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Global Existence and Uniqueness of The Inviscid Velocity-Vorticity Model of the g-Navier-Stokes Equations

Özge KAZAR*[1], Meryem KAYA[2]

Abstract

In this paper, we prove the global existence and uniqueness of the weak solutions to the inviscid velocity-vorticity model of the g-Navier-Stokes equations. The system is performed by entegrating the velocity-pressure system which is involved by using the rotational formulation of the nonlinearity and the vorticity equation for the g-Navier-Stokes equations without viscosity term. In this study we particularly interest the inviscid velocity-vorticity system of the g-Navier-Stokes equations over the two dimensional periodic box $\Omega = (0,1)^2 \subset R^2$.

Keywords: Existence and uniqueness, g-Navier-Stokes equations, inviscid velocity-vorticity model

1. INTRODUCTION

Velocity-vorticity formulation have been considered extensively by many scientists for example [1-4]. In [2] Gardner et al. studied continuous data assimilation to a velocity-vorticity formulation of the 2D Navier-Stokes equations. In [1, 3, 4] researchers studied in velocity-vorticity formulation of the Navier-Stokes equations by numerically. In recent years, the velocity-vorticity formulation and Voigt regularization combined for some fluid dynamical models. In [5] Larios et al. suggested the velocity-vorticity model for the Navier-Stokes-Voigt equations and they studied the global well-posedness of this system. Pei [6] studied velocity-vorticity-Voigt model for 3D Boussinesq equations and he considered global well-posedness and also higher order regularity. Inviscid form of the models in computational fluid dynamics have been attracted and extensively studied by many researchers [7-9]. Using classical Picard iteration method Cao, Lunasin and Titi prove global existence and uniqueness of inviscid Bardina model [7]. Larios and Titi have studied the inviscid Navier-Stokes-Voigt equations. They proved the global existence and uniqueness of weak solutions and higher order regularity of the solutions of this system [8]. In this study we are particularly interest the following velocity-vorticity system of the g-Navier-Stokes (gNS) equations over the two dimensional periodic box $\Omega = (0,1)^2 \subset R^2$:

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$\frac{\partial u}{\partial t} - \nu\Delta_g u + \nu\frac{1}{g}(\nabla g.\nabla)u + w \times u + \nabla P = f$, (1)

$\frac{\partial w}{\partial t} - \nu\Delta_g w + \nu\frac{1}{g}(\nabla g.\nabla)w + (u.\nabla)w = \nabla \times f + w\left(\frac{\nabla g}{g}.u\right)$. (2)

In this system $P = p + \frac{1}{2}|u^2|$, u represent velocity, w which play the role of vorticity, f is an external forcing term. We consider this problem under the periodic boundary conditions. We assume u, p and w and the first derivative of u, w to be spatially periodic. The existence and uniqueness of the weak and strong solutions of this system with the viscosity term is proved in [10]. Now we consider the following inviscid form. The inviscid velocity-vorticity model of the g-Navier-Stokes is equivalent to the functional differential equations

$\frac{du}{dt} + P_g(w \times u) = P_g f$, (3)

$\frac{dw}{dt} + B_g(u, w) = P_g(\nabla \times f) + P_g\left(w\left(\frac{\nabla g}{g}.u\right)\right)$, (4)

$\nabla.(gu) = 0$, $\nabla.(gw) = 0$, (5)

$u(x, 0) = u_0$, $w(x, 0) = w_0$, (6)

where, for simplicity, we assume f to be time independent. We rewrite $B_g = P_g\big((u.\nabla)w\big)$ and $P_g: L^2(\Omega, g) \longrightarrow H_g$ is Helmholtz-Leray orthogonal projection. The function $g = g(x_1, x_2)$ is positive real-valued smooth function. We assume that g satisfies the following conditions,

i. $g(x_1, x_2) \in C^\infty(\Omega)$.

ii. $0 < m_o \le g(x_1, x_2) \le M_o$ where m_o and M_o are positive constants for all $(x_1, x_2) \in \Omega$.

iii. $\|\nabla g\|_\infty = \sup_{(x_1,x_2)\in\Omega} |\nabla g(x_1, x_2)| < \infty$.

Throughout in this study c will denote a generic positive constant. It can be different from line to line. This study is organized as follows. In section 2 we give some notations and present the mathematical spaces. We also give some preliminary results [11, 12]. In section 3, we investigate global existence and uniqueness of the inviscid velocity-vorticity model of the gNS equations using the classical Picard iteration method.

2. PRELIMINARIES AND FUNCTIONAL SETTING

In this section we introduce the usual notation used in the context [11, 12]. $L^2(\Omega, g)$ denotes the Hilbert space with the inner product and norm

$(u, v)_g = \int_\Omega (u, v) g dx$ and

$\|u\|^2_{L^2(\Omega,g)} = (u, u)_g$,

respectively. The inner product and norm in H_g are the same of $L^2(\Omega, g)$. The norm in $H^1(\Omega, g)$

$\|u\|^2_{H^1(\Omega,g)} = \left[(u, u)_g + \sum_{i=1}^2 (D_i u, D_i u)_g\right]^{\frac{1}{2}}$,

where $D_i = \frac{\partial}{\partial x_i}$. The norm in V_g are the same of $H^1(\Omega, g)$. The two spaces $L^2(\Omega)$ and $L^2(\Omega, g)$ have equivalent norms in the following inequalities

$m_0\|u\|^2_{L^2(\Omega)} \le \|u\|^2_{L^2(\Omega,g)} \le M_0\|u\|^2_{L^2(\Omega)}$,

where m_0 and M_0 positive constants. We define spaces in the periodic setting for the gNS equations are

$\mathcal{V}_1 = \left\{u \in \left(C^{\infty}_{per}(\Omega)\right)^2 : \nabla(gu) = 0,\ \int_\Omega udx = 0\right\}$,

$\boldsymbol{H_g}$ = the closure of $\mathcal{V}_1$ in $L^2(\Omega, g)$,

$\boldsymbol{V_g}$ = the closure of $\mathcal{V}_1$ in $H^1(\Omega, g)$,

in two dimensions. Vorticity is considered as a scalar, we define vorticity space as

$\mathcal{V}_2 = \left\{u \in C^{\infty}_{per}(\Omega): \nabla(gu) = 0, \int_\Omega udx = 0\right\}$,

H_g = the closure of $\mathcal{V}_2$ in $L^2(\Omega, g)$,

V_g = the closure of $\mathcal{V}_2$ in $H^1(\Omega, g)$,

$H_{gcurl} = \left\{f \in H_g:\ \nabla \times f \in L^2(\Omega, g)\ \right\}$.

Now we rewrite g-Laplacian operator and g-Stokes operator and some notations in the following

$$-\Delta_g u \coloneqq -\frac{1}{g}(\nabla . g\nabla u) = -\Delta u - \frac{1}{g}(\nabla \mathrm{g}. \nabla)u,$$

$$A_g u = P_g\left[-\frac{1}{g}(\nabla . g\nabla \mathrm{u})\right],$$

respectively. A_g have countable eigenvalues which are satisfying as in the below;

$$0 < \lambda_g \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

where $\lambda_g = \frac{4\pi^2 m_0}{M_0}$. The Poincare inequality

$$\sqrt{\lambda_g}\|\phi\|_{L^2} \leq \|\nabla\phi\|_{L^2},$$

satisfy for all $\phi \in V_g$. Since the operators A_g and P_g are self adjoint, using integration by parts we have

$$\langle A_g u, u\rangle_g = \int_\Omega (\nabla u, \nabla u) g dx = \langle \nabla u, \nabla u\rangle_g = \|\nabla u\|_g.$$

The bilinear operator $B_g: V_g \times V_g \to V_g'$

$$B_g(u, v) = P_g(u.\nabla)v$$

and for this term the inner product of $w \in V_g$, we get

$$\langle B_g(u, v), w\rangle_{V_g'} = b_g(u, v, w).$$

The trilinear form b_g defined as

$$b_g(u, v, w) = \sum_{i,j=1}^{n} \int_\Omega u_i\,(D_i v_j) w_j g dx = (P_g(u.\nabla)v, w)_g.$$

We have the following properties

i. $b_g(u, v, w) = -b_g(u, w, v)$,
ii. $b_g(u, v, v) = 0$.

The function Cu defined by

$$Cu = P_g\left[\frac{1}{g}(\nabla g.\nabla)u\right]$$

and the inner product of $v \in V_g$ we write

$$\langle Cu, v\rangle_g = \langle\frac{1}{g}(\nabla g.\nabla)u, v\rangle_g = b_g\left(\frac{\nabla g}{g}, u, v\right).$$

It is easy to show this term belong to $L^2(0, T; H_g)$ and hence belong to $L^2\left(0, T; V_g'\right)$.

2.1. Lemma

The bilinear operator B_g satisfies the following inequality;

$$\left|\langle B_g(u,v),w\rangle_{V_g'}\right|$$

$$\leq c\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla v\|_{L^2}\|w\|_{L^2}^{1/2}\|\nabla w\|_{L^2}^{1/2},$$

$$for\ all\ \mathrm{u},\mathrm{v},\mathrm{w}\in V_g\ [11,12]. \tag{7}$$

3. GLOBAL EXISTENCE AND UNIQUENESS OF THE INVISCID VELOCITY-VORTICITY MODEL OF THE g-NAVIER-STOKES EQUATIONS

In this section, we will established the global existence and uniqueness of the inviscid velocity-vorticity model of the gNS equations using the classical Picard iteration method. Note that the $(3)-(4)$ equality is understood to hold in the sense of $\boldsymbol{V_g'}\times V_g'$.

3.1. Theorem

Let $u_0\in\boldsymbol{V_g}$, $w_o\in V_g$ and $f\in\boldsymbol{V}_g'$, $\nabla\times f\in V_g'$. There exist a short time $T^*\left(\|u_0\|_{\boldsymbol{V_g}},\|w_0\|_{V_g}\right)$ such that the equations $(3)-(6)$ has a unique solution $u,w\in C^1([-T^*,T^*],\boldsymbol{V_g}\times V_g)$.

Proof

We will use the classical Picard iteration principle to prove the short time existence and uniqueness theorem. Namely, it is enough to show that the vector field $N(u)=f-w\times u$ and $N(w)=\nabla\times f+\mathrm{w}\left(\frac{\nabla g}{g}.u\right)-B_g(u,w)$ is locally Lipschitz in the Hilbert Space $\boldsymbol{V_g}$ to $\boldsymbol{V}_g'$ and V_g to V_g' respectively. From the classical theory of ordinary differential equations we consider the equivalent equation for $(3)-(4)$ respectively.

$$u(t)=u_0-\int_0^t w(s)\times u(s)ds+ft, \tag{8}$$

$$w(t)=w_0-\int_0^t B_g\big(u(s),w(s)\big)ds+\int_0^t\left(\mathrm{w}(s)\left(\frac{\nabla g}{g}.u(s)\right)\right)ds+(\nabla\times f)t\,. \tag{9}$$

Let $u_1,u_2\in\boldsymbol{V_g}$ and $w_1,w_2\in V_g$, $\boldsymbol{\phi}\in\boldsymbol{V_g}$ and $\phi\in V_g$

$$\|N(u_1)-N(u_2)\|_{\boldsymbol{V}_g'}=\|w_1\times u_1-w_2\times u_2\|_{\boldsymbol{V}_g'}=\|w_1\times(u_1-u_2)+(w_1-w_2)\times u_2\|_{\boldsymbol{V}_g'}=\sup_{\substack{\phi\in\boldsymbol{V_g}\\ \|\boldsymbol{\phi}\|_{\boldsymbol{V_g}}=1}}\left|\langle w_1\times(u_1-u_2)+(w_1-w_2)\times u_2,\boldsymbol{\phi}\rangle_{\boldsymbol{V}_g'}\right|, \tag{10}$$

Applying Poincare inequality and (7) for (10) we write

$$\|N(u_1)-N(u_2)\|_{\boldsymbol{V}_g'}\leq c\|w_1\|_{L^2}^{1/2}\|\nabla w_1\|_{L^2}^{1/2}\|u_1-u_2\|_{L^2}^{1/2}\|\nabla(u_1-u_2)\|_{L^2}^{1/2}+c\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|w_1-\mathrm{w}_2\|_{L^2}^{1/2}\|\nabla(w_1-\mathrm{w}_2)\|_{L^2}^{1/2}\leq c\frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2}\|\nabla(u_1-u_2)\|_{L^2}+c\frac{1}{\lambda_g^{1/2}}\|\nabla u_2\|_{L^2}\|\nabla(w_1-w_2)\|_{L^2}. \tag{11}$$

Similar estimates can be obtained for $N(w)$ as in the following

$$\|N(w_1)-N(w_2)\|_{V_g'}=\left\|\mathrm{w}_1\left(\frac{\nabla g}{g}.u_1\right)-B_g(u_1,w_1)-\mathrm{w}_2\left(\frac{\nabla g}{g}.u_2\right)+B_g(u_2,w_2)\right\|_{V_g'}\leq\left\|\mathrm{w}_1\left(\frac{\nabla g}{g}.(u_1-u_2)\right)+(\mathrm{w}_1-\mathrm{w}_2)\left(\frac{\nabla g}{g}.u_2\right)\right\|_{V_g'}+\left\|B_g(u_1,w_1-\mathrm{w}_2)+B_g(u_1-u_2,w_2)\right\|_{V_g'}\leq\sup_{\substack{\phi\in V_g\\ \|\phi\|_{V_g}=1}}\left|\langle\mathrm{w}_1\left(\frac{\nabla g}{g}.(u_1-u_2)\right)+(\mathrm{w}_1-\mathrm{w}_2)\left(\frac{\nabla g}{g}.u_2\right),\phi\rangle_{V_g'}\right|+\sup_{\substack{\phi\in V_g\\ \|\phi\|_{V_g}=1}}\left|\langle B_g(u_1,w_1-\mathrm{w}_2)+B_g(u_1-u_2,w_2),\phi\rangle_{V_g'}\right|. \tag{12}$$

Again using Poincare inequality and (7) for (12) we obtained

$$\|N(w_1)-N(w_2)\|_{V_g'}\leq c\|\nabla g\|_\infty\|w_1\|_{L^2}^{1/2}\|\nabla w_1\|_{L^2}^{1/2}\|u_1-u_2\|_{L^2}^{1/2}\|\nabla(u_1-u_2)\|_{L^2}^{1/2}+c\|\nabla g\|_\infty\|w_1-\mathrm{w}_2\|_{L^2}^{1/2}\|\nabla(w_1-\mathrm{w}_2)\|_{L^2}^{1/2}\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}+c\|u_1\|_{L^2}^{1/2}\|\nabla u_1\|_{L^2}^{1/2}\|w_1-\mathrm{w}_2\|_{L^2}^{1/2}\|\nabla(w_1-$$

$$\mathrm{w}_2)\|_{L^2}^{1/2} + c\|u_1 - u_2\|_{L^2}^{1/2}\|\nabla(u_1 - u_2)\|_{L^2}^{1/2}\|w_2\|_{L^2}^{1/2}\|\nabla w_2\|_{L^2}^{1/2} \leq c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2}\|\nabla(u_1 - u_2)\|_{L^2} + c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}} \|\nabla(w_1 - \mathrm{w}_2)\|_{L^2}\|\nabla u_2\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla u_1\|_{L^2}\|\nabla(w_1 - w_2)\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla(u_1 - u_2)\|_{L^2}\|\nabla w_2\|_{L^2}. \tag{13}$$

Now adding the inequalities (11) and (13), we have

$$\|N(u_1) - N(u_2)\|_{\boldsymbol{V}'_g} + \|N(w_1) - N(w_2)\|_{V'_g} \leq c\frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2}\|\nabla(u_1 - u_2)\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla u_2\|_{L^2}\|\nabla(w_1 - w_2)\|_{L^2} + c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2}\|\nabla(u_1 - u_2)\|_{L^2} + c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}} \|\nabla(w_1 - \mathrm{w}_2)\|_{L^2}\|\nabla u_2\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla u_1\|_{L^2}\|\nabla(w_1 - w_2)\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla(u_1 - u_2)\|_{L^2}\|\nabla w_2\|_{L^2}.$$

Then, after rearranging the right hand side of the above inequality, it follows that

$$\|N(u_1) - N(u_2)\|_{\boldsymbol{V}'_g} + \|N(w_1) - N(w_2)\|_{V'_g} \leq \left(c\frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2} + c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}}\|\nabla w_1\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla w_2\|_{L^2}\right)\|\nabla(u_1 - u_2)\|_{L^2} + \left(c\frac{1}{\lambda_g^{1/2}}\|\nabla u_2\|_{L^2} + c\|\nabla g\|_\infty \frac{1}{\lambda_g^{1/2}} \|\nabla u_2\|_{L^2} + c\frac{1}{\lambda_g^{1/2}}\|\nabla u_1\|_{L^2}\right)\|\nabla(w_1 - w_2)\|_{L^2}.$$

We have

$$\|N(u_1) - N(u_2)\|_{\boldsymbol{V}'_g} + \|N(w_1) - N(w_2)\|_{V'_g} \leq \leq \frac{2c}{\lambda_g^{1/2}} \|u_1 - u_2\|_{\boldsymbol{V_g}}\left(\|w_1\|_{V_g} + \|w_2\|_{V_g}\right) + \frac{2c}{\lambda_g^{1/2}} \|w_1 - w_2\|_{V_g}\left(\|u_1\|_{\boldsymbol{V_g}} + \|u_2\|_{\boldsymbol{V_g}}\right). \tag{14}$$

For any large enough R such that $\|u_1\|_{\boldsymbol{V_g}}$, $\|u_2\|_{\boldsymbol{V_g}}$, $\|w_1\|_{V_g}$, $\|w_2\|_{V_g} \leq R$, we have

$$\|N(u_1) - N(u_2)\|_{\boldsymbol{V}'_g} + \|N(w_1) - N(w_2)\|_{V'_g} \leq \frac{4cR}{\lambda_g^{1/2}} \|u_1 - u_2\|_{\boldsymbol{V_g}} + \frac{4cR}{\lambda_g^{1/2}}\|w_1 - w_2\|_{V_g} \leq \frac{4cR}{\lambda_g^{1/2}}\left(\|u_1 - u_2\|_{\boldsymbol{V_g}} + \|w_1 - w_2\|_{V_g}\right) \tag{15}$$

From the inequality (15) we say that $N(u)$ and $N(w)$ is locally Lipschitz continuous function from the Hilbert Space $\boldsymbol{V_g}$ to $\boldsymbol{V'_g}$ and V_g to V'_g respectively. Therefore by the classical theory of ordinary differential equation $(8) - (9)$ has a unique fixed point in a small interval $[-T^*, T^*]$ and $u \in C\left([-T^*, T^*]\,; \boldsymbol{V_g}\right)$, $w \in C\left([-T^*, T^*]\,; V_g\right)$ (see, e.g., [13]). In particular, the forcing term f assume to be time independent and since the terms under the integral sign to the right of (8) and (9) are continuous functions with valued in $\boldsymbol{V'_g}$ and V'_g so left hand side $u(t)$ and $w(t)$ are differentiable and (3) and (4) satisfied with $u(0) = u_0$ and $w(0) = w_0$. From these results give us the local-in-time existence and uniqueness of solutions.

3.2. Theorem

Let $f \in \boldsymbol{V_g}$, $\nabla \times f \in V_g$ and $u_0 \in \boldsymbol{V_g}$, $w_o \in V_g$. Then the system in $(3) - (6)$ has a unique solution $u, w \in C^1((-\infty, \infty), \boldsymbol{V_g} \times V_g)$.

Proof

Let's show that global existence for the equations $(3) - (6)$. To do this we need to show that on the maximal interval of existence, $\|u(t)\|_{\boldsymbol{V_g}}$ ve $\|w(t)\|_{V_g}$ remain finite. Let $[0, T_{max}]$ be the maximal interval of existence. If $T_{max} = \infty$ in this case nothing need to prove. Let's admit

$$T_{max} < \infty. \tag{16}$$

This implies that

$$\limsup_{t \to T_{max}^-}\|u(t)\|_{\boldsymbol{V_g}} = \infty \; and \; \limsup_{t \to T_{max}^-}\|w(t)\|_{V_g} = \infty. \tag{17}$$

However we will provide a contradiction to the result in (17) We take inner product (3) with the $A_g u(t)$. We get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 \le 2\left|\left(P_g(w\times u), A_g u\right)_g\right| + 2\left|\left(P_g f, A_g u\right)_g\right|. \qquad (18)$$

Cauchy-Schwarz and Young inequalities are applied for each term on the right hand side for (18), we write

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 \le \frac{1}{2}\left\|A_g u\right\|_{L^2}^2 + \frac{27}{4}c\|\mathrm{w}\|_{L^2}^2\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + 4\|\mathrm{f}\|_{L^2}^2. \qquad (19)$$

We take inner product (4) with the $A_g w(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|_{L^2}^2 + b_g(u,w,A_g w) \le \left(P_g\left(w\left(\frac{\nabla g}{g}.u\right)\right), A_g w\right)_g + \left(\nabla\times P_g f, A_g w\right)_g.$$

And then we write

$$\frac{d}{dt}\|\nabla w\|_{L^2}^2 \le 2\left|b_g(u,w,A_g w)\right| + 2\left|\left(P_g\left(w\left(\frac{\nabla g}{g}.u\right)\right), A_g w\right)_g\right| + 2\left|\left(\nabla\times P_g f, A_g w\right)_g\right|. \qquad (20)$$

Cauchy-Schwarz and Young inequalities are applied for each term on the right hand side for (20), we have

$$\frac{d}{dt}\|\nabla w\|_{L^2}^2 \le \frac{3}{4}\left\|A_g w\right\|_{L^2}^2 + \frac{27}{4}c\|\mathrm{u}\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^2 + \frac{4c\|\nabla g\|_\infty^2}{m_0^2\lambda_g}\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + 4\|\nabla\times \mathrm{f}\|_{L^2}^2. \qquad (21)$$

Adding the inequalities (19) and (21), we obtain

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \le \frac{1}{2}\left\|A_g u\right\|_{L^2}^2 + \frac{3}{4}\left\|A_g w\right\|_{L^2}^2 + \frac{27}{4}c\|\mathrm{w}\|_{L^2}^2\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + \frac{27}{4}c\|\mathrm{u}\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla w\|_{L^2}^2 + \frac{4c\|\nabla g\|_\infty^2}{m_0^2\lambda_g}\|\nabla w\|_{L^2}^2\|\nabla u\|_{L^2}^2 + 4\|\mathrm{f}\|_{L^2}^2 + 4\|\nabla\times \mathrm{f}\|_{L^2}^2. \qquad (22)$$

In [10] we proved that $u\in L^\infty(0,T;\boldsymbol{H_g})$, $w\in L^\infty(0,T;H_g)$ and $u\in L^\infty(0,T;\boldsymbol{V_g})$, $w\in L^\infty(0,T;V_g)$ because of u and w weak and strong solution of the velocity-vorticity model of gNS equations. So we have

$$\sup_{s\in[0,T]}\|u(s)\|_{\boldsymbol{H_g}}^2 \le K_1,\ \sup_{s\in[0,T]}\|w(s)\|_{H_g}^2 \le K_2$$

and

$$\sup_{s\in[0,T]}\|\nabla u(s)\|_{\boldsymbol{H_g}}^2 \le K_8,\ \sup_{s\in[0,T]}\|\nabla w(s)\|_{H_g}^2 \le K_9,$$

where K_1, K_8 depend on u_0, f, ν, T and K_2, K_9 depend on w_0, f, ν, T. Using the above results in (22) we get

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \le \frac{1}{2}\left\|A_g u\right\|_{L^2}^2 + \frac{3}{4}\left\|A_g w\right\|_{L^2}^2 + \left(\frac{27}{4}cK_2K_9 + \frac{27}{4}cK_1K_9\right)\|\nabla u\|_{L^2}^2 + \frac{4c\|\nabla g\|_\infty^2}{m_0^2\lambda_g}K_8\|\nabla w\|_{L^2}^2 + 4\|\mathrm{f}\|_{L^2}^2 + 4\|\nabla\times \mathrm{f}\|_{L^2}^2. \qquad (23)$$

After some arrangement right hand side of the (23) we have the following inequality.

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) - \alpha\left(\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2\right) \le \frac{1}{2}\left\|A_g u\right\|_{L^2}^2 + \frac{3}{4}\left\|A_g w\right\|_{L^2}^2 + 4\|\mathrm{f}\|_{L^2}^2 + 4\|\nabla\times \mathrm{f}\|_{L^2}^2, \qquad (24)$$

where

$$\alpha = max\left\{\frac{27}{4}cK_2K_9 + \frac{27}{4}cK_1K_9, \frac{4c\|\nabla g\|_\infty^2}{m_0^2\lambda_g}K_8\right\}.$$

Using Gronwall inequality for (24), we get

$$\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \le e^{\alpha t}\left[\|\nabla u(0)\|_{L^2}^2 + \|\nabla w(0)\|_{L^2}^2\right] + e^{\alpha t}\left[\frac{1}{2}\int_0^t\left\|A_g u\right\|_{L^2}^2 ds + \frac{3}{4}\int_0^t\left\|A_g w\right\|_{L^2}^2 ds\right] + 4te^{\alpha t}\|\mathrm{f}\|_{L^2}^2 + 4te^{\alpha t}\|\nabla\times \mathrm{f}\|_{L^2}^2.$$

Since u and w are strong solutions of the velocity-vorticity model of gNS equations in [10], we have $u \in L^2(0,T;D(\boldsymbol{A_g}))$, $w \in L^2(0,T;D(A_g))$. Thus

$$\int_0^T \left\|A_g u\right\|_{L^2}^2 dt \leq K_{10}, \int_0^T \left\|A_g u\right\|_{L^2}^2 dt \leq K_{11}.$$

Using these results, we get

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 &\leq e^{\alpha t}\left[\|\nabla u(0)\|_{L^2}^2 + \|\nabla w(0)\|_{L^2}^2\right] \\ &+ e^{\alpha t}\left[\frac{1}{2}K_{10} + \frac{3}{4}K_{11} + 4t\|\mathrm{f}\|_{L^2}^2 \right. \\ &\left. + 4t\|\nabla \times \mathrm{f}\|_{L^2}^2\right].\end{aligned}$$

For all $t < T_{max}$. Hence we obtain

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 &\leq e^{\alpha T_{max}}\left[\|\nabla u(0)\|_{L^2}^2 \right. \\ &\left. + \|\nabla w(0)\|_{L^2}^2\right] \\ &+ e^{\alpha T_{max}}\left[\frac{1}{2}K_{10} + \frac{3}{4}K_{11} \right. \\ &+ 4T_{max}\|\mathrm{f}\|_{L^2}^2 \\ &\left. + 4T_{max}\|\nabla \times \mathrm{f}\|_{L^2}^2\right].\end{aligned}$$

This gives us

$$\limsup_{t \to T_{max}^-} \|u(t)\|_{V_g}^2 + \|w(t)\|_{V_g}^2 \leq K$$

This is a contradiction to conclusion (17) The proof is completed.

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This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

In the writing process of this study, international scientific, ethical and citation rules were followed, and no falsification was made on the collected data. Sakarya University Journal of Science and its editorial board have no responsibility for all ethical violations. All responsibility belongs to the responsible author and this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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