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Research Article

Increasing property and logarithmic convexity concerning Dirichlet beta function, Euler numbers, and their ratios

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Dedicated to Professor Ravi P. Agarwal

Abstract

In the paper, by virtue of an integral representation of the Dirichlet beta function, with the aid of a relation between the Dirichlet beta function and the Euler numbers, and by means of a monotonicity rule for the ratio of two definite integrals with a parameter, the author finds increasing property and logarithmic convexity of two functions and two sequences involving the Dirichlet beta function, the Euler numbers, and their ratios.

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1. Motivations and main results

Let

$$\mathbb{Z} = \{0, \pm1, \pm2, \dots\}, \quad \mathbb{N} = \{1, 2, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \dots\}, \quad \mathbb{N}_- = \{-1, -2, \dots\}.$$

Let

$$\langle z\rangle_n = \prod_{k=0}^{n-1}(z-k) = \begin{cases} z(z-1)\cdots(z-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

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for $z \in \mathbb{C}$, which is called the falling factorial, and

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_-, \quad w, z-w \notin \mathbb{N}_- \\ 0, & z \notin \mathbb{N}_-, \quad w \in \mathbb{N}_- \text{ or } z-w \in \mathbb{N}_- \\ \dfrac{\langle z\rangle_w}{w!}, & z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0 \\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_0 \\ 0, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_- \\ \infty, & z \in \mathbb{N}_-, \quad w \notin \mathbb{Z} \end{cases} \tag{1.1}$$

for $z, w \in \mathbb{C}$, which is called in [13] the extended binomial coefficient.

In analytic number theory and mathematical physics, the Dirichlet beta function

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^z}, \quad \Re(z) > 0$$

is a member of the family of functions including the Riemann zeta function

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \Re(z) > 1,$$

the Dirichlet eta function

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad \Re(z) > 0,$$

and the Dirichlet lambda function

$$\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z}, \quad \Re(z) > 1.$$

It was written in [3, Section 3.3] that the function $\beta(z)$ is sometimes known as the Catalan zeta function, since $\beta(2) = G$ is Catalan's constant, perhaps the simplest number whose irrationality is unproven.

In [2, Theorem 1.1], the function

$$T_a(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(an+1)^x}, \quad a, x \in (0, \infty)$$

was proved to be concave in $x \in (0, \infty)$ and to satisfy the limits

$$\lim_{x\to 0^+} \sum_{n=0}^{\infty} \frac{(-1)^n}{(an+1)^x} = \frac{1}{2} \quad \text{and} \quad \lim_{x\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(an+1)^x} = 1.$$

In particular, the Dirichlet eta function $\eta(x) = T_1(x)$ and the Dirichlet beta function $\beta(x) = T_2(x)$ are both concave on $(0, \infty)$. For more information on study of the Dirichlet beta function $\beta(x)$, please refer to [5] and closely-related references therein.

The Dirichlet beta function $\beta(x)$ has an integral representation

$$\beta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t + e^{-t}} \,d\,t, \quad x > 0 \tag{1.2}$$

and connects with the Euler numbers E_{2n} by a relation

$$\beta(2n+1) = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1} \frac{|E_{2n}|}{(2n)!}, \quad n = 0, 1, 2, \dots \tag{1.3}$$

in [1, p. 807, 23.2.22] and [3, p. 141, (3.29)], where the classical Euler gamma function $\Gamma(z)$ can be defined [12, Chapter 3] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

and the Euler numbers E_{2n} can be generated [3, p. 140, (3.26)] by

$$\sec z = \sum_{n=0}^{\infty} |E_{2n}| \frac{z^{2n}}{(2n)!}, \quad |z| \le \frac{\pi}{2}.$$

In this paper, motivated by the papers [7, 8, 14, 15], with the help of an integral representation (1.2), by the aid of the relation (1.3), and by means of a monotonicity rule in [9, Lemma 2.8 and Remark 6.3] and [10, Remark 7.2] for the ratio of two definite integrals with a parameter, we mainly find the following monotonicity and logarithmic convexity.

Theorem 1.1. *Let $\alpha > 0$ be a constant. Then*

(1) *the function*

$$\binom{x+\alpha+1}{\alpha} \frac{\beta(x+\alpha)}{\beta(x)} \tag{1.4}$$

is increasing in $x \in (0, \infty)$;

(2) *the sequence*

$$\frac{2n+3}{2n+1} \frac{|E_{2(n+1)}|}{|E_{2n}|} \tag{1.5}$$

is increasing in $n \ge 0$;

(3) *the function $\Gamma(x+1)\beta(x)$ is logarithmically convex in $x \in (0, \infty)$;*

(4) *the sequence*

$$\left(\frac{\pi}{2}\right)^{2n+1} (2n+1)|E_{2n}| \tag{1.6}$$

is logarithmically convex in $n \ge 0$.

2. A lemma

For proving our main results in this paper, we need the following monotonicity rule for the ratio of two definite integrals with a parameter.

Lemma 2.1 ([9, Lemma 2.8 and Remark 6.3] and [10, Remark 7.2])**.** *Let $U(t), V(t) > 0$, and $W(t,x) > 0$ be integrable in $t \in (a, b)$,*

(1) *if the ratios $\frac{\partial W(t,x)/\partial x}{W(t,x)}$ and $\frac{U(t)}{V(t)}$ are both increasing or both decreasing in $t \in (a, b)$, then the ratio*

$$R(x) = \frac{\int_a^b W(t,x)U(t)\,\mathrm{d}\,t}{\int_a^b W(t,x)V(t)\,\mathrm{d}\,t}$$

is increasing in x;

(2) *if one of the ratios $\frac{\partial W(t,x)/\partial x}{W(t,x)}$ and $\frac{U(t)}{V(t)}$ is increasing and another one of them is decreasing in $t \in (a, b)$, then the ratio $R(x)$ is decreasing in x.*

3. Proofs of main results

We are now in a position to prove our main results in this paper.

With the recurrence $\Gamma(z+1)=z\Gamma(z)$ and the integral representation (1.2), integrating by parts yields

$$\begin{aligned}
\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\beta(x+\alpha)}{\beta(x)} &= \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\frac{1}{\Gamma(x+\alpha)}\int_0^\infty \frac{t^{x+\alpha-1}}{\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}\,t}{\frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}\,t}\\
&= \frac{(x+\alpha)\int_0^\infty \frac{t^{x+\alpha-1}}{\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}\,t}{x\int_0^\infty \frac{t^{x-1}}{\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}\,t}\\
&= \frac{\int_0^\infty \frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}(t^{x+\alpha})'_t\,\mathrm{d}\,t}{\int_0^\infty \frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}(t^{x})'_t\,\mathrm{d}\,t}\\
&= \frac{\left(\frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}t^{x+\alpha}\right)\Big|_{t\to0^+}^{t\to\infty}-\int_0^\infty\left(\frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}\right)'t^{x+\alpha}\,\mathrm{d}\,t}{\left(\frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}t^{x}\right)\Big|_{t\to0^+}^{t\to\infty}-\int_0^\infty\left(\frac{1}{\mathrm{e}^t+\mathrm{e}^{-t}}\right)'t^{x}\,\mathrm{d}\,t}\\
&= \frac{\int_0^\infty\left[\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}t^\alpha\right]t^x\,\mathrm{d}\,t}{\int_0^\infty \frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}t^x\,\mathrm{d}\,t}.
\end{aligned}$$

Applying Lemma 2.1 to

$$U(t)=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}t^\alpha,\quad V(t)=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}>0,\quad W(t,x)=t^x>0,$$

and $(a,b)=(0,\infty)$ for $x>0$, since $\frac{U(t)}{V(t)}=t^\alpha$ and $\frac{\partial W(t,x)/\partial x}{W(t,x)}=\ln t$ are both increasing in $x\in(0,\infty)$, we conclude that the ratio

$$\frac{\int_0^\infty\left[\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}t^\alpha\right]t^x\,\mathrm{d}\,t}{\int_0^\infty \frac{\mathrm{e}^t-\mathrm{e}^{-t}}{(\mathrm{e}^t+\mathrm{e}^{-t})^2}t^x\,\mathrm{d}\,t}=\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\beta(x+\alpha)}{\beta(x)}=\Gamma(\alpha+1)\binom{x+\alpha}{\alpha}\frac{\beta(x+\alpha)}{\beta(x)}$$

is increasing in $x\in(0,\infty)$. The increasing property of the function in (1.4) is proved.

Letting $x=2n+1$ and $\alpha=2$ in (1.4) and using the relation (1.3) reveal that the sequence

$$\frac{\Gamma(2n+4)}{\Gamma(2n+2)}\frac{\beta(2n+3)}{\beta(2n+1)}=\frac{(2n+3)!}{(2n+1)!}\frac{\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+3}\frac{|E_{2(n+1)}|}{(2n+2)!}}{\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1}\frac{|E_{2n}|}{(2n)!}}=\left(\frac{\pi}{2}\right)^2\frac{2n+3}{2n+1}\frac{|E_{2(n+1)}|}{|E_{2n}|}$$

is increasing for $n\ge0$. The increasing property of the sequence in (1.5) is proved.

Because the function $\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\beta(x+\alpha)}{\beta(x)}$ is increasing in $x\in(0,\infty)$, its first derivative

$$\left[\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\beta(x+\alpha)}{\beta(x)}\right]'=\frac{\begin{pmatrix}[\Gamma(x+\alpha+1)\beta(x+\alpha)]'[\Gamma(x+1)\beta(x)]\\-[\Gamma(x+\alpha+1)\beta(x+\alpha)][\Gamma(x+1)\beta(x)]'\end{pmatrix}}{[\Gamma(x+1)\beta(x)]^2}$$

is positive for $x\in(0,\infty)$. Hence, we have

$$\frac{[\Gamma(x+\alpha+1)\beta(x+\alpha)]'}{\Gamma(x+\alpha+1)\beta(x+\alpha)}>\frac{[\Gamma(x+1)\beta(x)]'}{[\Gamma(x+1)\beta(x)]},$$

that is, the logarithmic derivative

$$(\ln[\Gamma(x+1)\beta(x)])'=\frac{[\Gamma(x+1)\beta(x)]'}{[\Gamma(x+1)\beta(x)]}$$

is increasing in $x\in(0,\infty)$. Consequently, the function $\Gamma(x+1)\beta(x)$ is logarithmically convex in $(0,\infty)$.

Taking $x = 2n+1$ in the function $\Gamma(x+1)\beta(x)$ and using the relation (1.3) reveal that the sequence

$$\Gamma(2n+2)\beta(2n+1) = (2n+1)!\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1}\frac{|E_{2n}|}{(2n)!} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+1}(2n+1)|E_{2n}|$$

is logarithmically convex for $n \ge 0$. The logarithmic convexity of the sequence in (1.6) is proved. The proof of Theorem 1.1 is complete.

4. Remarks

Finally, we list several remarks on our main results and related stuffs.

Remark 4.1. By the relation (1.3) and the integral representation (1.2), we acquire

$$\begin{aligned}\frac{|E_{2(n+1)}|}{|E_{2n}|} &= \left(\frac{2}{\pi}\right)^2 \frac{(2n+2)(2n+1)\beta(2n+3)}{\beta(2n+1)} \\ &= \left(\frac{2}{\pi}\right)^2 \frac{(2n+2)(2n+1)\frac{1}{\Gamma(2n+3)}\int_0^\infty \frac{t^{2n+2}}{e^t+e^{-t}}\,d\,t}{\frac{1}{\Gamma(2n+1)}\int_0^\infty \frac{t^{2n}}{e^t+e^{-t}}\,d\,t} \\ &= \left(\frac{2}{\pi}\right)^2 \frac{\int_0^\infty \frac{t^2}{e^t+e^{-t}} t^{2n}\,d\,t}{\int_0^\infty \frac{1}{e^t+e^{-t}} t^{2n}\,d\,t}.\end{aligned}$$

Applying Lemma 2.1 to

$$U(t) = \frac{t^2}{e^t+e^{-t}}, \quad V(t) = \frac{1}{e^t+e^{-t}} > 0, \quad W(t,x) = t^x, \quad (a,b) = (0,\infty)$$

for $x > 0$, since $\frac{U(t)}{V(t)} = t^2$ and $\frac{\partial W(t,x)/\partial x}{W(t,x)} = \ln t$ are both increasing in $t \in (0,\infty)$, we conclude that the function $\frac{\int_0^\infty \frac{t^{x+2}}{e^t+e^{-t}}\,d\,t}{\int_0^\infty \frac{t^x}{e^t+e^{-t}}\,d\,t}$ is increasing on $(0,\infty)$. Hence, the sequence $\frac{|E_{2(n+1)}|}{|E_{2n}|}$ is increasing for $n \in \mathbb{N}$. Further considering $\frac{|E_2|}{|E_0|} = 1 < \frac{|E_4|}{|E_2|} = 5$, we finally acquire that the sequence $\frac{|E_{2(n+1)}|}{|E_{2n}|}$ is increasing for $n \ge 0$.

Remark 4.2. Since the sequence $\frac{2n+3}{2n+1}$ in (1.5) is decreasing for $n \ge 0$, so the increasing property of the sequence in (1.5) is stronger than the increasing property of $\frac{|E_{2(n+1)}|}{|E_{2n}|}$ for $n \ge 0$.

Remark 4.3. This paper is a companion of the papers [4, 6, 11].

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