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The Class of Demi-Order Norm Continuous Operators

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Highlights

- This paper focuses on the class of the demi-order norm continuous operators.
- Some properties of the demi-order norm continuous operators are obtained.
- Examples of demi-order norm continuous operators are given.

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Abstract

In this paper, we introduce the class of demi-order norm continuous operator on a normed Riesz space. We study the relationship between order-to-norm continuous operator and demi-order norm continuous operator. We also investigate some properties of the class of demi-order norm continuous operator, and it is given a characterization of a normed Riesz space with order continuous norm by the term of the demi-order norm continuous operator.

1. INTRODUCTION

The demi notation was used firstly in the article named "Construction of fixed points of demicompact mappings in Hilbert space" by Petryshyn in 1966 [1]. Krichen B. and O'Regan D. studied some results of the class of weakly demicompact linear operators in 2019 [2]. After, in [3], Benkhaled H., Hajji M., and Jeribi A. introduced the class of demi Dunford-Pettis operators which are a generalization of Dunford Pettis operators. The class of order weakly demicompact operators was introduced by Benkhaled H., Elleuch A., and Jeribi A. in [4].

In this study, we will introduce the class of demi-order-norm continuous operators which are a generalization order-to-norm continuous operators on a Banach lattice, given by Jalili, Haghnejad Azar, and Moghimi in [5].

A net $\{x_\alpha\}$ in a Riesz space E is said to be order convergent to $x \in E$ if there is a net $\{y_\beta\}$ in E^+ with $y_\beta \downarrow 0$ and that for every β, there is $a_0 = a_0(\beta)$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq a_0$. It is denoted by $x_\alpha \xrightarrow{o} x$. Let E and F be two Riesz spaces, every linear mapping from E into F is called operator (linear operator). Briefly the net $\{x_\alpha : \alpha \in \Lambda\}$ is denoted by $\{x_\alpha\}$ where Λ is a nonempty directed set. Recall from [5] that let E be a Banach lattice, a bounded operator T on E is said to be an order-to-norm continuous operator if $x_\alpha \xrightarrow{o} 0$, then $T(x_\alpha) \xrightarrow{\|.\|} 0$ for all net x_α in E. The class of all order-to-norm continuous operators will be denoted $L_{on}(E)$. E has order continuous norm if and only if $x_\alpha \downarrow 0$, then $x_\alpha \xrightarrow{\|.\|} 0$ [6]. Let E, F be two Banach lattices and two operators S, T from E into F. $S \leq T$ means that $S(x) \leq T(x)$ for all $x \in E^+$ [6]. The class of all continuous operators on E is denoted by $L(E)$. A norm $\|.\|$ on a Riesz space is said to be a

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lattice norm whenever $|x| \leq |y|$, then $\|x\| \leq \|y\|$ [6]. A Riesz space equipped with a lattice norm is known as a normed Riesz space, and a subset A of a Riesz space is said to be order closed whenever $\{x_\alpha\} \subseteq A$ and $x_\alpha \xrightarrow{o} x$, imply $x \in A$ [6].

Throughout this study, the identity operator is denoted by I. In this study, for all other undefined terms and notation, we will adhere to the conventions in [6].

2. MAIN RESULTS

Definition 2.1. Let M be a normed Riesz space, a bounded operator $H:M \to M$. It is said to be a demi-order norm continuous operator (d-onco) if for every net $\{x_\alpha\}$ in M^+ whenever $x_\alpha \xrightarrow{o} 0$ and $x_\alpha - H(x_\alpha) \xrightarrow{\|.\|} 0$, implies $x_\alpha \xrightarrow{\|.\|} 0$, and the class of all demi-order norm continuous operators is denoted by $\mathfrak{D}L_{on}(M)$.

Example 2.1. Let M be a normed Riesz space. βI is a demi-order norm continuous operator on M for all $\beta \neq 1$.

Assume that $x_\alpha \xrightarrow{o} 0$ and $\|x_\alpha - \beta I(x_\alpha)\| \to 0$. Therefore, we obtain

$$\|x_\alpha - \beta I(x_\alpha)\| \to 0 \Rightarrow |1-\beta|\|x_\alpha\| \to 0 \Rightarrow \|x_\alpha\| \to 0.$$

Thus, βI is a demi-order norm continuous operator on M.

Generally, the following example shows that the above example is not true in case $\beta = 1$.

Example 2.2. Let c be the set of all convergent sequence of $\mathbb{R}$. Consider the sequence u_n; its first n terms are one, and others are zero and, $u = (1,1,\ldots)$. It is clear that $0 \leq u_n \uparrow u$ in c. Therefore, we get $u - u_n \downarrow 0$. Hence, $u - u_n \xrightarrow{o} 0$. On the other hand $(u - u_n)$ does not convergence to zero in norm, since $\|u - u_n\| = 1$ so, identity operator does not belong to a demi-order norm continuous operator.

The next example gives us that the set of all demi-order norm continuous operator on E is a proper subset of $L(E)$ in general.

Example 2.3. Let $k \in \mathbb{N}$ and $T_k: c \to c$ be an operator defined by $T_k(x) = \sum_{i=1}^{k} x_i e_i$ for each $x = (x_i) \in c$. Consider the sequence s_n, its first n terms are one, and others are zero and $s = (1,1,\ldots)$. It is obvious that $0 \leq s_n \uparrow s$ and $(s - s_n) \downarrow 0$. We obtain that $s - s_n \xrightarrow{o} 0$. Hence, $\|T_k(s - s_n)\| \to 0$ in c. Define $S_k = I + T_k$ for each $k \in \mathbb{N}$.

$(s - s_n) \xrightarrow{o} 0$ and $\|(I - S_k)(s - s_n)\| = \|(I - I - T_k)(s - s_n)\| = \|T_k(s - s_n)\| \to 0$. Since $\|s - s_n\| = 1$, $(s - s_n)$ convergence is not zero in norm. Therefore, S_k does not belong to $\mathfrak{D}L_{on}(c)$ for each $k \in \mathbb{N}$.

Theorem 2.1. Every order-to-norm continuous operator is a d-onco.

Proof. Let M be a normed Riesz space, $H \in L_{on}(M)$, (x_α) in M^+ such that $x_\alpha \xrightarrow{o} 0$ and $\|(x_\alpha - H(x_\alpha)\| \to 0$. Since $H \in L_{on}(M)$, satisfies $\|H(x_\alpha)\| \to 0$. We can write

$$\begin{aligned} \|x_\alpha\| &= \|(x_\alpha - H(x_\alpha) + H(x_\alpha)\| \\ &\leq \|(x_\alpha - H(x_\alpha)\| + \|H(x_\alpha)\| \end{aligned}$$

and then we know that $\|(x_\alpha - H(x_\alpha)\| \to 0$ and $\|H(x_\alpha)\| \to 0$. Therefore, $\|x_\alpha\| \to 0$. Hence, H is a demi-order norm continuous operator on M.

In the next example, it is shown that the inverse of the theorem is not generally true.

Example 2.4. Let H be an operator on $M = C[0,1]$ and $H = \frac{1}{2}I$. Since the norm on M is not order continuous norm (see [6]), then the operator H is not in $L_{on}(M)$, but H is a demi-order norm continuous operator on M from Example 2.1.

Theorem 2.2. Let N be a normed Riesz space, $H: N \to N$ be an order-to-norm continuous operator, $S: N \to N$ be a d-onco, then $H + S$ is a d-onco.

Proof. Let a net (x_α) in N^+ such that $x_\alpha \overset{o}{\to} 0$ and $\|x_\alpha - (H + S)(x_\alpha)\| \to 0$. We can write as

$$\begin{aligned}\|x_\alpha - S(x_\alpha)\| &= \|x_\alpha - S(x_\alpha) - H(x_\alpha) + H(x_\alpha)\| \\ &\leq \|x_\alpha - (H + S)(x_\alpha)\| + \|H(x_\alpha)\|.\end{aligned}$$

It is obvious that $\|H(x_\alpha)\| \to 0$, since $H \in L_{on}(N)$. Moreover, we know that $\|x_\alpha - (H + S)(x_\alpha)\| \to 0$. Thus, $\|x_\alpha - S(x_\alpha)\| \to 0$. We obtain that $\|x_\alpha\| \to 0$, since S belongs to $\mathfrak{D}L_{on}(N)$. Hence, $H + S$ is a d-onco.

The result of Theorem 2.2 is true for $S + H$ as well as for $S - H$.

However, as the next example shows that the sum of two d-onco is not a d-onco in general.

Example 2.5. Let T_1, T_2 be two operators on $M = C[0,1]$, defined as $T_1(f) = T_2(f) = \frac{1}{2}f$ for each $f \in M$. T_1 and T_2 are two demi-order norm continuous operators, but $T_1 + T_2 = I$ does not belong to $\mathfrak{D}L_{on}(M)$.

The following theorem gives that a characterization of a normed Riesz space having an order continuous norm.

Theorem 2.3. Let M be a normed Riesz space. Then the following statements are equivalent

(i) M has order continuous norm,

(ii) $L_{on}(M) = \mathfrak{D}L_{on}(M)$.

Proof. $(i) \Rightarrow (ii)$ It is clear that $L_{on}(M) \subset \mathfrak{D}L_{on}(M)$ from Theorem 2.1. We have to show that $\mathfrak{D}L_{on}(M) \subset L_{on}(M)$.

It is well-known $x_\alpha \overset{o}{\to} 0$ implies $x_\alpha \overset{\|.\|}{\to} 0$ if M has order continuous norm. Then, $H(x_\alpha) \overset{\|.\|}{\to} 0$ if $H \in L(M)$. It show that $L_{on}(M) = L(M)$, so it is clear that $\mathfrak{D}L_{on}(M) \subset L_{on}(M)$. The proof is complete.

$(ii) \Rightarrow (i)$ Let $L_{on}(M) = \mathfrak{D}L_{on}(M)$ and $x_\alpha \downarrow 0$. Since $\frac{1}{2}I$ is a d-onco, then $\frac{1}{2}I$ is in $L_{on}(M)$. Hence, I belongs to $L_{on}(M)$. It is obvious that

$$\begin{aligned}x_\alpha \downarrow 0 &\Rightarrow x_\alpha \overset{o}{\to} 0 \\ &\Rightarrow x_\alpha \overset{\|.\|}{\to} 0.\end{aligned}$$

Since $\|x_\alpha\| \downarrow$ and $x_\alpha \overset{\|.\|}{\to} 0$, we obtain that $\|x_\alpha\| \downarrow 0$, so M has order continuous norm.

Let M and N be two normed Riesz spaces and $\widehat{M}= M \oplus N=\{$ (a,b): $a \in M, b \in N\}$ if $\widehat{M}$ is equipped with the coordinatewise order that is $(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow a_1 \leq a_2$ and $b_1 \leq b_2$ for each $(a_1, b_1), (a_2, b_2) \in \widehat{M}$ and the norm $\|(a, b)\|_{\widehat{M}} = \|a\|_M + \|b\|_N$.

Theorem 2.4. Let M and N be two normed Riesz spaces. Then the following operators are d-onco.

(i)All operators H on M which $(I-H)^{-1}$ exists and is bounded.

(ii) $\widehat{(H_\alpha)}$ is the class of operator on $\widehat{M}$. $\widehat{(H_\alpha)}$ is defined by $\begin{bmatrix} 0 & 0 \\ H & \alpha I \end{bmatrix}$ and $\widehat{M} = M \oplus N$ $(\alpha \neq 1)$ for every operator H from M into N.

Proof. (i) Assume a net (x_α) in M^+such that $x_\alpha \xrightarrow{o} 0$ and $\|x_\alpha - H(x_\alpha)\| \xrightarrow{\|.\|} 0$. It is written as

$$\begin{aligned}\|x_\alpha\| &= \|(I-H)^{-1}\,(I-H)x_\alpha\| \\ &\leq \|(I-H)^{-1}\|\|(I-H)x_\alpha\|.\end{aligned}$$

Since $(I-H)^{-1}$ exists, is bounded, and there are inequalities, we obtain that $\|x_\alpha\| \to 0$. Hence, H belongs to $\mathfrak{D}L_{on}(M)$.

(ii) Let $\{\widehat{x_\alpha}\}$ be a net in $\widehat{M}^+$ such that $\{\widehat{x_\alpha} = (x_\alpha, y_\alpha)\}$, $x_\alpha \in M$, $y_\alpha \in N$ for $\alpha \neq 1$, $\widehat{x_\alpha} \xrightarrow{o} 0$ and $\left\|\widehat{x_\alpha} - \widehat{H}\widehat{x_\alpha})\right\|_{\widehat{M}} \to 0$. It will be shown that $\|\widehat{x_\alpha}\|_{\widehat{M}} \to 0$. We know that $\|\widehat{x_\alpha}\|_{\widehat{M}} = \|x_\alpha\|_M + \|y_\alpha\|_N$. Hence, to show that $\|\widehat{x_\alpha}\|_{\widehat{M}} \to 0$, we have to show that $\|x_\alpha\|_M \to 0$ and $\|y_\alpha\|_N \to 0$.

$$\begin{aligned}\left\|\widehat{x_\alpha} - \widehat{H}(\widehat{x_\alpha})\right\|_{\widehat{M}} &= \left\|(x_\alpha, y_\alpha) - \widehat{H}(x_\alpha, y_\alpha)\right\|_{\widehat{M}} \\ &= \|(x_\alpha, y_\alpha) - (0, Hx_\alpha + \alpha y_\alpha)\|_{\widehat{M}} \\ &= \|(x_\alpha, y_\alpha - Hx_\alpha - \alpha y_\alpha\|_{\widehat{M}} \\ &= \|(x_\alpha, y_\alpha(1-\alpha) - Hx_\alpha\|_{\widehat{M}} \\ &= \|(x_\alpha)\|_M + \|y_\alpha(1-\alpha) - Hx_\alpha\|_N.\end{aligned}$$

From the assumption that $\left\|\widehat{x_\alpha} - \widehat{H}\widehat{x_\alpha})\right\|_{\widehat{M}} \to 0$. Therefore, it is obtained $\|(x_\alpha)\|_M \to 0$ and $\|y_\alpha(1-\alpha) - Hx_\alpha\|_N \to 0$. $\|(x_\alpha)\|_M \to 0$ implies $\|Hx_\alpha\|_N \to 0$, since H is continuous. Moreover, we can write as

$$\begin{aligned}|(1-\alpha)|\|y_\alpha\|_N &= \|y_\alpha(1-\alpha) - Hx_\alpha + Hx_\alpha\|_N \\ &\leq \|y_\alpha(1-\alpha) - Hx_\alpha\|_N + \|Hx_\alpha\|_N.\end{aligned}$$

We get $|(1-\alpha)|\|y_\alpha\|_N \to 0$ so, $\|\widehat{x_\alpha}\|_{\widehat{M}} \to 0$. Thus, $\widehat{(H_\alpha)}$ belongs to $\mathfrak{D}L_{on}(\widehat{M})$.

The following example gives us that Theorem 2.4 (ii) may not be valid in case α =1

Example 2.6. Let an operator $H: \ell_1 \to \ell_\infty$, $\widehat{M} = \ell_1 \oplus \ell_\infty$ equipped with coordinatewise order and operator $\widehat{H}$. is defined by $\begin{bmatrix} 0 & 0 \\ H & I \end{bmatrix}$ $\widehat{H}$ does not belong to $\mathfrak{D}L_{on}(\widehat{M})$. An order bounded sequence $\{\widehat{x_n}\}$ in $\widehat{M}^+$such that $\widehat{x_n} = (0, e_n)$ and e_n the nth. term equals one and the others are zero. Since (e_n) is order convergent in ℓ_∞, then $\widehat{(x_n)}$ is order convergent and $\left\|\widehat{x_n} - \widehat{H}\,\widehat{x_n}\right\| = 0 \to 0$. Since $\|e_n\|_\infty = 1$, then $\|\widehat{x_n}\|_{\widehat{M}} = 1$, so $\widehat{H}$ does not belong to $\mathfrak{D}L_{on}(\widehat{M})$.

The next example shows that the set of all demi-order norm continuous operators on a normed Riesz space is not closed according to multiplication with scalar.
Note that, if H is a d-onco and $\alpha \in \mathbb{R}$, then αH may not be d-onco in general. For example, $M = C[0,1]$, $H = \frac{1}{2}I$: M $\to M$ is a d-onco, but $2H = I: M \to M$ is not a d-onco.

Theorem 2.5. Let M be normed Riesz space. Then the following assertions are equivalent

(i) All operator $H: M \to M$ is a d-onco.

(ii) $I: M \to M$ is a d-onco,

(iii) M has order continuous norm.

Proof. $(i) \Rightarrow (ii)$ It is obvious.

$(ii) \Rightarrow (iii)$ Assume that a net (x_α) in M^+such that $I \in \mathfrak{D}L_{on}(M)$ and $x_\alpha \downarrow 0$. Since $\|x_\alpha - I(x_\alpha)\| = 0 \to 0$, and I is in $\mathfrak{D}L_{on}(M)$, we get $\|x_\alpha\| \to 0$. We know x_α is decreasing. Hence, it is clear that $\|x_\alpha\|$ is decreasing. Since $\|x_\alpha\| \downarrow$ and $\|x_\alpha\| \to 0$, then we get $\|x_\alpha\| \downarrow 0$, so M has order continuous norm.

$(iii) \Rightarrow (i)$ It is obvious from Teorem 2.3.

The next example shows that if H is a d-onco and, $0 \leq S \leq H$, then S is not a d-onco in general.

Example 2.7. Let H, S be two operators on $M = C[0,1]$, $S = I$ and $H = 2I$. It holds $0 \leq S \leq H$. H belongs to $\mathfrak{D}L_{on}(M)$, but S does not belong to $\mathfrak{D}L_{on}(M)$.

The following theorem gives us that the domination property is satisfied under the some special conditions.

Theorem 2.6. Let S and H be two positive operators on the normed Riesz space M and $0 \leq S \leq H \leq I$. If H is the d-onco, then S is also the d-onco.

Proof. Assume that a net (x_α) in M^+ such that $H \in \mathfrak{D}L_{on}(M)$, $x_\alpha \xrightarrow{o} 0$ and $\|x_\alpha - S(x_\alpha)\| \to 0$.

Since $0 \leq (I - H)(x_\alpha) \leq (I - S)(x_\alpha)$, we obtain that

$$\|(I - H)(x_\alpha)\| \leq \|(I - S)(x_\alpha)\|.$$

Thus, $\|x_\alpha - H(x_\alpha)\| \to 0$. Since H is in $\mathfrak{D}L_{on}(M)$, then $\|x_\alpha\| \to 0$. Therefore, we get S is also a d-onco.

Theorem 2.7. Let M be a normed Riesz space, S and H two operators on M and $I \leq S \leq H$. If S is in $\mathcal{D}L_{on}(M)$, then H is in $\mathcal{D}L_{on}(M)$.

Proof. Assume that a net (x_α) in M^+such that $S \in \mathfrak{D}L_{on}(M)$, $x_\alpha \xrightarrow{o} 0$ and $\|(H - I)(x_\alpha)\| \to 0$. We know that $0 \leq (S - I)(x_\alpha) \leq (H - I)(x_\alpha)$. Hence,

$$\|(H - I)(x_\alpha)\| \to 0 \Rightarrow \|(S - I)(x_\alpha)\| \to 0 .$$

Since S is in $\mathfrak{D}L_{on}(M)$, we obtain that $\|x_\alpha\| \to 0$, so H belongs to $\mathfrak{D}L_{on}(M)$.

Theorem 2.8. Let M be a normed Riesz space, $H, S, N: M \to M$ be three operators and $N \leq S \leq H \leq I + N$. If N is in $L_{on}(M)$ and H is in $\mathfrak{D}L_{on}(M)$, then S is in $\mathfrak{D}L_{on}(M)$.

Proof. We obtain from the hypothesis $0 \leq S - N \leq H - N \leq I$. $H - N$ is a d-onco from Theorem 2.2, and $S - N$ is a d-onco from Theorem 2.6. Since $S = S - N + N$, $S - N$ is a d-onco, N is in $L_{on}(M)$, and from Theorem 2.2, we obtain that S is a d-onco.

Note that if a continuous operator belongs to $\mathfrak{D}L_{on}(M)$, then its adjoint does not generally belong to $\mathfrak{D}L_{on}(M)$. For example, $I: l_1 \to l_1$ a d-onco, but its adjoint $I^* = I \coloneqq l_\infty \to l_\infty$ is not a d-onco.

Similarly, if the adjoint of a continuous operator is d-onco, then it may not be a d-onco in general; for example, choice $M = l_\infty$. Since M' is AL-space, then M' has order continuous norm [6]. Hence, $I': M' \to M'$ is a d-onco, but $I: l_\infty \to l_\infty$ is not a d-onco.

The following example gives us that the set of all demi-order norm continuous operators on M does not form a lattice in general.

Example 2.8. Let $M = L^1([0,1]) \times c_0$, $H: M \to M$ be an operator, and defined as $H(f,x) = (0, (\int_0^1 f(x)sinxdx, \int_0^1 f(x)sin2xdx, \int_0^1 f(x)sinx3dx, \cdots))$, for each $f \in L^1([0,1])$. Since the norm on the M is order continuous, then H is in $L_{on}(M)$. Therefore, H is a d-onco, but it does not have modulus. Since this operator is not order bounded [6], so $\mathfrak{D}L_{on}(M)$ is not a lattice.

Let $L(M)$ be a Riesz space. $\mathfrak{D}L_{on}(M)$ is not order closed in $L(M)$ in general.

Example 2.9. Let T: $l_\infty \to l_\infty$ be an operator, $x = (x_i)$ and defined as $T_n(x) = \sum_{i=1}^{n} x_i e_i$. We get $0 \leq T_n \uparrow I$. Therefore, it is clear that $T_n \overset{o}{\to} I$. $\mathfrak{D}L_{on}(M)$ is not order closed, since I is not a d-onco.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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