeficiency virus infection(Boukhouima *et al.* 2017). Mousfata et al consider a fractional-order eco-epidemiological system of prey

Manuscript received: 31 March 2022,
Revised: 21 May 2022,
Accepted: 5 June 2022.

[1] 13437617328@163.com
[2] liguoy@scau.edu.cn
[3] weizhouchao@163.com (**Corresponding Author**)

population with disease. Moreover, the dynamics of this model was analyzed(Moustafa *et al.* 2020).

Delay plays an important part in ecosystem and it exists universally. Different models take different biological delays into account(Tao *et al.* 2018; Shi *et al.* 2022; Chinnathambi and Rihan 2018; Fernández-Carreón *et al.* 2022; Rihan and Rajivganthi 2020; Xu and Zhang 2013; Huang *et al.* 2019; Mahmoud *et al.* 2017; Pu 2020; Alidousti and Mostafavi Ghahfarokhi 2019; Huang *et al.* 2020; Deng *et al.* 2007; Kashkynbayev and Rihan 2021; Yuan *et al.* 2013). Compared with the systems without delay, the systems with delays will show more complex nonlinear dynamic behavior. Delay may cause the equilibrium points instability. Moreover, spreading of disease is not happen immediately. In general, infectious disease has an incubation period. Therefore, it is important to take delay into account for biological model, and it will describe real life more accurately.

In (Zhou *et al.* 2010), Zhou et al formulated the following system

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)(1-\frac{S(t)+I(t)}{K}) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t)I(t) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1}, \\ \frac{dy(t)}{dt} &= y(t)(a_2 - \frac{c_2 y(t)}{I(t)+K_2}), \end{aligned} \tag{1}$$

and studied the dynamics of (1). Based on the importance of delay, Adak et al considered delay into account and formulated the following system(Adak *et al.* 2020)

$$\begin{aligned} \frac{dS(t)}{dt} &= rS(t)(1-\frac{S(t)+I(t)}{K}) - \beta S(t)I(t), \\ \frac{dI(t)}{dt} &= \beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1}, \\ \frac{dy(t)}{dt} &= y(t)(a_2 - \frac{c_2 y(t)}{I(t)+K_2}). \end{aligned} \tag{2}$$

They exhibited the dynamic behavior of system (2), such as chaos and Hopf bifurcation. However, Zhou and Adak et al did not take the good memory characteristics of fractional derivative into account, which can well describe long-range temporal memory. Hence, we consider the fractional derivative into account for system (2) and establish a fractional-order Leslie-Gower prey-predator-parasite system with delay

$$\begin{aligned} D^{\alpha}S(t) &= rS(t)(1-\frac{S(t)+I(t)}{K}) - \beta S(t)I(t), \\ D^{\alpha}I(t) &= \beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1}, \\ D^{\alpha}y(t) &= y(t)(a_2 - \frac{c_2 y(t)}{I(t)+K_2}). \end{aligned} \tag{3}$$

The initial conditions of (3) are as follows:

$$S(t) = \eta_1(t), I(t) = \eta_2(t), y(t) = \eta_3(t), t \in [-\tau, 0], \tag{4}$$

where $S(t), I(t), y(t)$ represent the growth rates of susceptible prey, infected prey and predator population at time t respectively. r represents intrinsic growth rate of susceptible prey. K represents environmental prey carrying capacity. β represents infection rate. c represents predation-independent death rate of infectious prey. c_1 represents maximum predation rate of predator on an infectious prey. K_1 represents half-saturation density. a_2 represents intrinsic growth rate of predator. c_2 and K_2 are positive constants. D^{α} denotes α-order Caputo differential derivative, $\alpha \in (0, 1]$, and $r, K, \beta, c, c_1, K_1, a_2, c_2, K_2$ are all nonnegetive. Label R^3_+ as the non-negative cone, $\eta = (\eta_1(t), \eta_2(t), \eta_3(t)) \in C([-\tau, 0], R^3_+)$, the Banach space of continuous real-valued functions on the interval $[-\tau, 0]$ with norm $||\eta|| = \sup_{-\tau \le t \le 0} |\eta(t)|$, and $\eta_1(t) \ge 0, \eta_2(t) \ge 0, \eta_3(t) \ge 0, \eta_1(0) > 0, \eta_2(0) > 0, \eta_3(0) > 0$.

We aim to investigate the stability of system (3) and how the delay affects the dynamics of this system. Firstly, we investigate the existence and uniqueness of the solutions, as well as their non-negativity and boundedness. Furthermore, we derive the local asymptotic stability of every equilibrium point. Then, we demonstrate the global asymptotic stability of the infection-free equilibrium point by formulating a Lyapunov function. Moreover, we choose delay as the bifurcation parameter to show interior equilibrium point occurs Hopf bifurcation under some conditions. Lastly, we give the numerical examples to back up our results.

The structure of this article is as follows. We describe basic concepts in section 2. The existence and uniqueness of the solutions, as well as their non-negativity and boundedness are investigated in section 3. Besides, we derive equilibrium points and the local asymptotic stability corresponding to each equilibrium point. Then, we analyze the Hopf bifurcation of the interior equilibrium point. We provide two illustrative examples to back up our findings in section 4. Finally, we close the paper in last section.

MATHEMATICAL PRELIMINARIES

Definition 1. (Kilbas *et al.* 2006) The Riemann-Liouville's fractional integral of order $\alpha > 0$ for a function f is defined as

$$D^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2. (Kilbas *et al.* 2006) The Caputo's fractional derivative of order α for a function f is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)ds, t > 0,$$

where $0 \le m-1 \le \alpha < m, m \in Z^+$.

.

Lemma 1. (Wang *et al.* 2011) Consider the following nonlinear differential equation with Caputo fractional derivative

$$\begin{aligned} D^{\alpha}X(t) &= f(X(t)) + g(X(t-\tau)), \\ X(t) &= \Phi(t), t \in [-\tau, 0], \end{aligned} \tag{5}$$

where $\alpha \in (0, 1], X(t) \in R^n, \tau \ge 0$, then the characteristic equation of system is

$$|s^{\alpha}E - A - Be^{-s\tau}| = 0,$$

where A and B is the Jacobian matrix of the function $f(X(t))$ and $g(X(t))$ at the equilibrium point of the system (5). The zero solution of system (5) is locally asymptotically stable if all the roots of the characteristic equation restricted to $arg(\lambda) > \frac{\pi\alpha}{2}$ have negative real parts.

Lemma 2. (Odibat and Shawagfeh 2007) Suppose that $f(t) \in C[a, b]$ and $D^{\alpha}f(t) \in C[a, b]$ for $0 < \alpha \le 1$. If $D^{\alpha}f(t) \ge 0, \forall t \in [a, b]$, then $f(t)$ is non-decreasing for each $t \in [a, b]$. If $D^{\alpha}f(t) \le 0, \forall t \in (a, b)$, then $f(t)$ is non-increasing for each $t \in [a, b]$.

Lemma 3. (Li *et al.* 2010) Consider the system

$$D^{\alpha}x(t) = f(t,x), t > t_0, \tag{6}$$

with initial condition $x(t_0)$, where $\alpha \in (0,1], f : [t_0,\infty) \times \Omega \to R^n, \Omega \subseteq R^n$, if $f(t,x)$ satisfies the locally lipschitz condition with respect to x, then there exists a unique solution of (6) on $[t_0,\infty) \times \Omega$.

Lemma 4. (Cruz 2015) Let $x(t) \in R^+$ be a continuous and derivative function. Then, for any time instant $t \geq t_0$,

$${}_{t_0}D_t^{\alpha}(x(t) - x^* - x^* \ln \frac{x(t)}{x^*}) \leq (1 - \frac{x^*}{x(t)}){}_{t_0}D_t^{\alpha}x(t), \tag{7}$$

where $\forall\alpha \in (0,1), x^* \in R^+$.

Lemma 5. (Li *et al.* 2017b) Let $u(t) \in C([0,+\infty))$. If $u(t)$ satisfies $D^{\alpha}u(t) \leq a - bu(t)$, $u(0) = u_0$, where $\alpha \in (0,1], (a,b) \in R^2$ and $b \neq 0$, then

$$u(t) \leq (u_0 - \frac{a}{b})E_{\alpha}(-bt^{\alpha}) + \frac{a}{b}.$$

MAIN RESULTS

Existence and Uniqueness of solutions

Theorem 6. For any non-negative initial conditions the fractional-order system (3) has a unique solution.

Proof. Consider the region $\Pi = \{(S,I,y) \in R^3 : \max\{|S|,|I|,|y|\} \leq M\}$, and denote $X = (S,I,y)$, $\widehat{X} = (\widehat{S},\widehat{I},\widehat{y})$, then define a mapping $f(X) = (f_1(X), f_2(X), f_3(X))$, where

$$f_1(X) = rS(t)(1 - \frac{S(t)+I(t)}{K}) - \beta S(t)I(t),$$
$$f_2(X) = \beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1},$$
$$f_3(X) = y(t)(a_2 - \frac{c_2 y(t)}{I(t)+K_2}).$$

For $X, \widehat{X} \in \Pi$, then

$$\begin{aligned}
&\|f(X) - f(\widehat{X})\| = |f_1(X) - f_1(\widehat{X})| + |f_2(X) - f_2(\widehat{X})| \\
&+ |f_3(X) - f_3(\widehat{X})| \\
&= |r(S-\widehat{S}) - \frac{r}{K}(S^2 - \widehat{S}^2) - (\frac{r}{K}+\beta)(SI - \widehat{S}\widehat{I})| \\
&+ |\beta I(t-\tau)(S(t-\tau) - \widehat{S}(t-\tau)) + \beta\widehat{S}(t-\tau)\times \\
&(I(t-\tau) - \widehat{I}(t-\tau)) - c(I-\widehat{I}) - \frac{c_1 I\widehat{I}(y-\widehat{y})}{(I+K_1)(\widehat{I}+K_1)} \\
&- \frac{c_1K_1 I(y-\widehat{y}) + c_1K_1\widehat{y}(I-\widehat{I})}{(I+K_1)(\widehat{I}+K_1)}| + |a_2(y-\widehat{y}) \\
&- \frac{K_2c_2(y+\widehat{y})(y-\widehat{y}) + c_2I(y^2-\widehat{y}^2) - c_2y^2(I-\widehat{I})}{(I+K_2)(\widehat{I}+K_2)}| \\
&\leq (r + 3\frac{Mr}{K} + \beta M)|S-\widehat{S}| + M(\frac{r}{K}+\beta)|I-\widehat{I}| \\
&+ \beta M|S(t-\tau) - \widehat{S}(t-\tau)| + \beta M|I(t-\tau) - \widehat{I}(t-\tau)| \\
&+ (c + \frac{c_1K_1M}{K_1^2})|I-\widehat{I}| \\
&+ (\frac{c_1M^2 + c_1K_1M}{K_1^2} + a_2 + \frac{2MK_2c_2 + 2M^2c_2}{K_2^2})|y-\widehat{y}| \\
&+ \frac{c_2M^2}{K_2^2}|I-\widehat{I}| \\
&= (r + 3\frac{Mr}{K} + 2\beta M)|S-\widehat{S}| + (M\frac{r}{K} + 2\beta M + c \\
&+ \frac{c_1K_1M}{K_1^2} + \frac{c_2M^2}{K_2^2})|I-\widehat{I}| + (\frac{c_1M^2 + c_1K_1M}{K_1^2} + a_2 \\
&+ \frac{2MK_2c_2 + 2M^2c_2}{K_2^2})|y-\widehat{y}| \\
&\leq L\|X - \widehat{X}\|,
\end{aligned}$$

where $L = \max\{(r + 3\frac{Mr}{K} + 2\beta M), (\frac{Mr}{K} + 2\beta M + c + \frac{c_1K_1M}{K_1^2} + \frac{c_2M^2}{K_2^2}), (\frac{c_1M^2+c_1K_1M}{K_1^2} + a_2 + \frac{2MK_2c_2+2M^2c_2}{K_2^2})\}$. Hence, Lipschitz condition is satisfied for $f(X)$. There exist a unique solution of system (3) on the basis of Lemma 3. □

Non-negativity of solutions

Theorem 7. All the solutions of system (3) starting from

$$D_+ = \{(S,I,y) \in R^3 : S,I,y \in R^+\},$$

are non-negative.

Proof. Above all, we derive that the solution $S(t)$ starting from D_+ is non-negative, i.e. $S(t) \geq 0$ for $t \geq t_0$. Suppose that is not true, then there exist $t_1 > t_0$ such that $S(t) > 0, t_0 \leq t < t_1, S(t_1) = 0, S({t_1}^+) < 0$. From the first equation of system (3), we get

$$D^{\alpha}S(t_1)|_{S(t_1)=0} = 0.$$

Based on the Lemma 2, there exsits $S({t_1}^+) = 0$ and it contradicts with $S({t_1}^+) < 0$. Hence, we can get $S(t) \geq 0$ for $t \geq t_0$.

If there exist $t_2 > t_0$ such that $I(t) > 0, t_0 \leq t < t_2, I(t_2) = 0, I({t_2}^+) < 0$, then we get

$$D^{\alpha}I(t_2)|_{I(t_2)=0} = \beta S(t_2-\tau)I(t_2-\tau) > 0.$$

Based on the Lemma 2, there exsits $I({t_2}^+) > 0$ and it contradicts with $I({t_2}^+) < 0$. So, we get $I(t) \geq 0$ for $t \geq t_0$.

If there exist a constant $t_3 > t_0$ such that $y(t) > 0, t_0 \leq t < t_3, y(t_3) = 0, y(t_3{}^+) < 0$, then we get

$$D^\alpha y(t_3)|_{y(t_3)=0} = 0.$$

Similarly, we have $y(t_3{}^+) = 0$, which contradicts with $y(t_3{}^+) < 0$. Hence, we obtain $y(t) \geq 0$ for $t \geq t_0$. □

Boundedness of solutions

Theorem 8. All solutions of system (3) starting from R^3_+ are bounded.

Proof. Denote

$$f(S(t)) = rS(t)(1 - \frac{S(t) + I(t)}{K}) - \beta S(t)I(t),$$
$$F(S(t)) = rS(t)(1 - \frac{S(t)}{K}),$$

and let

$$D^\alpha S(t) = f(S(t)), \tag{8}$$
$$D^\alpha S(t) = F(S(t)). \tag{9}$$

Assume $h(t)$ is the solution of (8) and $H(t)$ is the solution of (9). Since $f(S(t)) \leq F(S(t))$, we can derive $h(t) \leq H(t)$ according to the comparison theorems of fractional-order differential equations(Hu *et al.* 2009). Let $z_1(t) = \frac{rS(t)}{K}$, then (9) become

$$D^\alpha z_1(t) = z_1(t)(r - z_1(t)). \tag{10}$$

Denote $\bar{H}(t)$ is the solution of (10), then $\bar{H}(t) = \frac{rH(t)}{K}$. Based on the methods in (Li *et al.* 2019), we can get $\limsup_{t\to\infty} z_1(t) \leq \hat{m}$, thus we can derive $\limsup_{t\to\infty} S(t) \leq \frac{K\hat{m}}{r}$, denote $m = \frac{K\hat{m}}{r}$, then $\limsup_{t\to\infty} S(t) \leq m$. Define a function $W(t) = S(t-\tau) + I(t)$. Then

$$\begin{aligned} D^\alpha W(t) &= D^\alpha S(t-\tau) + D^\alpha I(t) \\ &= rS(t-\tau)(1 - \frac{S(t-\tau) + I(t-\tau)}{K}) \\ &\quad - cI(t) - \frac{c_1 I(t) y(t)}{I(t) + K_1} \\ &\leq rS(t-\tau) - cI(t) \\ &= 2rS(t-\tau) - dW(t) \\ &\leq 2rm - dW(t), \end{aligned}$$

where $d = \min\{r, c\}$. From Lemma 5, we can get

$$0 \leq W(t) \leq (W(0) - \frac{2rm}{d})E_\alpha(-dt^\alpha) + \frac{2rm}{d},$$

where E_α is the Mittag-Leffler function. Hence, we can obtain $\limsup_{t\to\infty} W(t) \leq \frac{2rm}{d}$. Then $\limsup_{t\to\infty} I(t) \leq \frac{2rm}{d}$. For the third equation of system (3), we can obtain

$$D^\alpha y(t) \leq y(t)(a_2 - \frac{dc_2 y(t)}{2rm + dK_2}). \tag{11}$$

Denote $\frac{dc_2}{2rm+dK_2} = a_1$, and let $z_2(t) = a_1 y(t)$, then (11) become

$$D^\alpha z_2(t) = z_2(t)(a_2 - z_2(t)). \tag{12}$$

Based on the methods in (Li *et al.* 2019), we also can get $\limsup_{t\to\infty} y(t) \leq \hat{m}$. Hence, the proof is completed and the region is $\Omega' = \{(S, I, y) \in R^3_+ : S(t) \leq m, I(t) \leq \frac{2rm}{d}, y(t) \leq \hat{m}\}$, where $d = \min\{r, c\}$. □

Equilibrium points

Set

$$D^\alpha S(t) = 0, D^\alpha I(t) = 0, D^\alpha y(t) = 0,$$

then the equilibrium points can be determined.
(1)The trivial equilibrium point is $E_0(0,0,0)$.
(2)The infection-free and predator-free equilibrium point is $E_1(S_1, 0, 0)$, where $S_1 = K$.
(3)The predator-only equilibrium point is $E_2(0,0,y_2)$, where $y_2 = \frac{a_2 K_2}{c_2}$.
(4)The predator-free equilibrium point is $E_3(S_3, I_3, 0)$, where $S_3 = \frac{c}{\beta}, I_3 = \frac{r(\beta K - c)}{\beta(r+\beta K)}$. E_3 exists if $\beta > \beta_1$, where $\beta_1 = \frac{c}{K}$.
(5)The infection-free equilibrium point is $E_4(S_4, 0, y_4)$, where $S_4 = K, y_4 = \frac{a_2 K_2}{c_2}$.
(6)The interior equilibrium point is $E'(S', I', y')$, where $S' = \frac{1}{\beta}[c + \frac{c_1 a_2}{c_2}\frac{K_2+I'}{K_1+I'}], y' = \frac{a_2(I'+K_2)}{c_2}, I' = \frac{-\Delta_2 + \sqrt{\Delta_2^2 - 4\Delta_1\Delta_3}}{2\Delta_1}$, Δ_1, Δ_2 and Δ_3 are the coefficients of the equation $\Delta_1 I'^2 + \Delta_2 I' + \Delta_3 = 0$, and $\Delta_1 = \frac{r+\beta K}{K} > 0, \Delta_2 = \frac{rc_1 a_2}{K\beta c_2} + \frac{K_1(r+\beta K)}{K} + \frac{r(c-\beta K)}{\beta K}, \Delta_3 = \frac{r}{\beta K}[\frac{c_1 a_2 K_2}{c_2} + (c - \beta K)K_1]$. E' exists if $\beta > \beta_2$, where $\beta_2 = \beta_1 + \frac{c_1 a_2 K_2}{c_2 K K_1}, \beta_1 = \frac{c}{K}$.

Suppose $E^*(S^*, I^*, y^*)$ is arbitrary equilibrium point, we transform E^* into the origin. Let

$$U_1(t) = S(t) - S^*, U_2(t) = I(t) - I^*, U_3(t) = y(t) - y^*,$$

then we can rewrite system (3) as

$$\begin{aligned} D^\alpha U_1(t) &= r(U_1(t) + S^*)(1 - \frac{U_1(t) + S^* + U_2(t) + I^*}{K}) \\ &\quad - \beta(U_1(t) + S^*)(U_2(t) + I^*), \\ D^\alpha U_2(t) &= \beta(U_1(t-\tau) + S^*)(U_2(t-\tau) + I^*) \\ &\quad - c(U_2(t) + I^*) - \frac{c_1(U_2(t) + I^*)(U_3(t) + y^*)}{U_2(t) + I^* + K_1}, \\ D^\alpha U_3(t) &= (U_3(t) + y^*)(a_2 - \frac{c_2(U_3(t) + y^*)}{U_2(t) + I^* + K_2}). \end{aligned} \tag{13}$$

Taking advantage of Taylor expansion formula and linearizing the system (13), we can get

$$\begin{aligned} D^\alpha U_1(t) &= (r - \frac{2rS^*}{K} - \frac{rI^*}{K} - \beta I^*)U_1(t) \\ &\quad - (\frac{r}{K} + \beta)S^* U_2(t), \\ D^\alpha U_2(t) &= -(c + \frac{c_1 K_1 y^*}{(I^* + K_1)^2})U_2(t) - \frac{c_1 I^*}{I^* + K_1}U_3(t) \\ &\quad + \beta I^* U_1(t-\tau) + \beta S^* U_2(t-\tau), \\ D^\alpha U_3(t) &= \frac{c_2(y^*)^2}{(I^* + K_2)^2}U_2(t) + (a_2 - \frac{2c_2 y^*}{I^* + K_2})U_3(t). \end{aligned} \tag{14}$$

Stability

According to Lemma 1, we obtain

$$V_1 = \begin{pmatrix} m_{11} & m_{12} & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix}, V_2 = \begin{pmatrix} 0 & 0 & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

where

$$\begin{aligned}
m_{11} &= r - \frac{2rS^*}{K} - \frac{rI^*}{K} - \beta I^*, m_{12} = -(\frac{r}{K} + \beta)S^*, \\
m_{22} &= -(c + \frac{c_1 K_1 y^*}{(I^* + K_1)^2}), m_{23} = -\frac{c_1 I^*}{I^* + K_1}, \\
m_{32} &= \frac{c_2 (y^*)^2}{(I^* + K_2)^2}, m_{33} = a_2 - \frac{2c_2 y^*}{I^* + K_2}, \\
n_{21} &= \beta I^*, n_{22} = \beta S^*.
\end{aligned} \tag{16}$$

Denote $V = V_1 + V_2 e^{-s\tau}$, then the Jacobi Matrix of the system (14) is

$$V = \begin{pmatrix} m_{11} & m_{12} & 0 \\ n_{21}e^{-s\tau} & m_{22} + n_{22}e^{-s\tau} & m_{23} \\ 0 & m_{32} & m_{33} \end{pmatrix}, \tag{17}$$

thus the characteristic equation of (14) can be obtained as:

$$det \begin{pmatrix} s^\alpha - m_{11} & -m_{12} & 0 \\ -n_{21}e^{-s\tau} & s^\alpha - m_{22} - n_{22}e^{-s\tau} & -m_{23} \\ 0 & -m_{32} & s^\alpha - m_{33} \end{pmatrix} = 0, \tag{18}$$

i.e.$(s^\alpha - m_{11})(s^\alpha - m_{22} - n_{22}e^{-s\tau})(s^\alpha - m_{33}) - m_{12}n_{21}e^{-s\tau}(s^\alpha - m_{33}) - m_{23}m_{32}(s^\alpha - m_{11}) = 0$.

(i) For equilibrium point $E_0(0,0,0)$, (18) becomes

$$(s^\alpha - r)(s^\alpha + c)(s^\alpha - a_2) = 0. \tag{19}$$

Suppose $s^\alpha = \lambda$, then (19) has eigenvalues $\lambda_1 = r > 0, \lambda_2 = -c < 0, \lambda_3 = a_2 > 0$, thus $|arg(\lambda_i)| = 0 < \frac{\pi\alpha}{2}, i = 1,3$. According to Lemma 1, equilibrium point E_0 is unstable.

(ii) For equilibrium point $E_1(S_1,0,0)$, (18) becomes

$$(s^\alpha + r)(s^\alpha + c - \beta K e^{-s\tau})(s^\alpha - a_2) = 0. \tag{20}$$

Let $s^\alpha = \lambda$, then (20) has a positive eigenvalue $\lambda_1 = a_2 > 0$, thus $|arg(\lambda_1)| = 0 < \frac{\pi\alpha}{2}$. According to Lemma 1, equilibrium point E_1 is unstable.

(iii) For equilibrium point $E_2(0,0,y_2)$, (18) reduces to

$$(s^\alpha - r)(s^\alpha + (c + \frac{c_1 a_2 K_2}{c_2 K_1}))(s^\alpha + a_2) = 0. \tag{21}$$

Let $s^\alpha = \lambda$, then (21) has a positive eigenvalue $\lambda_1 = r > 0$, thus $|arg(\lambda_1)| = 0 < \frac{\pi\alpha}{2}$. According to Lemma 1, equilibrium point E_2 is unstable.

(iv) For equilibrium point $E_3(S_3, I_3, 0)$, (18) reduces to

$$(s^\alpha - a_2)[(s^\alpha - m_{11})(s^\alpha - m_{22} - n_{22}e^{-s\tau}) - m_{12}n_{21}e^{-s\tau}] = 0, \tag{22}$$

where $m_{11}|_{E_3} = r - \frac{2rS_3}{K} - \frac{rI_3}{K} - \beta I_3, m_{12}|_{E_3} = -(\frac{r}{K} + \beta)S_3, m_{22}|_{E_3} = -c, m_{23}|_{E_3} = -\frac{c_1 I_3}{I_3 + K_1}, m_{32}|_{E_3} = 0, m_{33}|_{E_3} = a_2, n_{21}|_{E_3} = \beta I_3, n_{22}|_{E_3} = \beta S_3$, and $S_3 = \frac{c}{\beta}, I_3 = \frac{r(\beta K - c)}{\beta(r + \beta K)}$. Let $s^\alpha = \lambda$, then (22) has a positive eigenvalue $\lambda_1 = a_2 > 0$, thus $|arg(\lambda_1)| = 0 < \frac{\pi\alpha}{2}$. According to Lemma 1, equilibrium point E_3 is unstable.

We derive the following theorem based on the above analysis.

Theorem 9. E_0, E_1, E_2, E_3 are unstable for all $\tau \geq 0$.

(v) For equilibrium point $E_4(S_4, 0, y_4)$, (18) reduces to

$$(s^\alpha + r)(s^\alpha - m_{22} - n_{22}e^{-s\tau})(s^\alpha + a_2) = 0. \tag{23}$$

where $m_{11}|_{E_4} = -r, m_{12}|_{E_4} = -(\frac{r}{K} + \beta)K, m_{22}|_{E_4} = -(c + \frac{c_1 a_2 K_2}{c_2 K_1}), m_{23}|_{E_4} = 0, m_{32}|_{E_4} = \frac{a_2^2}{c_2}, m_{33}|_{E_4} = -a_2, n_{21}|_{E_4} = 0, n_{22}|_{E_4} = \beta K$. Let $s^\alpha = \lambda$, then two eigenvalues of (23) are $\lambda_1 = -r < 0, \lambda_2 = -a_2 < 0$, thus $|arg(\lambda_i)| = \pi > \frac{\alpha\pi}{2}, i = 1,2$. By solving the following equation

$$s^\alpha - m_{22} - n_{22}e^{-s\tau} = 0, \tag{24}$$

we can gain other eigenvalues.

When $\tau = 0$, the other eigenvalue is $\lambda_3 = (\beta K - c) - \frac{c_1 a_2 K_2}{c_2 K_1}$. $\lambda_3 < 0$ if $\beta < \beta_2 = \frac{c}{K} + \frac{c_1 a_2 K_2}{c_2 K K_1}$. Then we acquire $|arg(\lambda_i)| > \frac{\alpha\pi}{2}, i = 1,2,3$, thus all characteristic roots of (23) have negative real parts. E_4 is locally asymptotically stable on the basic of Lemma 1.

When $\tau > 0$, assume that $s = \mathrm{i}\omega = \omega(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2})(\omega > 0)$ is a root of (24). Separating real and imaginary parts

$$|\omega|^\alpha \cos\frac{\pi}{2}\alpha - n_{22}\cos\omega\tau - m_{22} = 0, |\omega|^\alpha \sin\frac{\pi}{2}\alpha + n_{22}\sin\omega\tau = 0. \tag{25}$$

From (25) we can obtain

$$\cos\omega\tau = \frac{1}{n_{22}}|\omega|^\alpha \cos\frac{\pi}{2}\alpha - \frac{m_{22}}{n_{22}}, \sin\omega\tau = -\frac{1}{n_{22}}|\omega|^\alpha \sin\frac{\pi}{2}\alpha. \tag{26}$$

Add up the squares of both equations of (26)

$$|\omega|^{2\alpha} - 2m_{22}\cos(\frac{\pi}{2}\alpha)|\omega|^\alpha + m_{22}^2 - n_{22}^2 = 0, \tag{27}$$

Let $\omega^\alpha = t$, then we can get

$$t^2 - 2m_{22}\cos(\frac{\pi}{2}\alpha)t + m_{22}^2 - n_{22}^2 = 0. \tag{28}$$

Since $\alpha \in (0,1], m_{22}|_{E_4} = -(c + \frac{c_1 a_2 K_2}{c_2 K_1}) < 0, n_{22}|_{E_4} = \beta K$, then $-2m_{22}\cos\frac{\pi}{2}\alpha > 0, m_{22}^2 - n_{22}^2 = (c + \frac{c_1 a_2 K_2}{c_2 K_1})^2 - \beta^2 K^2 = (K(\frac{c}{K} + \frac{c_1 a_2 K_1}{c_2 K_1 K}))^2 - \beta^2 K^2 = K^2(\beta_2)^2 - K^2\beta^2 = K^2(\beta + \beta_2)(\beta_2 - \beta)$. We derive $m_{22}^2 - n_{22}^2 > 0$ if $\beta < \beta_2$. According to Routh-Hurwitz theorem, (28) has no positive real part. Then (24) has no pure imaginary root. Therefore, equilibrium point E_4 is locally asymptotically stable. We derive the following theorem based on the above analysis.

Theorem 10. E_4 is locally asymptotically stable for $\tau \geq 0$ if $\beta < \beta_2 = \frac{c}{K} + \frac{c_1 a_2 K_2}{c_2 K K_1}$.

Furthermore, we obtain the globally asymptotically stable of system (3) at E_4. To investigate the globally asymptotically stable of system (3) at E_4, we introduce the following assumption.
(H1) $(\frac{r}{K} + \beta)S_4 - c \leq 0$,
(H2) $(c_2 y_4 - K_2 c_1)I + K_1 c_2 y_4 - K_2^2 c_1 \leq 0$.
Motivated by (Sene 2021), we define a Lyapunov functional as

$$V(t) = S(t) - S_4 - S_4 \ln\frac{S(t)}{S_4} + I(t) + y(t) - y_4 - y_4 \ln\frac{y(t)}{y_4}.$$

Taking fractional-order derivative on both sides, according to Lemma 4, we get

$$
\begin{aligned}
D^{\alpha}V(t) \leq & (\frac{S(t)-S_4}{S(t)})D^{\alpha}S(t) + D^{\alpha}I(t) + \frac{y(t)-y_4}{y(t)}D^{\alpha}y(t) \\
= & (S(t)-S_4)(r(1-\frac{S(t)+I(t)}{K}) - \beta I(t)) + \\
& (\beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1}) + \\
& (y(t)-y_4)(a_2 - \frac{c_2 y(t)}{I(t)+K_2}) \\
= & (S(t)-S_4)(-\frac{r}{K}(S(t)-S_4) - (\frac{r}{K}+\beta)I(t)) \\
& + (\beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1}) \\
& + (y(t)-y_4)(\frac{c_2 y_4}{K_2} - \frac{c_2 y(t)}{I(t)+K_2}) \\
= & -\frac{r}{K}(S(t)-S_4)^2 - (\frac{r}{K}+\beta)I(t)(S(t)-S_4) \\
& + \beta S(t-\tau)I(t-\tau) - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1} + \\
& c_2(y(t)-y_4)(\frac{y_4}{K_2} - \frac{y(t)}{I(t)+K_2}) \\
= & -\frac{r}{K}(S(t)-S_4)^2 - (\frac{r}{K}+\beta)S(t)I(t) + \\
& (\frac{r}{K}+\beta)S_4 I(t) + \beta S(t-\tau)I(t-\tau) \\
& - cI(t) - \frac{c_1 I(t)y(t)}{I(t)+K_1} + c_2(y(t)-y_4)\times \\
& (-\frac{y(t)-y_4}{I(t)+K_2} + \frac{I(t)y_4}{K_2(I(t)+K_2)}) \\
= & -\frac{r}{K}(S(t)-S_4)^2 + (-(\frac{r}{K}+\beta)S(t)I(t) + \\
& \beta S(t-\tau)I(t-\tau)) + ((\frac{r}{K}+\beta)S_4 - c)I(t) \\
& - \frac{c_2}{K_2+I(t)}(y(t)-y_4)^2 - \frac{c_2 I(t)(y_4)^2}{K_2(I(t)+K_2)} \\
& + \frac{c_2 y_4 I(t)y(t)}{K_2(I(t)+K_2)} - \frac{c_1 I(t)y(t)}{I(t)+K_1} \\
= & -\frac{r}{K}(S(t)-S_4)^2 + (\beta - (\frac{r}{K}+\beta))S(t)I(t) \\
& + ((\frac{r}{K}+\beta)S_4 - c)I(t) - \frac{c_2}{K_2+I(t)}\times \\
& (y(t)-y_4)^2 - \frac{c_2 I(t)(y_4)^2}{K_2(I(t)+K_2)} + \\
& \frac{(c_2y_4 - K_2c_1)I(t) + (K_1c_2y_4 - K_2^2c_1)}{K_2(I(t)+K_2)(I(t)+K_1)}I(t)y(t) \\
= & -\frac{r}{K}(S(t)-S_4)^2 - \frac{r}{K}S(t)I(t) \\
& + ((\frac{r}{K}+\beta)S_4 - c)I(t) - \frac{c_2}{K_2+I(t)}(y(t)-y_4)^2 \\
& - \frac{c_2 I(t)(y_4)^2}{K_2(I(t)+K_2)} + \\
& \frac{(c_2y_4 - K_2c_1)I(t) + (K_1c_2y_4 - K_2^2c_1)}{K_2(I(t)+K_2)(I(t)+K_1)}I(t)y(t).
\end{aligned}
$$

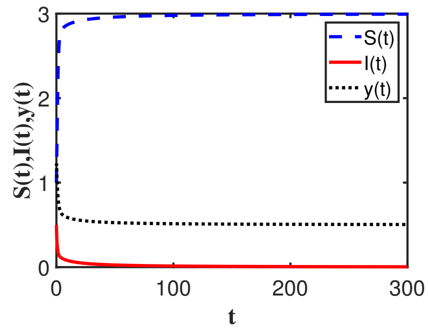


Figure 1 Waveform plots of system (49) with $\tau = 0.4$.

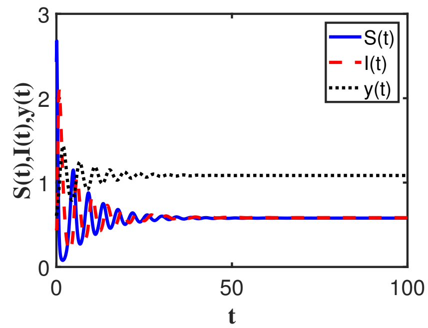


Figure 2 Waveform plots of system (50) with $\tau = 0 < \tau_0$.

Based on the assumption $(\frac{r}{K}+\beta)S_4 - c \leq 0$ and $(c_2y_4 - K_2c_1)I + K_1c_2y_4 - K_2^2c_1 \leq 0$, we can get $D^{\alpha}V(t) \leq 0$. According to (Huo *et al.* 2015), we can derive the system (3) is globally asymptotically stable at E_4.

Therefore, We derive the following theorem.

Theorem 11. Assume that $(\frac{r}{K}+\beta)S_4 - c \leq 0$ and $(c_2y_4 - K_2c_1)I + K_1c_2y_4 - K_2^2c_1 \leq 0$, then the system (3) is globally asymptotically stable at E_4.

(vi) For equilibrium point $E'(S', I', y')$, the characteristic equation at E' is:

$$s^{3\alpha} + \delta_2 s^{2\alpha} + \delta_1 s^{\alpha} + \delta_0 + e^{-s\tau}(\vartheta_2 s^{2\alpha} + \vartheta_1 s^{\alpha} + \vartheta_0) = 0, \qquad (29)$$

where

$$
\begin{aligned}
\delta_2 &= -(m_{11} + m_{22} + m_{33}), \\
\delta_1 &= m_{11}m_{22} + m_{22}m_{33} + m_{11}m_{33} - m_{23}m_{32}, \\
\delta_0 &= m_{11}m_{23}m_{32} - m_{11}m_{22}m_{33},
\end{aligned}
$$

$$
\begin{aligned}
\vartheta_2 &= -n_{22}, \\
\vartheta_1 &= m_{11}n_{22} - m_{12}n_{21} + m_{33}n_{22}, \\
\vartheta_0 &= m_{12}m_{33}n_{21} - m_{11}m_{33}n_{22}.
\end{aligned}
$$

When $\tau = 0$, (29) can be expressed as

$$s^{3\alpha} + (\delta_2 + \vartheta_2)s^{2\alpha} + (\delta_1 + \vartheta_1)s^{\alpha} + \delta_0 + \vartheta_0 = 0, \qquad (30)$$

Let $z = s^{\alpha}$, then

$$z^3 + (\delta_2 + \vartheta_2)z^2 + (\delta_1 + \vartheta_1)z + \delta_0 + \vartheta_0 = 0. \qquad (31)$$

According to the Routh-Hurwitz theorem, (30) has no positive real part if $\delta_2 + \vartheta_2 > 0$ and $(\delta_2 + \vartheta_2)(\delta_1 + \vartheta_1) - \delta_0 + \vartheta_0 > 0$. Thus (29) has no pure imaginary root. Hence, E' is locally asymptotically stable.

We obtain the following theorem on the basic of our analysis.

Theorem 12. The equilibrium point E' is locally asymptotically stable for $\tau = 0$ if $\delta_2 + \vartheta_2 > 0$ and $(\delta_2 + \vartheta_2)(\delta_1 + \vartheta_1) - \delta_0 + \vartheta_0 > 0$.

Assume that $s = \mathrm{i}\xi = \xi(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2})(\xi > 0)$ is a root of (29). Separating real and imaginary parts,

$$\Psi\cos\xi\tau + \Omega\sin\xi\tau = \Phi_1, \tag{32}$$

$$\Omega\cos\xi\tau - \Psi\sin\xi\tau = \Phi_2, \tag{33}$$

where

$$\begin{aligned}
\Psi &= \vartheta_0 + \vartheta_1\xi^{\alpha}\cos(\alpha\frac{\pi}{2}) + \vartheta_2\xi^{2\alpha}\cos(2\alpha\frac{\pi}{2}),\\
\Omega &= \vartheta_1\xi^{\alpha}\sin(\alpha\frac{\pi}{2}) + \vartheta_2\xi^{2\alpha}\sin(2\alpha\frac{\pi}{2}),\\
\Phi_1 &= -(\delta_0 + \delta_1\xi^{\alpha}\cos(\alpha\frac{\pi}{2}) + \delta_2\xi^{2\alpha}\cos(2\alpha\frac{\pi}{2}) +\\
&\quad \xi^{3\alpha}\cos(3\alpha\frac{\pi}{2})),\\
\Phi_2 &= -(\delta_1\xi^{\alpha}\sin(\alpha\frac{\pi}{2}) + \delta_2\xi^{2\alpha}\sin(2\alpha\frac{\pi}{2}) +\\
&\quad \xi^{3\alpha}\sin(3\alpha\frac{\pi}{2})).
\end{aligned}$$

Add up the squares of both equations (32) and (33),

$$\begin{aligned}
G(\xi^{\alpha}) &= \xi^{6\alpha} + H_5\xi^{5\alpha} + H_4\xi^{4\alpha} + H_3\xi^{3\alpha}\\
&\quad + H_2\xi^{2\alpha} + H_1\xi^{\alpha} + H_0\\
&= 0,
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
H_5 &= 2\delta_2\cos(\alpha\frac{\pi}{2}),\\
H_4 &= \delta_2^2 - \vartheta_2^2 + 2\delta_1\cos(2\alpha\frac{\pi}{2}),\\
H_3 &= (2\delta_1\delta_2 - 2\vartheta_1\vartheta_2)\cos(\alpha\frac{\pi}{2}) + 2\delta_0\cos(3\alpha\frac{\pi}{2}),\\
H_2 &= \delta_1^2 - \vartheta_1^2 + (2\delta_0\delta_2 - 2\vartheta_0\vartheta_2)\cos(2\alpha\frac{\pi}{2}),\\
H_1 &= (2\delta_0\delta_1 - 2\vartheta_0\vartheta_1)\cos(\alpha\frac{\pi}{2}),\\
H_0 &= \delta_0^2 - \vartheta_0^2.
\end{aligned}$$

According to the Routh-Hurwitz theorem, we can get the routh list

$$\begin{array}{cccc}
1 & H_4 & H_2 & H_0\\
H_5 & H_3 & H_1 & 0\\
b_5 & b_3 & b_1 & 0\\
d_5 & d_3 & 0 & 0\\
u_5 & u_3 & 0 & 0\\
v_5 & 0 & 0 & 0\\
h_5 & & &
\end{array} \tag{35}$$

where $b_5 = -\frac{H_3 - H_4H_5}{H_5}, b_3 = -\frac{H_1 - H_2H_5}{H_5}, b_1 = H_0, d_5 = -\frac{H_5b_3 - H_3b_5}{b_5}, d_3 = -\frac{H_5b_1 - H_1b_5}{b_5}, u_5 = -\frac{b_5d_3 - b_3d_5}{d_5}, u_3 = b_1, v_5 = -\frac{d_5u_3 - d_3u_5}{u_5}, h_5 = u_3$.

When (35) satisfies some conditions(Li *et al.* 2021), there will be a change of sign, then (34) at least has one positive root. Thus, there exists a pair of purely imaginary roots of (29), which satisfy one of the conditions of Hopf bifurcation.

From (32) and (33), we can derive

$$\begin{aligned}
\cos\xi\tau &= \frac{\Psi\Phi_1 + \Omega\Phi_2}{\Omega^2 + \Psi^2},\\
\sin\xi\tau &= \frac{\Omega\Phi_1 - \Psi\Phi_2}{\Omega^2 + \Psi^2}.
\end{aligned} \tag{36}$$

According to (36), we can get

$$\tau^{(k)} = \frac{1}{\xi}(\arctan\frac{\Omega\Phi_1 - \Psi\Phi_2}{\Psi\Phi_1 + \Omega\Phi_2} + k\pi), k = 0, 1, 2, \ldots, \tag{37}$$

then we define the bifurcation point

$$\tau_0 = \min\tau^{(k)}, k = 0, 1, 2, \ldots \tag{38}$$

We introduce the following assumption to obtain the conditions of Hopf bifurcation.

(H3) $\frac{A_1N_1 + A_2N_2}{N_1^2 + N_2^2} \neq 0$,

where A_1, A_2 are defined by (43), and N_1, N_2 are defined by (48).

Lemma 13. Let $s(\tau) = \gamma(\tau) + \mathrm{i}\omega(\tau)$ be the root of (29) near $\tau = \tau_j$ meeting $\gamma(\tau_j) = 0$ and $\omega(\tau_j) = \omega_0$, then the following transversality condition meets

$$Re[\frac{ds}{d\tau}]|_{\tau=\tau_0,\omega=\omega_0} \neq 0. \tag{39}$$

Proof. Let $P_1(s) = s^{3\alpha} + \delta_2 s^{2\alpha} + \delta_1 s^{\alpha} + \delta_0, P_2(s) = \vartheta_2 s^{2\alpha} + \vartheta_1 s^{\alpha} + \vartheta_0$, then (29) can be rewritten as

$$P_1(s) + P_2(s)e^{-s\tau} = 0. \tag{40}$$

Derivation on both sides of (40) respect to τ,

$$P_1'(s)\frac{ds}{d\tau} + P_2'(s)e^{-s\tau}\frac{ds}{d\tau} + P_2(s)e^{-s\tau}(-\tau\frac{ds}{d\tau} - s) = 0, \tag{41}$$

where $P_i'(s)$ are the derivatives of $P_i(s)(i = 1, 2)$.

Then,

$$\frac{ds}{d\tau} = \frac{M(s)}{N(s)}, \tag{42}$$

where

$$\begin{aligned}
M(s) &= s(\vartheta_2 s^{2\alpha} + \vartheta_1 s^{\alpha} + \vartheta_0)e^{-s\tau},\\
N(s) &= 3\alpha s^{3\alpha-1} + 2\alpha\delta_2 s^{2\alpha-1} + \alpha\delta_1 s^{\alpha-1}\\
&\quad - \tau e^{-s\tau}(\vartheta_2 s^{2\alpha} + \vartheta_1 s^{\alpha} + \vartheta_0)\\
&\quad + e^{-s\tau}(2\alpha\vartheta_2 s^{2\alpha-1} + \alpha\vartheta_1 s^{\alpha-1}).
\end{aligned}$$

By straightforward computation,

$$[\frac{ds}{d\tau}]|_{\tau=\tau_0,\omega=\omega_0} = \frac{A_1 + \mathrm{i}A_2}{(B_1 + C_1 + D_1) + \mathrm{i}(B_2 + C_2 + D_2)},$$

where

$$\begin{aligned}
A_1 &= (-\vartheta_2\omega_0^{2\alpha+1}\sin(\frac{\pi}{2}2\alpha) - \vartheta_1\omega_0^{\alpha+1}sin(\frac{\pi}{2}\alpha))\times\\
&\quad \cos(\omega_0\tau_0) + (\vartheta_2\omega_0^{2\alpha+1}\cos(\frac{\pi}{2}2\alpha) +\\
&\quad \vartheta_1\omega_0^{\alpha+1}\cos(\frac{\pi}{2}\alpha) + \omega_0\vartheta_0)\sin(\omega_0\tau_0),\\
A_2 &= (\vartheta_2\omega_0^{2\alpha+1}\sin(\frac{\pi}{2}2\alpha) + \vartheta_1\omega_0^{\alpha+1}\sin(\frac{\pi}{2}\alpha))\times\\
&\quad \sin(\omega_0\tau_0) + (\vartheta_2\omega_0^{2\alpha+1}\cos(\frac{\pi}{2}2\alpha) +\\
&\quad \vartheta_1\omega_0^{\alpha+1}\cos(\frac{\pi}{2}\alpha) + \omega_0\vartheta_0)\cos(\omega_0\tau_0),
\end{aligned} \tag{43}$$

$$
\begin{aligned}
B_1 &= 3\alpha\omega_0^{3\alpha-1}\cos(\frac{(3\alpha-1)\pi}{2}) + 2\alpha\delta_2\omega_0^{2\alpha-1}\times \\
&\quad \cos(\frac{(2\alpha-1)\pi}{2}) + \alpha\delta_1\omega_0^{\alpha-1}\cos(\frac{(\alpha-1)\pi}{2}), \\
B_2 &= 3\alpha\omega_0^{3\alpha-1}sin(\frac{(3\alpha-1)\pi}{2}) + 2\alpha\delta_2\omega_0^{2\alpha-1}\times \\
&\quad \sin(\frac{(2\alpha-1)\pi}{2}) + \alpha\delta_1\omega_0^{\alpha-1}\sin(\frac{(\alpha-1)\pi}{2}),
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
C_1 &= -\tau sin\omega_0\tau_0(\vartheta_2\omega_0^{2\alpha}\sin(\frac{\pi}{2}2\alpha) + \vartheta_1\omega_0^{\alpha}\sin(\frac{\pi}{2}\alpha)) \\
&\quad - \tau\cos\omega_0\tau_0(\vartheta_2\omega_0^{2\alpha}\cos(\frac{\pi}{2}2\alpha) + \vartheta_1\omega_0^{\alpha}\cos(\frac{\pi}{2}\alpha) \\
&\quad + \vartheta_0), \\
C_2 &= -\tau\cos\omega_0\tau_0(\vartheta_2\omega_0^{2\alpha}\sin(\frac{\pi}{2}2\alpha) + \vartheta_1\omega_0^{\alpha}\sin(\frac{\pi}{2}\alpha)) \\
&\quad + \tau sin\omega_0\tau_0(\vartheta_2\omega_0^{2\alpha}\cos(\frac{\pi}{2}2\alpha) + \vartheta_1\omega_0^{\alpha}\cos(\frac{\pi}{2}\alpha) \\
&\quad + \vartheta_0),
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
D_1 &= \cos(\omega_0\tau_0)(2\alpha\vartheta_2\omega_0^{2\alpha-1}\times \\
&\quad \cos\frac{(2\alpha-1)\pi}{2} + \alpha\vartheta_1\omega_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}) + \\
&\quad \sin(\omega_0\tau_0)(2\alpha\vartheta_2\omega_0^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2} + \\
&\quad \alpha\vartheta_1\omega_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}), \\
D_2 &= -\sin(\omega_0\tau_0)(2\alpha\vartheta_2\omega_0^{2\alpha-1}\cos\frac{(2\alpha-1)\pi}{2} + \\
&\quad \alpha\vartheta_1\omega_0^{\alpha-1}\cos\frac{(\alpha-1)\pi}{2}) + \cos(\omega_0\tau_0)\times \\
&\quad (2\alpha\vartheta_2\omega_0^{2\alpha-1}\sin\frac{(2\alpha-1)\pi}{2} + \\
&\quad \alpha\vartheta_1\omega_0^{\alpha-1}\sin\frac{(\alpha-1)\pi}{2}).
\end{aligned} \tag{46}
$$

Hence,

$$
Re[\frac{ds}{d\tau}]|_{\tau=\tau_0,\omega=\omega_0} = \frac{A_1N_1 + A_2N_2}{N_1^2 + N_2^2}, \tag{47}
$$

where

$$
N_1 = B_1 + C_1 + D_1, N_2 = B_2 + C_2 + D_2. \tag{48}
$$

The proof is completed. □

Hence, we obtain the following theorem.

Theorem 14. Suppose that (H3) holds, we can gain the following results:
(i) E' is locally asymptotically stable for $\tau \in [0, \tau_0)$.
(ii) System (3) undergoes a Hopf bifurcation at E' when $\tau = \tau_0$.

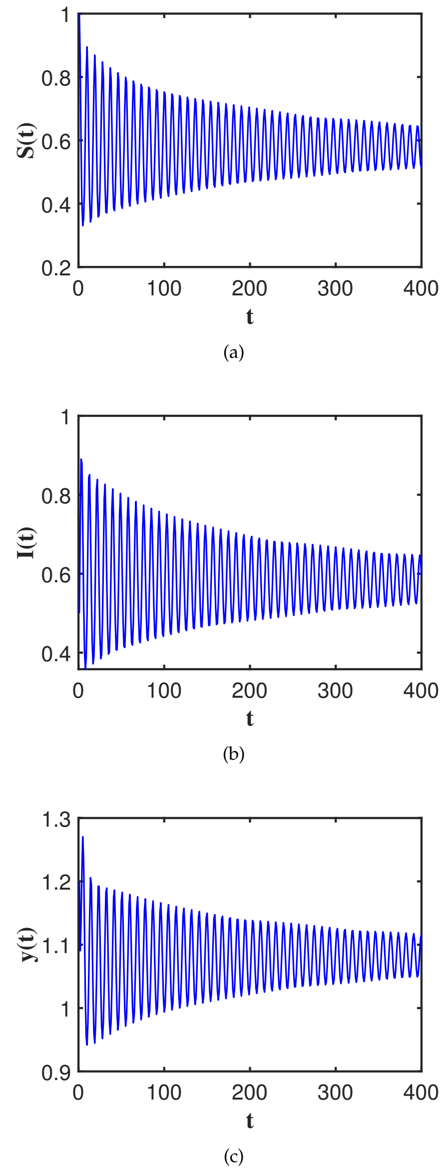


Figure 3 Waveform plots of system (50) with $\tau = 0.1 < \tau_0$.

NUMERICAL SIMULATIONS

Diethem et al proposed the Adams-Bashforth-Moulton prediction-correction numerical algorithm of fractional differential equations defined by Caputo(Kai *et al.* 2002), and Bhalekar et al extended it to fractional differential equations with delay(Bhalekar and Daftardar-Gejji 2011). Here, the modified Adams-Bashforth-Moulton prediction-correction numerical algorithm is used to verify our theoretical analysis(Bhalekar and Daftardar-Gejji 2011).

Example 1

According to the numerical simulations of (Zhou *et al.* 2010) and (Adak *et al.* 2020), we make two examples and set the following values for the parameters. When the order is close to 1, the dynamic properties of fractional-order system will be close to the dynamic properties of integer-order system. Hence, we choose the order $\alpha = 0.96$ and the other parameters are taken from (Zhou *et al.* 2010),$r = 2, a_2 = 1, c = 0.3, c_1 = 1, c_2 = 1, K = 3, K_1 = 0.6, K_2 =$

0.5. Then, we choose $\beta = 0.37 < \beta_2$, which satisfy the Theorem 10, then system (3) is

$$
\begin{aligned}
D^{0.96}S(t) &= 2S(t)(1 - \frac{S(t)+I(t)}{3}) - 0.37S(t)I(t),\\
D^{0.96}I(t) &= 0.37S(t-\tau)I(t-\tau) - 0.3I(t) - \frac{I(t)y(t)}{I(t)+0.6}, \qquad (49)\\
D^{0.96}y(t) &= y(t)(1 - \frac{y(t)}{I(t)+0.5}).
\end{aligned}
$$

It is not difficult to get equilibrium point $E_4(S_4, I_4, y_4) = (3, 0, 0.5)$. Fig. 1 exhibits that E_4 is locally asymptotically stable.

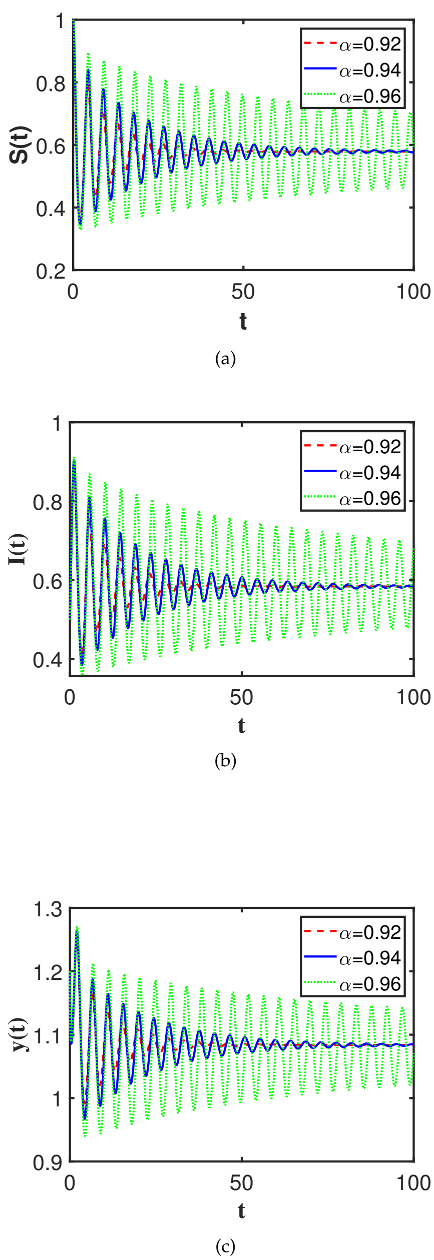


Figure 4 Waveform plots of system (50) with $\tau = 0.1$ for $\alpha = 0.92, \alpha = 0.94, \alpha = 0.96$.

Example 2

Choose $\beta = 2.1 > \beta_2$, thus E' exists. The system (3) is

$$
\begin{aligned}
D^{0.96}S(t) &= 2S(t)(1 - \frac{S(t)+I(t)}{3}) - 2.1S(t)I(t),\\
D^{0.96}I(t) &= 2.1S(t-\tau)I(t-\tau) - 0.3I(t) - \frac{I(t)y(t)}{I(t)+0.6}, \qquad (50)\\
D^{0.96}y(t) &= y(t)(1 - \frac{y(t)}{I(t)+0.5}).
\end{aligned}
$$

We acquire $E'(S', I', y') = (0.5788, 0.5834, 1.0834)$. It is not difficult to check system (50) satisfys $\delta_2 + \vartheta_2 = 0.9345 > 0$ and $(\delta_2 + \vartheta_2)(\delta_1 + \vartheta_1) - (\delta_0 + \vartheta_0) = 0.0923 > 0$. Thus, system (50) at E' is locally asymptotically stable for $\tau = 0$. We calculate that $\omega_0 = 1.3490, \tau_0 = 0.1689$. Fig. 2 and Fig. 3 show that E' is locally asymptotically stable when $\tau = 0 < \tau_0$ and $\tau = 0.1 < \tau_0$. For Fig. 3, we draw waveform plots every 20 points as a point. Motivated by the investigation on the different orders in (Sene 2019) and (Sene 2022), we show that the waveform plots of system (50) with $\tau = 0.1$ for different orders α in Fig. 4. The numerical simulation results implies that the lower values of α, the oscillating behavior is suppressed. E' is unstable of system (50) when $\tau = 0.2 > \tau_0$, which is shown in Fig. 5. Here, we give the waveform plot of $S(t)$. The waveform plots of $I(t)$ and $y(t)$ are omitted. Furthermore, we give the phase portraits in I-y plane for $\tau = 1, \tau = 3$ and $\tau = 6$. Fig. 6 exhibits the development of chaos.

Remark. In system (3), the order is $0 < \alpha \leq 1$. When $\alpha = 1$, this system is reduced to system (2). Therefore, our research extends the results of system (2).

Remark. The difference between the integer-order system (2) and the fractional-order system (3) are as follows. E_0, E_1, E_2, E_3 of system (3) are unstable for all $\tau \geq 0$, and if $\beta \leq \beta_2$, equilibrium point E_4 is locally asymptotically stable for $\tau \geq 0$. In integer-order system (2), it also has the same results. However, the conditions of the global asymptotically stability for equilibrium point E_4 is different from system (2). And the conditions of the occurrence of Hopf bifurcation of equilibrium point E' are related to the order α, which is different from integer-order system (2). Besides, the numerical results indicate that the oscillation behavior is suppressed when the order α is lower. And the chaos gradually arise when the delay τ increases. These results are not shown in the integer-order system (2).

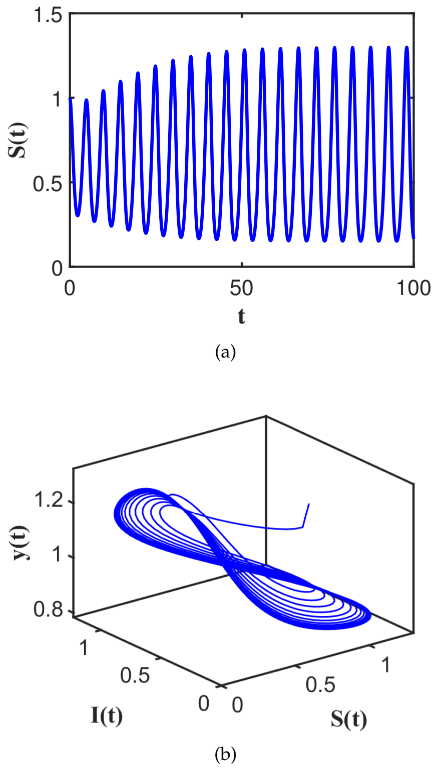
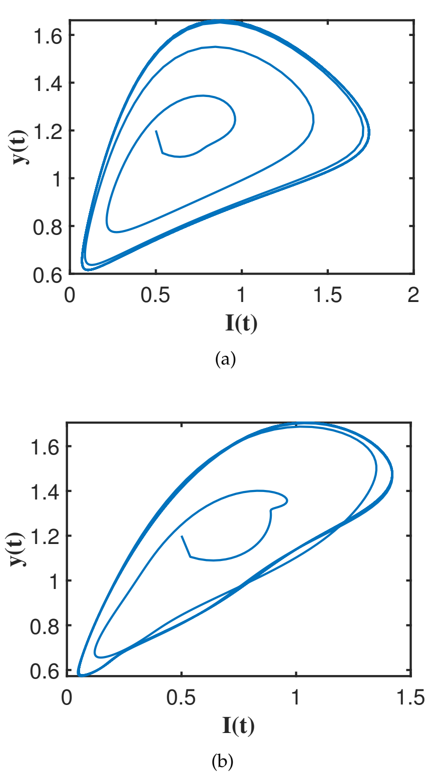


Figure 5 Waveform plots of system (50) with $\tau = 0.2 > \tau_0$.



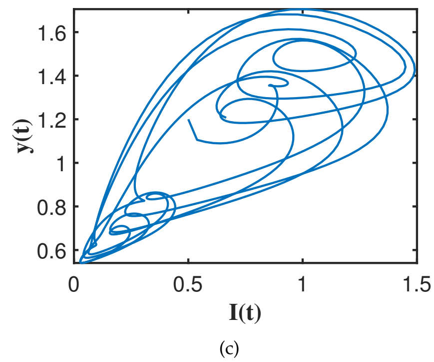


Figure 6 Phase portraits of system (50) in I-y plane for $\tau = 1, \tau = 3, \tau = 6$ respectively.

CONCLUSION

A fractional-order Leslie-Gower prey-predator-parasite system with delay is considered in this article. We investigate the existence and uniqueness of the solutions, as well as non-negativity and boundedness. We also show E_0, E_1, E_2, E_3 are unstable for $\tau \geq 0$ and if $\beta < \beta_2$, E_4 is locally asymptotically stable for $\tau \geq 0$. If the conditions of Theorem 10 are meeted, the system (3) at E_4 is globally asymptotically stable. If the conditions of Theorem 12 are satisfied, E' is locally asymptotically stable for $\tau = 0$ by Routh-Hurwitz theorem. In addition, E' occurs Hopf bifurcation when the conditions of Theorem 14 are meeted. We can change the critical value τ_0 to control the stability of system. Moreover, the system exhibits different results for different order α. The numerical results indicate that the oscillation behavior is suppressed for $\tau = 0.1$ when the order α is lower. The chaos gradually arise when the delay τ increases. Finally, we hope to explore chaos of this system.

Acknowledgments

The research is supported by National Natural Science Foundation of China (No.12172340). Liguo Yuan is a visiting scholar of Anhui University from September 2021 to August 2022.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Availability of data and material

Not applicable.

LITERATURE CITED

Adak, D., N. Bairagi, and H. Robert, 2020 Chaos in delay-induced leslie–gower prey–predator–parasite model and its control through prey harvesting. Nonlinear Analysis: Real World Applications **51**: 102998.

Alidousti, J. and M. Mostafavi Ghahfarokhi, 2019 Stability and bifurcation for time delay fractional predator prey system by incorporating the dispersal of prey. Applied Mathematical Modelling **72**: 385–402.

Anderson, R. M. and R. M. May, 1980 Infectious diseases and population cycles of forest insects. Science **210**: 658–661.

Bhalekar, S. and V. Daftardar-Gejji, 2011 A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. Fractional Calculus and Applications **5**: 1–9.

Boukhouima, A., K. Hattaf, and N. Yousfi, 2017 Dynamics of a fractional order hiv infection model with specific functional response and cure rate. International Journal of Differential Equations **2017**: 8372140.

Chinnathambi, R. and F. A. Rihan, 2018 Stability of fractional-order prey–predator system with time-delay and monod–haldane functional response. Nonlinear Dynamics **92**: 1637–1648.

Cruz, V.-D.-L., 2015 Volterra-type lyapunov functions for fractional-order epidemic systems. Communications in Nonlinear Science and Numerical Simulation **24**: 75–85.

Deng, W., C. Li, and J. Lü, 2007 Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dynamics **48**: 409–416.

Fernández-Carreón, B., J. Munoz-Pacheco, E. Zambrano-Serrano, and O. Félix-Beltrán, 2022 Analysis of a fractional-order glucose-insulin biological system with time delay. Chaos Theory and Applications **4**: 10–18.

Hu, T. C., D. L. Qian, and C. P. Li, 2009 Comparison theorems for fractional differential equations. Communication on Applied Mathematics and Computation **23**: 97–103.

Huang, C., H. Li, T. Li, and S. Chen, 2019 Stability and bifurcation control in a fractional predator–prey model via extended delay feedback. International Journal of Bifurcation and Chaos **29**: 1950150.

Huang, C., H. Liu, X. Chen, M. Zhang, L. Ding, *et al.*, 2020 Dynamic optimal control of enhancing feedback treatment for a delayed fractional order predator–prey model. Physica A: Statistical Mechanics and its Applications **554**: 124136.

Huo, J., H. Zhao, and L. Zhu, 2015 The effect of vaccines on backward bifurcation in a fractional order hiv model. Nonlinear Analysis: Real World Applications **26**: 289–305.

Kai, D., N. J. Ford, and A. D. Freed, 2002 A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics **29**: 3–22.

Kashkynbayev, A. and F. A. Rihan, 2021 Dynamics of fractional-order epidemic models with general nonlinear incidence rate and time-delay. Mathematics **9**.

Kilbas, A. A., H. M. Srivastava, and J. J. Trujillo, 2006 *Theory and Applications of Fractional Differential Equations*. Elsevier.

Li, C. and G. Chen, 2004 Chaos in the fractional order chen system and its control. Chaos, Solitons & Fractals **22**: 549–554.

Li, H., L. Zhang, C. Hu, Y. Jiang, and Z. Teng, 2017a Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing **54**: 435–449.

Li, H., L. Zhang, C. Hu, Y. Jiang, and Z. Teng, 2017b Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing **54**: 435–449.

Li, S., C. Huang, and X. Song, 2019 Bifurcation based-delay feedback control strategy for a fractional-order two-prey one-predator system. Complexity **2019**: 1–13.

Li, X. L., F. Gao, and W. Q. Li, 2021 Stability analysis of fractional-oeder hepatitis b virus infection model with immune delay(in Chinese). Acta Mathematica Scientia **41**: 562–576.

Li, Y., Y. Chen, and I. Podlubny, 2010 Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized mittag–leffler stability. Computers & Mathematics with Applications **59**: 1810–1821, Fractional Differentiation and Its Applications.

Mahmoud, G. M., A. A. Arafa, T. M. Abed-Elhameed, and E. E. Mahmoud, 2017 Chaos control of integer and fractional orders of chaotic burke–shaw system using time delayed feedback control. Chaos, Solitons & Fractals **104**: 680–692.

Mbava, W., J. Mugisha, and J. Gonsalves, 2017 Prey, predator and super-predator model with disease in the super-predator. Applied Mathematics and Computation **297**: 92–114.

Moustafa, M., M. H. Mohd, A. I. Ismail, and F. A. Abdullah, 2020 Dynamical analysis of a fractional-order eco-epidemiological model with disease in prey population. Advances in Difference Equations **2020**: 1–24.

Odibat, Z. M. and N. T. Shawagfeh, 2007 Generalized taylor's formula. Applied Mathematics and Computation **186**: 286–293.

Pu, W., 2020 Stability analysis of a class of fractional sis models with time delay (in chinese). Journal of Tonghua Normal University **41**: 18–22.

Rajagopal, K., N. Hasanzadeh, and F. Parastesh, 2020 A fractional-order model for the novel coronavirus (covid-19) outbreak. Nonlinear Dynamics **101**: 711–718.

Rihan, F. and C. Rajivganthi, 2020 Dynamics of fractional-order delay differential model of prey-predator system with holling-type iii and infection among predators. Chaos, Solitons & Fractals **141**: 110365.

Sene, N., 2019 Stability analysis of the generalized fractional differential equations with and without exogenous inputs. Journal of Nonlinear Sciences and Applications **12**: 562–572.

Sene, N., 2021 Qualitative analysis of class of fractional-order chaotic system via bifurcation and lyapunov exponents notions. Journal of Mathematics **2021**.

Sene, N., 2022 Fractional model and exact solutions of convection flow of an incompressible viscous fluid under the newtonian heating and mass diffusion. Journal of Mathematics **2022**: 1–20.

Shaikh, A. A., H. Das, and N. Ali, 2018 Study of lg-holling type iii predator-prey model with disease in predator. Journal of Applied Mathematics and Computing **58**: 235–255.

Shi, J., K. He, and H. Fang, 2022 Chaos, hopf bifurcation and control of a fractional-order delay financial system. Mathematics and Computers in Simulation **194**: 348–364.

Tao, B., M. Xiao, Q. Sun, and J. Cao, 2018 Hopf bifurcation analysis of a delayed fractional-order genetic regulatory network model. Neurocomputing **275**: 677–686.

Wang, Z., M. Du, and M. Shi, 2011 Stability test of fractional delay systems via integration. Science China: Physics, Mechanics and Astronomy **54**: 1839–1846.

Xu, R. and S. Zhang, 2013 Modelling and analysis of a delayed predator–prey model with disease in the predator. Applied Mathematics and Computation **224**: 372–386.

Yousef, F. B., A. Yousef, and C. Maji, 2021 Effects of fear in a fractional-order predator-prey system with predator density-dependent prey mortality. Chaos, Solitons & Fractals **145**: 110711.

Yuan, L., Q. Yang, and C. Zeng, 2013 Chaos detection and parameter identification in fractional-order chaotic systems with delay. Nonlinear Dynamics **73**: 439–448.

Zhou, X., J. Cui, X. Shi, and X. Song, 2010 A modified leslie–gower predator–prey model with prey infection. Journal of Applied Mathematics and Computing **33**: 471–487.

How to cite this article: Yang, X., Yuan, L., and Wei, Z. Stability and Hopf Bifurcation Analysis of a Fractional-order Leslie-Gower Prey-predator-parasite System with Delay. Chaos Theory and Applications, 4(2), 71-81, 2022.