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Some Common Fixed Point Theorems in Bipolar Metric Spaces

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Abstract. In this article, we introduce the notion of commutativity for covariant and contravariant mappings in bipolar metric spaces. Afterwards, by using this notion, we prove some common fixed point theorems which show the existence and uniqueness of common fixed point for covariant and contravariant mappings satisfying contractive type conditions.

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1. Introduction

Fixed point theorems, especially Banach's fixed point theorem, have been an interesting subject of research in literature. There are many generalization of the Banach fixed point theorem. In 1976, Jungck [13] added a new one to them which is refered as common fixed point theorem for commuting mappings. This theorem have important applications to nonlinear integral equations, dynamic programming and systems of Urysohn integral equations [18, 22, 25]. After that, many authors as Sessa and Pant [21, 24] contributed to the development of this result for some different types of mappings as discontinuous commuting mappings, weakly commuting mappings, R-weakly commuting mappings. In recent years, several researchers have extended these theorems to some types of generalized metric spaces [1–12, 14–17, 23, 26].

Metric spaces have many generalizations as partial metric spaces, G-metric spaces, modular metric spaces and rectangular metric spaces. One of such generalizations is bipolar metric spaces, which are considered as a new framework to study distances between classes of dissimilar objects. These spaces that are assumed to have many applications in various areas are introduced by Mutlu and Gürdal [19] in 2016. They both expressed the link between metric spaces and bipolar metric spaces, and proved some extensions of well-known fixed point theorems as Banach's, Kannan's. Afterwards, Mutlu, Özkan and Gürdal proved coupled fixed point theorems in complete bipolar metric spaces [20].

The aim of this paper is to introduce the notion of commutativity for covariant and contravariant mappings in bipolar metric spaces. Afterwards, using this notion, some common fixed point theorems which show the existence and uniqueness of common fixed point for covariant and contravariant mappings satisfying contractive type conditions are proved.

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2. Bipolar Metric Spaces

Definition 2.1. ([19]) Let $X, Y \neq \varnothing$ and $d : X \times Y \to \mathbb{R}^+$ be a function. d is called a bipolar metric on (X, Y) if the following properties are satisfied

(B0) $x = y$ if $d(x, y) = 0$,

(B1) $d(x, y) = 0$ if $x = y$,

(B2) $d(x, y) = d(y, x)$ if $x, y \in X \cap Y$,

(B3) $d(x, y) \leq d(x, y') + d(x', y') + d(x', y)$,

for all $(x, y), (x', y') \in X \times Y$. Then, the triple (X, Y, d) is called a bipolar metric space.

Definition 2.2. ([19]) Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a covariant map if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$. Similarly, a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a contravariant map if $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$. These maps are denoted as $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$, respectively.

Definition 2.3. ([19]) In a bipolar metric space (X, Y, d);

(1) (a) The points of the set X are called left points,

(b) The points of the set Y are called right points,

(c) The points of the set $X \cap Y$ are called central points,

(2) (a) A sequence of left points is called a left sequence,

(b) A sequence of right points is called a right sequence,

(c) The term "sequence" is commonly used for left sequences and right sequences,

(3) (a) If $\lim_{n\to\infty} d(a_n, y) = 0$ for a left sequence (a_n) and a right point y, then (a_n) is called convergent to y,

(b) If $\lim_{n\to\infty} d(x, b_n) = 0$ for a right sequence (b_n) and a left point x, then (b_n) is called convergent to x,

(4) A sequence (x_n, y_n) on the set $X \times Y$ is called a bisequence on (X, Y, d),

(5) A bisequence is called convergent, if both the left sequence (x_n) and the right sequence (y_n) converge,

(6) If (x_n) and (y_n) converge to a common point, then (x_n, y_n) is called biconvergent,

(7) A Cauchy bisequence is a bisequence (x_n, y_n) such that $\lim_{n,m\to\infty} d(x_n, y_m) = 0$,

(8) A bipolar metric space in which every Cauchy bisequence converges, is called a complete bipolar metric space.

It is shown in [19] that convergence of Cauchy bisequences implies biconvergence.

Definition 2.4. ([19]) (1) A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at $x_0 \in X_1$ if and only if there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that $d_1(x_0, y) < \delta \Rightarrow d_2(f(x_0), f(y)) < \varepsilon$ for every $\varepsilon > 0$ and all $y \in Y_1$.

(2) A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is right-continuous at $y_0 \in Y_1$ if and only if there exists a $\delta = \delta(y_0, \varepsilon) > 0$ such that $d_1(x, y_0) < \delta \Rightarrow d_2(f(x), f(y_0)) < \varepsilon$ for every $\varepsilon > 0$ and all $x \in X_1$.

(3) If a covariant map f is left-continuous at each $x \in X_1$ and right-continuous at each $y \in Y_1$, then it is called continuous.

(4) A contravariant map $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, right-continuous at a point $y_0 \in Y_1$ or continuous, if and only if the corresponding covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, d_2)$ is left-continuous at x_0, right-continuous at y_0 or continuous, respectively.

This definition implies that a contravariant or a covariant map f, which is defined from (X_1, Y_1, d_1) to (X_2, Y_2, d_2), is continuous, if and only if $(a_n) \to v$ on (X_1, Y_1, d_1) implies $f(a_n) \to f(v)$ on (X_2, Y_2, d_2).

3. Main Results

Definition 3.1. Let (X, Y, d) be a bipolar metric space and S, T be covariant or contravariant selfmappings on (X, Y). A point $z \in X \cup Y$ is called a common fixed point of S and T if $Sz = Tz = z$.

Definition 3.2. Let f and g be covariant or contravariant selfmappings on (X, Y). If

$$g(f(x)) = f(g(x)) \text{ for all } x \in X \cup Y,$$

it is said that g commutes with f.

Theorem 3.3. *Let (X, Y, d) be a complete bipolar metric space and $f : (X, Y) \rightrightarrows (X, Y)$ be a continuous covariant mapping on (X, Y). If a covariant mapping $g : (X, Y) \rightrightarrows (X, Y)$ which commutes with f such that*

$$d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \tag{3.1}$$

for all $x \in X$ and $y \in Y$ where $\alpha \in (0, 1)$, $g(X) \subset f(X)$ and $g(Y) \subset f(Y)$, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, $y_0 \in Y$ and $f(x_1) = g(x_0)$, $f(y_1) = g(y_0)$. In general, chosen (x_n, y_n) so that

$$f(x_n) = g(x_{n-1}) \text{ and } f(y_n) = g(y_{n-1}). \tag{3.2}$$

Then, $(f(x_n), f(y_n))$ and $(g(x_n), g(y_n))$ are bisequences on (X, Y). Since $g(X) \subset f(X)$ and $g(Y) \subset f(Y)$, we can make this choice. From (3.1) and (3.2), we get

$$\begin{aligned} d(g(x_n), g(y_n)) &\leq \alpha d(f(x_n), f(y_n)) \\ &= \alpha d(g(x_{n-1}), g(y_{n-1})) \\ &\leq \alpha^2 d(f(x_{n-1}), f(y_{n-1})) \\ &= \alpha^2 d(g(x_{n-2}), g(y_{n-2})) \\ &\vdots \\ &\leq \alpha^n d(g(x_0), g(y_0)) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}$.

On the other hand, using (3.1) and (3.2), we have

$$\begin{aligned} d(g(x_n), g(y_{n+1})) &\leq \alpha d(f(x_n), f(y_{n+1})) \\ &= \alpha d(g(x_{n-1}), g(y_n)) \\ &\leq \alpha^2 d(f(x_{n-1}), f(y_n)) \\ &\vdots \\ &\leq \alpha^n d(g(x_0), g(y_1)). \end{aligned} \tag{3.4}$$

For $n, m \in \mathbb{N}$ with $n > m$, from (3.3) and (3.4), we get

$$\begin{aligned} d(g(x_n), g(y_m)) &\leq d(g(x_n), g(y_n)) + d(g(x_{n-1}), g(y_n)) + d(g(x_{n-1}), g(y_{n-1})) \\ &\quad + d(g(x_{n-2}), g(y_{n-1})) + \cdots + d(g(x_m), g(y_{m+1})) \\ &\quad + d(g(x_m), g(y_m)) \\ &\leq \alpha^n d(g(x_0), g(y_0)) + \alpha^{n-1} d(g(x_0), g(y_1)) + \alpha^{n-1} d(g(x_0), g(y_0)) \\ &\quad + \alpha^{n-2} d(g(x_0), g(y_1)) + \cdots + \alpha^m d(g(x_0), g(y_1)) \\ &\quad + \alpha^m d(g(x_0), g(y_0)) \\ &= (\alpha^n + \alpha^{n-1} + \cdots + \alpha^m) d(g(x_0), g(y_1)) + \\ &\quad (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m) d(g(x_0), g(y_0)) \\ &\leq \frac{\alpha^m}{1-\alpha} (d(g(x_0), g(y_0)) + d(g(x_0), g(y_1))). \end{aligned}$$

We take $d(g(x_0), f(g(y_0))) + d(g(x_0), g(y_1)) = K$ such that $K > 0$. Since $\alpha \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\alpha^m}{1-\alpha} K < \epsilon$ for every $\epsilon > 0$ with $n_0 \leq n$. Similarly, for $n, m \in \mathbb{N}$ with $m > n \geq n_1$, there exists $n_1 \in \mathbb{N}$ such that $d(g(x_n), g(y_m)) < \epsilon$. On the other hand, we obtain in a similar way that $d(f(x_n), f(y_m)) < \epsilon$ for all $n, m \in \mathbb{N}$. Then we get the conclusion that $(f(x_n), f(y_n))$ and $(g(x_n), g(y_n))$ are Cauchy bisequences on (X, Y, d). Since (X, Y, d) is a complete bipolar metric space, $(f(x_n), f(y_n))$ and $(g(x_n), g(y_n))$ biconverge. Then, there exists $z \in X \cap Y$ such that $f(x_n) \to z$, $f(y_n) \to z$ as $n \to \infty$. Using (3.2), we say that $g(x_n) \to z$, $g(y_n) \to z$ as $n \to \infty$. Since the covariant mapping f is continuous, from (3.1), g is also continuous. Using continuity and commutativity of f and g, we get

$$\begin{aligned} f(z) &= f(\lim_{n\to\infty} f(x_n)) = \lim_{n\to\infty} f^2(x_n) \\ f(z) &= f(\lim_{n\to\infty} g(x_n)) = \lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} g(f(x_n)). \end{aligned} \tag{3.5}$$

From (3.1), we obtain

$$d(g(f(x_n)), g(z)) \leq \alpha d(f^2(x_n), f(z)).$$

Taking the limit as $n \to \infty$, from (3.5), we get

$$d(f(z), g(z)) \leq \alpha d(f(z), f(z)).$$

Then, we get $0 \leq d(f(z), g(z)) \leq 0$ which implies $f(z) = g(z)$. Again from (3.1), we get

$$d(g(x_n), g(z)) \leq \alpha d(f(x_n), f(z)).$$

Letting n tend to infinity, we obtain

$$d(z, g(z)) \leq \alpha d(z, f(z)) = \alpha d(z, g(z)).$$

Then, we get $g(z) = z$. Hence $f(z) = g(z) = z$. Therefore, z is a common fixed point of f and g.

Now, we show that the common fixed point is unique. We suppose that t is another common fixed points of f and g with $z \neq t$. Then, we obtain $z = g(z) = f(z)$ and $t = g(t) = f(t)$.

$$\begin{aligned} d(z, t) &= d(g(z), g(t)) \\ &\leq \alpha d(f(z), f(t)) \\ &= \alpha d(z, t), \end{aligned}$$

where $\alpha \in (0, 1)$. This implies $d(z, t) = 0$. Hence, $z = t$. Then, f and g have a unique common fixed point. □

Theorem 3.4. *Let (X, Y, d) be a complete bipolar metric space and $f : (X, Y) \rightleftarrows (X, Y)$ be a continuous contravariant mapping on (X, Y). If a contravariant mapping $g : (X, Y) \rightleftarrows (X, Y)$ which commutes with f such that*

$$d(g(y), g(x)) \leq \alpha d(f(y), f(x))$$

for all $x \in X$ and $y \in Y$ where $\alpha \in (0, 1)$, $g(X) \subset f(X)$ and $g(Y) \subset f(Y)$, then f and g have a unique common fixed point.

Proof. The proof is similar to the proof of Theorem 3.3. □

Theorem 3.5. *Let (X, Y, d) be a complete bipolar metric space and $f : (X, Y) \rightleftarrows (X, Y)$ be a continuous contravariant mapping on (X, Y). If a covariant mapping $g : (X, Y) \rightrightarrows (X, Y)$ which commutes with f satisfies*

$$d(g(x), g(y)) \leq \alpha d(f(y), f(x)) \tag{3.6}$$

for all $x \in X$ and $y \in Y$ where $\alpha \in (0, 1)$, $g(X) \subset f(Y)$ and $g(Y) \subset f(X)$, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, $y_0 \in Y$ and $f(x_0) = g(y_0)$, $f(y_1) = g(x_0)$. More generally, chosen (x_n, y_n) so that

$$f(x_n) = g(y_n) \text{ and } f(y_n) = g(x_{n-1}). \tag{3.7}$$

Then, $(f(y_n), f(x_n))$ and $(g(x_n), g(y_n))$ are bisequences on (X, Y). Since $g(X) \subset f(Y)$ and $g(Y) \subset f(X)$, we can make this choice. From (3.6) and (3.7), we get

$$\begin{aligned} d(g(x_n), g(y_n)) &\leq \alpha d(f(y_n), f(x_n)) \\ &= \alpha d(g(x_{n-1}), g(y_n)) \\ &\leq \alpha^2 d(f(y_n), f(x_{n-1})) \\ &= \alpha^2 d(g(x_{n-1}), g(y_{n-1})) \\ &\vdots \\ &\leq \alpha^{2n} d(g(x_0), g(y_0)) \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$.

On the other hand, from (3.6) and (3.7), we have

$$\begin{aligned} d(g(x_{n+1}), g(y_n)) &\leq \alpha d(f(y_n), f(x_{n+1})) \\ &= \alpha d(g(x_{n-1}), g(y_{n+1})) \\ &\leq \alpha^2 d(f(y_{n+1}), f(x_{n-1})) \\ &= \alpha^2 d(g(x_n), g(y_{n-1})) \\ &\vdots \\ &\leq \alpha^{2n} d(g(x_1), g(y_0)). \end{aligned} \tag{3.9}$$

For $n, m \in \mathbb{N}$ with $m > n$, from (3.8) and (3.9), we get

$$\begin{aligned} d(g(x_n), g(y_m)) &\leq d(g(x_n), g(y_n)) + d(g(x_{n+1}), g(y_n)) + d(g(x_{n+1}), g(y_{n+1})) \\ &\quad + d(g(x_{n+2}), g(y_{n+1})) + \cdots + d(g(x_m), g(y_{m-1})) \\ &\quad + d(g(x_m), g(y_m)) \\ &\leq \alpha^{2n} d(g(x_0), g(y_0)) + \alpha^{2n} d(g(x_1), g(y_0)) + \alpha^{2n+2} d(g(x_0), g(y_0)) \\ &\quad + \alpha^{2n+2} d(g(x_1), g(y_0)) + \cdots + \alpha^{2m-2} d(g(x_1), g(y_0)) \\ &\quad + \alpha^{2m} d(g(x_0), g(y_0)) \\ &= (\alpha^{2n} + \alpha^{2n+2} + \cdots + \alpha^{2m}) d(g(x_0), g(y_0)) \\ &\quad + (\alpha^{2n} + \alpha^{2n+2} + \cdots + \alpha^{2m-2}) d(g(x_1), g(y_0)) \\ &\leq \frac{\alpha^{2n}}{1-\alpha} (d(g(x_0), g(y_0)) + d(g(x_1), g(y_0))). \end{aligned}$$

We take $d(g(x_0), g(y_0)) + d(g(x_1), g(y_0)) = K$ so that $K > 0$. Since $\alpha \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{2n}}{1-\alpha} K < \epsilon$ for each $\epsilon > 0$ with $n_0 \leq n$. Similarly, for $n, m \in \mathbb{N}$ with $m > n \geq n_1$, there exists $\epsilon > 0$ such that $d(g(x_n), g(y_m)) < \epsilon$. On the other hand, we obtain with similar way that $d(f(y_m), f(x_n)) < \epsilon$ for all $n, m \in \mathbb{N}$. Then, we get the conclusion that $(f(y_n), f(x_n))$ and $(g(x_n), g(y_n))$ are Cauchy bisequences on (X, Y). Since (X, Y, d) is a complete bipolar metric spaces, $(f(y_n), f(x_n))$ and $(g(x_n), g(y_n))$ are biconvergent. Then, there exists $z \in X \cap Y$ such that $f(x_n) \to z$, $f(y_n) \to z$ as $n \to \infty$. Using (3.7), we say that $g(x_n) \to z$, $g(y_n) \to z$ as $n \to \infty$. Since the contravariant mapping f is continuous, from (3.6), g is also continuous. Using continuity and commutativity of f and g, we get

$$\begin{aligned} f(z) &= f(\lim_{n\to\infty} f(x_n)) = \lim_{n\to\infty} f^2(x_n) \\ f(z) &= f(\lim_{n\to\infty} g(x_n)) = \lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} g(f(x_n)). \end{aligned} \tag{3.10}$$

From (3.1), we obtain

$$d(g(f(x_n)), g(z)) \leq \alpha d(f^2(x_n), f(z)).$$

Taking the limit as $n \to \infty$, from (3.10), we get

$$d(f(z), g(z)) \leq \alpha d(f(z), f(z)).$$

Then, we get $0 \leq d(f(z), g(z)) \leq 0$ implies $f(z) = g(z)$. Again from (3.6), we get

$$d(g(x_n), g(z)) \leq \alpha d(f(x_n), f(z)).$$

Letting n tend to infinity, we obtain

$$d(z, g(z)) \leq \alpha d(z, f(z)) = \alpha d(z, g(z)).$$

Then, we get $g(z) = z$. Hence $f(z) = g(z) = z$. Therefore, z is a common fixed point of f and g.

Now, we show that the common fixed point is unique. We suppose that t is another common fixed points of f and g where $z \neq t$. Then we obtain $z = f(z) = g(z)$ and $t = f(t) = g(t)$.

$$\begin{aligned} d(z, t) &= d(g(z), g(t)) \\ &\leq \alpha d(f(z), f(t)) \\ &= \alpha d(z, t), \end{aligned}$$

where $\alpha \in (0, 1)$. This implies $d(z, t) = 0$. Hence, $z = t$. Then, f and g have a unique common fixed point. □

Theorem 3.6. *Let (X, Y, d) be a complete bipolar metric space and $f : (X, Y) \rightrightarrows (X, Y)$ be a continuous contravariant mapping on (X, Y). If a contravariant mapping $g : (X, Y) \rightrightarrows (X, Y)$ which commutes with f such that*

$$d(g(y), g(x)) \leq \alpha d(f(x), f(y))$$

for all $x \in X$ and $y \in Y$ where $\alpha \in (0,1)$, $g(Y) \subset f(X)$, $g(X) \subset f(Y)$, then f and g have a unique common fixed point.

Proof. This theorem can be proved to be similar to the proof of the Theorem 3.5. □

Corollary 3.7. *Let f and g be commuting covariant mappings of a complete bipolar metric space (X, Y, d). Suppose that f is continuous and $g(X) \subset f(X)$, $g(Y) \subset f(Y)$. If there exists $\alpha \in (0,1)$ and positive integer k such that*

$$d(g^k(x), g^k(y)) \leq \alpha d(f(x), f(y))$$

for all $x \in X$ and $y \in Y$, then f and g have a unique common fixed point.

Proof. g^k commutes with f and

$$g^k(X) \subset g(X) \subset f(X),$$
$$g^k(Y) \subset g(Y) \subset f(Y).$$

Then, from Theorem 3.3, f and g^k have a unique common fixed point. Let z be this fixed point. Then,

$$z = f(z) = g^k(z). \tag{3.11}$$

On the other hand, since f and g commute, using equality (3.11), we obtain that

$$g(z) = f(g(z)) = g^k(g(z)).$$

Then, $g(z)$ is a common fixed point of f and g^k. That contradicts with uniqueness of the common fixed point z. Therefore, $z = g(z) = f(z)$. Then, f and g have a unique common fixed point. □

Corollary 3.8. *Let f and g be commuting contravariant and covariant mappings, respectively, of a complete bipolar metric space (X, Y, d). Suppose that f is continuous and $g(X) \subset f(Y)$, $g(Y) \subset f(X)$. If there exists $\alpha \in (0,1)$ and positive integer k such that*

$$d(g^k(x), g^k(y)) \leq \alpha d(f(y), f(x))$$

for all $x \in X$ and $y \in Y$, then f and g have a unique common fixed point.

Proof. g^k commutes with f and

$$g^k(X) \subset g(X) \subset f(Y),$$
$$g^k(Y) \subset g(Y) \subset f(X).$$

Then, from Theorem 3.5, f and g^k have a unique common fixed point. Let z be this fixed point. Then

$$z = f(z) = g^k(z). \tag{3.12}$$

On the other hand, since f and g commute, using equality (3.12), we obtain that

$$g(z) = f(g(z)) = g^k(g(z)).$$

Then, $g(z)$ is a common fixed point of f and g^k. That contradicts with uniqueness of the common fixed point z. Therefore, $z = g(z) = f(z)$. Then, f and g have a unique fixed point. □

Corollary 3.9. *Let f and g be two commuting contravariant mappings of a complete bipolar metric space (X, Y, d). Suppose that f is continuous and $g(X) \subset f(X)$, $g(Y) \subset f(Y)$. If there exists $\alpha \in (0,1)$ and positive integer k such that*

$$\begin{aligned} d(g^k(x), g^k(y)) &\leq \alpha d(f(y), f(x)) \text{ for } k = 2n, \\ d(g^k(y), g^k(x)) &\leq \alpha d(f(y), f(x)) \text{ for } k = 2n-1 \end{aligned}$$

for all $x \in X$ and $y \in Y$, then f and g have a unique common fixed point.

Proof. For $k = 2n$, the proof is the similar the proof of Corollary 3.8. We consider the proof for $k = 2n - 1$.
g^k commutes with f and

$$g^k(X) \subset g(X) \subset f(X),$$
$$g^k(Y) \subset g(Y) \subset f(Y).$$

Then, from Theorem 3.4, f and g^k have a unique common fixed point. Let z be this fixed point. Then,

$$z = f(z) = g^k(z). \tag{3.13}$$

On the other hand, since f and g commute, using equality (3.13), we obtain that

$$g(z) = f(g(z)) = g^k(g(z)).$$

Then, $g(z)$ is a common fixed point of f and g^k. That contradicts with uniqueness of the common fixed point z. Therefore, $z = g(z) = f(z)$. So that f and g have a unique fixed point. □

Corollary 3.10. *Let f and g be commuting covariant and contravariant mappings, respectively, of a complete bipolar metric space (X, Y, d). Suppose that f is continuous and $g(X) \subset f(Y)$, $g(Y) \subset f(X)$. If there exists $\alpha \in (0, 1)$ and positive integer k such that*

$$\begin{aligned} d(g^k(x), g^k(y)) &\leq \alpha d(f(x), f(y)) \text{ for } k = 2n, \\ d(g^k(y), g^k(x)) &\leq \alpha d(f(x), f(y)) \text{ for } k = 2n-1 \end{aligned}$$

for all $x \in X$ and $y \in Y$, then f and g have a unique common fixed point.

Proof. The proof is similar way with Corollary 3.9's. □

Theorem 3.11. *Let (X, Y, d) be a complete bipolar metric spaces. $f : (X, Y) \rightrightarrows (X, Y)$ be a covariant continuous mapping on (X, Y). If a contravariant mapping $g : (X, Y) \leftrightarrows (X, Y)$ which commutes with f such that*

$$d(g(y), g(x)) \leq \alpha(d(f(x), g(x)) + d(g(y), f(y))) \tag{3.14}$$

for all $x \in X$, $y \in Y$ where $\alpha \in \left[0, \frac{1}{2}\right)$, $g(X) \subset f(Y)$, $g(Y) \subset f(X)$, then f and g have a unique common fixed point on (X, Y).

Proof. We take $x_0 \in X$, $y_0 \in Y$. Let be $f(y_n) = g(x_n)$ and $f(x_{n+1}) = g(y_n)$.

$$\begin{aligned} d(f(x_{n+1}), f(y_n)) &= d(g(y_n), g(x_n)) \\ &\leq \alpha(d(f(x_n), g(x_n)) + d(g(y_n), f(y_n))) \\ &= \alpha(d(f(x_n), f(y_n)) + d(f(x_{n+1}), f(y_n))), \end{aligned}$$

$$d(f(x_{n+1}), f(y_n)) \leq \frac{\alpha}{1-\alpha}(d(f(x_n), f(y_n))). \tag{3.15}$$

$$\begin{aligned} d(f(x_n), f(y_n)) &= d(g(y_{n-1}), g(x_n)) \\ &\leq \alpha(d(f(x_n), g(x_n)) + d(g(y_{n-1}), f(y_{n-1}))) \\ &= \alpha(d(f(x_n), f(y_n)) + d(f(x_n), f(y_{n-1}))), \end{aligned}$$

$$d(f(x_n), f(y_n)) \leq \frac{\alpha}{1-\alpha}(d(f(x_n), f(y_{n-1}))). \tag{3.16}$$

We say $h = \frac{\alpha}{1-\alpha}$. If we combine inequalities (3.15) and (3.16), then we obtain that

$$\begin{aligned} d(f(x_{n+1}), f(y_n)) &= h(d(f(x_n), f(y_n))) \\ &\leq h^2(d(f(x_n), f(y_{n-1}))) \\ &\vdots \\ &\leq h^{2n+1}(d(f(x_0), f(y_0))) \end{aligned}$$

and

$$\begin{aligned} d(f(x_n), f(y_n)) &\leq h(d(f(x_n), f(y_{n-1}))) \\ &\leq h^2(d(f(x_{n-1}), f(y_{n-1}))) \\ &\vdots \\ &\leq h^{2n}(d(f(x_0), f(y_0))). \end{aligned}$$

For $n > m$

$$\begin{aligned} d(f(x_n), f(y_m)) &\leq d(f(x_n), f(y_{n-1})) + d(f(x_{n-1}), f(y_{n-1})) + \cdots \\ &\quad + d(f(x_{m+1}), f(y_{m+1})) + d(f(x_{m+1}), f(y_m)) \\ &\leq h^{2n-1} d(f(x_0), f(y_0)) + h^{2n-2} d(f(x_0), f(y_0)) + \cdots \\ &\quad + h^{2m+2} d(f(x_0), f(y_0)) + h^{2m+1} d(f(x_0), f(y_0)) \\ &= (h^{2n-1} + h^{2n-2} + \cdots + h^{2m+2} + h^{2m+1}) d(f(x_0), f(y_0)) \\ &\leq \frac{h^{2m+1}}{1-h} d(f(x_0), f(y_0)). \end{aligned}$$

For $K = \frac{h^{2m+1}}{1-h}$, we get $d(f(x_n), f(y_m)) \leq K d(f(x_0), f(y_0))$.

For $m > n$,

$$\begin{aligned} d(f(x_n), f(y_m)) &\leq d(f(x_n), f(y_n)) + d(f(x_{n+1}), f(y_n)) + \cdots \\ &\quad + d(f(x_m), f(y_{m-1})) + d(f(x_m), f(y_m)) \\ &\leq h^{2n} d(f(x_0), f(y_0)) + h^{2n+1} d(f(x_0), f(y_0)) + \cdots \\ &\quad + h^{2m-1} d(f(x_0), f(y_0)) + h^{2m} d(f(x_0), f(y_0)) \\ &= (h^{2n} + h^{2n+1} + \cdots + h^{2m-1} + h^{2m}) d(f(x_0), f(y_0)) \\ &\leq \frac{h^{2n}}{1-h} d(f(x_0), f(y_0)). \end{aligned}$$

For $K' = \frac{h^{2n}}{1-h}$, we get $d(f(x_n), f(y_m)) \leq K' d(f(x_0), f(y_0))$. Then, for all $n, m \in \mathbb{N}$, $d(f(x_n), f(y_m)) \to 0$ as $n, m \to \infty$. Similarly, $d(g(y_m), g(x_n)) \to 0$ as $n, m \to \infty$. Then $(g(y_n), g(x_n))$ and $(f(x_n), f(y_n))$ are Cauchy bisequences on (X, Y). Since (X, Y, d) is complete bipolar metric space, $(g(y_n), g(x_n))$ and $(f(x_n), f(y_n))$ are biconvergent. Then, there exists $z \in X \cap Y$ such that

$$f(x_n) \to z, \ f(y_n) \to z$$

as $n \to \infty$. Then, we obtain that

$$g(x_n) \to z, \ g(y_n) \to z$$

as $n \to \infty$. Since, the covariant mapping f is continuous, from inequality (3.14), g are also continuous. Therefore,

$$\begin{aligned} g(f(x_n)) \to g(z) \quad &\text{and} \quad g(f(y_n)) \to g(z) \\ f(g(x_n)) \to f(z) \quad &\text{and} \quad f(g(y_n)) \to f(z). \end{aligned}$$

Since f and g commute, we get

$$f(g(x_n)) = g(f(x_n)) \text{ and } f(g(y_n)) = g(f(y_n))$$

for all $n \in \mathbb{N}$. Then, $f(z) = g(z)$. On the other hand,

$$\begin{aligned} d(z, g(z)) &= \lim_{n\to\infty} d(g(x_n), g(g(x_n))) \\ &\leq \lim_{n\to\infty} \alpha(d(f(x_n), g(x_n)) + d(g(g(x_n)), f(g(x_n)))) \\ &= \lim_{n\to\infty} \alpha(d(z, z) + d(g(z), f(z))) \end{aligned}$$

implies $d(z, g(z)) = 0 \Rightarrow z = g(z) = f(z)$. Then, f and g have a common fixed point.

Now, we show that the common fixed point is unique. We assume that $t \in X \cap Y$ is another common fixed point of f and g such that $z \neq t$. That is, $f(t) = g(t) = t$. Then,

$$d(z, t) = d(g(z), g(t)) \leq \alpha(d(f(z), g(z)) + d(g(t), f(t)))$$

where $\alpha \in \left[0, \frac{1}{2}\right)$. Consequently, $d(z, t) = 0$ implies $z = t$. Then, f and g have a unique common fixed point. □

Theorem 3.12. *Let (X, Y, d) be a complete bipolar metric space. $f : (X, Y) \rightrightarrows (X, Y)$ be a covariant continuous mapping on (X, Y). If a contravariant mapping $g : (X, Y) \rightrightarrows (X, Y)$ which commutes with f such that*

$$d(g(x), g(y)) \leq \alpha(d(g(x), f(x)) + d(f(y), g(y)))$$

for all $x \in X$, $y \in Y$ where $\alpha \in \left[0, \frac{1}{2}\right)$, $g(X) \subset f(Y)$, $g(Y) \subset f(X)$, then f and g have a unique common fixed point on (X, Y).

Proof. This theorem can be proved in a similar way with Theorem 3.11. □

Example 3.13. Let $X = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$, $Y = \{(0, y) \in \mathbb{R}^2 : 0 \leq y \leq 1\}$ and a function $d : X \times Y \to \mathbb{R}^+$ be defined such that $d((x, 0), (0, y)) = |x - y|$ for $x \in X$, $y \in Y$. Then, (X, Y, d) is a complete bipolar metric space. Let $g : (X, Y) \rightrightarrows (X, Y)$ and $f : (X, Y) \rightrightarrows (X, Y)$ be defined as

$$\begin{aligned} g(x, 0) &= (0, \tfrac{x}{2}), & f(x, 0) &= (x, 0), \\ g(0, y) &= (\tfrac{y}{2}, 0), & f(0, y) &= (0, y) \end{aligned}$$

for all $x \in X$ and $y \in Y$. We obtain that

$$d(g(0, y), g(x, 0)) \leq \alpha d(f(x, 0), f(0, y))$$

is satisfied for all $x \in X$, $y \in Y$ where $\alpha = \frac{1}{2} \in (0, 1)$. And, it is obvious that $g(X) \subset f(Y)$, $g(Y) \subset f(X)$. And

$$g(f(x, 0)) = g(x, 0) = (0, \frac{x}{2}) = f(0, \frac{x}{2}) = f(g(x, 0)).$$

So, f and g are commuting mappings. Therefore, from Theorem 3.6, f and g has a unique common fixed point and it is $(0, 0)$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have read and agreed to the published version of the manuscript.

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