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Research Article

Generalized invertibility in two semigroups of Banach algebras

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Abstract

Motivated by the results involving Drazin inverses of Patrício and Puystjens, we investigate the relevant results for pseudo Drazin invertibility and generalized Drazin invertibility in two semigroups of Banach algebras. Given a Banach algebra $\mathcal{A}$ and $e^2 = e \in \mathcal{A}$, we firstly establish the relation between pseudo Drazin invertibility (resp., generalized Drazin invertibility) of elements in $e\mathcal{A}e$ and $e\mathcal{A}e + 1 - e$. Then this result leads to a remarkable behavior of pseudo Drazin invertibility (resp., generalized Drazin invertibility) between the operators in the semigroup $AA^-\mathscr{B}(Y)AA^- + I_Y - AA^-$ and the semigroup $A^=A\mathscr{B}(X)A^=A + I_X - A^=A$, where $A^-, A^= \in \mathscr{B}(Y,X)$ are inner inverses of $A \in \mathscr{B}(X,Y)$.

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1. Introduction

The motivation for this research appeared in [16]. Patrício and Puystjens gave the characterizations of the relation between Drazin invertibility of elements in two semigroups of rings. In such the case of Banach algebras, the corresponding versions of results can be described as follows. Let $\mathcal{A}$ be a complex Banach algebra with unit 1 and $e^2 = e \in \mathcal{A}$. Then $e\mathcal{A}e = \{exe : x \in \mathcal{A}\}$ is a closed subspace of $\mathcal{A}$, and hence is also a Banach algebra. Let $\mathscr{B}(X,Y)$ denote the set of all bounded linear operators from Banach space X to Banach space Y and $\mathscr{B}(X) = \mathscr{B}(X,X)$. First of all, they proved that $exe + 1 - e$ is Drazin invertible in $\mathcal{A}$ if and only if exe is Drazin invertible in $e\mathcal{A}e$, and gave their expressions of Drazin inverses. Then using previous result, they related Drazin invertibility between the operators in the semigroup $AA^-\mathscr{B}(Y)AA^- + I_Y - AA^-$ and the semigroup $A^=A\mathscr{B}(X)A^=A + I_X - A^=A$ in the case that $A^-, A^= \in \mathscr{B}(Y,X)$ are inner inverses of $A \in \mathscr{B}(X,Y)$.

The main theme of this article can be described as the relevant research of generalized Drazin invertibility and pseudo Drazin invertibility in two semigroups of Banach algebras. These two generalized inverses, as the extensions of the Drazin inverse, have attracted

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wide interest in the study of Banach algebras (for example, see [1–7, 9, 11–15, 21]). We use $\mathcal{A}^{qnil}$ to denote the set of all quasinilpotent elements of $\mathcal{A}$, which is equal to the set $\{a \in \mathcal{A} : \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0\}$. And we use $J(\mathcal{A})$ to denote the Jacobson radical of $\mathcal{A}$. First we recall the definition of generalized Drazin inverse, which was introduced and investigated in Banach algebra by Koliha [10] in 1996.

Definition 1.1 ([10]). Let $a \in \mathcal{A}$. Then a is called generalized Drazin invertible if there exists $x \in \mathcal{A}$ such that

$$xax = x,\ ax = xa,\ a - a^2x \in \mathcal{A}^{qnil}.$$

Any element $x \in \mathcal{A}$ satisfying the above conditions is called the generalized Drazin inverse of a, denoted by $x = a^{gD}$.

Note that such x is called the Drazin inverse [8] of a (denoted by a^D if it exists) when $\mathcal{A}^{qnil}$ is replaced by $\mathcal{A}^{nil}$ which is the set of all nilpotent elements of $\mathcal{A}$, in the conditions of Definition 1.1.

Afterwards, the pseudo Drazin inverse was introduced by Wang and Chen [17] in Banach algebra, which is listed as follows:

Definition 1.2 ([17]). Let $a \in \mathcal{A}$. Then a is said to be pseudo Drazin invertible if there exists $x \in \mathcal{A}$ such that

$$xax = x,\ ax = xa,\ a^k - a^{k+1}x \in J(\mathcal{A}) \text{ for some integer } k \geq 1.$$

Such x is called the pseudo Drazin inverse of a and unique when it exists, denoted by a^{pD}. The smallest positive integer k for which the above conditions hold is called the pseudo Drazin index of a, denoted by $\operatorname{ind}(a)$.

Throughout the paper, the symbols $\mathcal{A}^D$, $\mathcal{A}^{gD}$ and $\mathcal{A}^{pD}$ denote the sets of all Drazin invertible, generalized Drazin invertible and pseudo Drazin invertible elements of $\mathcal{A}$, respectively. Moreover, it was known that $\mathcal{A}^D \subseteq \mathcal{A}^{pD} \subseteq \mathcal{A}^{gD}$ [17].

The rest paper is built up as follows. In Section 2, we first characterize the relation between pseudo Drazin invertibility (resp., generalized Drazin invertibility) of elements in $e\mathcal{A}e$ and $e\mathcal{A}e + 1 - e$ in the case of $e^2 = e \in \mathcal{A}$. In Section 3, pseudo Drazin invertibility (resp., generalized Drazin invertibility) between the operators in the semigroup $AA^-\mathscr{B}(Y)AA^- + I_Y - AA^-$ and the semigroup $A^=A\mathscr{B}(X)A^=A + I_X - A^=A$ is related, when $A^-, A^= \in \mathscr{B}(Y, X)$ are inner inverses of $A \in \mathscr{B}(X, Y)$. In addition, we give an alternative proof of the relevant results for Drazin invertibility presented by Patrício and Puystjens [16].

2. The relation between generalized invertibility of elements in $e\mathcal{A}e$ and $e\mathcal{A}e + 1 - e$

In this section, $\mathcal{A}$ denotes a complex Banach algebra with unit 1 and $e^2 = e \in \mathcal{A}$. The notation $e\mathcal{A}e + 1 - e$ represents the set $\{exe + 1 - e : x \in \mathcal{A}\}$. First of all, we give some lemmas.

Lemma 2.1 ([17, 19]). *Let $a \in \mathcal{A}^{pD}$. Then the following statements hold.*

(1) *If $b \in \mathcal{A}$ with $ab = ba$, then $a^{pD}b = ba^{pD}$.*
(2) *If $b \in \mathcal{A}^{pD}$ with $ab = ba = 0$, then $(a + b)^{pD} = a^{pD} + b^{pD}$.*

Lemma 2.2 ([10]). *Let $a \in \mathcal{A}^{gD}$. Then the following statements hold.*

(1) *If $b \in \mathcal{A}$ with $ab = ba$, then $a^{gD}b = ba^{gD}$.*
(2) *If $b \in \mathcal{A}^{gD}$ with $ab = ba = 0$, then $(a + b)^{gD} = a^{gD} + b^{gD}$.*

Lemma 2.3 ([20]). *Let $a \in e\mathcal{A}e$. Then $a \in \mathcal{A}^{pD}$ with $\operatorname{ind}(a) = k$ if and only if $a \in (e\mathcal{A}e)^{pD}$ with $\operatorname{ind}(a) = k$. In this case, $a^{pD}_{\mathcal{A}} = a^{pD}_{e\mathcal{A}e}$.*

Lemma 2.4. *Let $a \in e\mathcal{A}e$. Then $a \in \mathcal{A}^{gD}$ if and only if $a \in (e\mathcal{A}e)^{gD}$. In this case, $a_{\mathcal{A}}^{gD} = a_{e\mathcal{A}e}^{gD}$.*

Proof. Suppose that $a \in (e\mathcal{A}e)^{gD}$ with $a_{e\mathcal{A}e}^{gD} = x$. Then we get $a(1-ax) = a(e-ax) \in (e\mathcal{A}e)^{qnil}$. Applying [18, Lemma 3.5], we have $a(1-ax) \in \mathcal{A}^{qnil}$, which has completed the proof.

On the contrary, assume that $a \in \mathcal{A}^{gD}$ with $a_{\mathcal{A}}^{gD} = x$. Then it suffices to prove that $a(e-ax) \in (e\mathcal{A}e)^{qnil}$ since $x = ax^3a \in e\mathcal{A}e$. Due to $a(1-ax) \in \mathcal{A}^{qnil}$, we can obtain that $a(e-ax) = a(1-ax) \in (e\mathcal{A}e) \cap \mathcal{A}^{qnil}$. Hence this completes the proof by [18, Lemma 3.5]. □

From Lemmas 2.3 and 2.4, we can still keep the usual notations as $(exe)^{gD}$ and $(exe)^{pD}$ which both belong to $e\mathcal{A}e$ if they exist in $\mathcal{A}$, for ease of notations.

Theorem 2.5. *Let $x \in \mathcal{A}$. Then $exe + 1 - e \in \mathcal{A}^{pD}$ with $\operatorname{ind}(exe+1-e) = k$ if and only if $exe \in (e\mathcal{A}e)^{pD}$ with $\operatorname{ind}(exe) = k$. In this case,*

$$(exe)^{pD} = e(exe+1-e)^{pD}e \in e\mathcal{A}e,$$

and

$$(exe+1-e)^{pD} = (exe)^{pD} + 1 - e \in e\mathcal{A}e + 1 - e.$$

Proof. Suppose that $exe + 1 - e \in \mathcal{A}^{pD}$. Then

$$(exe+1-e)(exe+1-e)^{pD} = (exe+1-e)^{pD}(exe+1-e), \tag{2.1}$$

$$(exe+1-e)^{pD}(exe+1-e)(exe+1-e)^{pD} = (exe+1-e)^{pD}. \tag{2.2}$$

Multiplying on the both sides of (2.1) and (2.2) by e, respectively, we can obtain that

$$(exe)e(exe+1-e)^{pD}e = e(exe+1-e)^{pD}e(exe),$$

$$e(exe+1-e)^{pD}(exe+1-e)(exe+1-e)^{gD}e = e(exe+1-e)^{pD}e. \tag{2.3}$$

Since $e(exe+1-e) = (exe+1-e)e$, it follows that $e(exe+1-e)^{pD} = (exe+1-e)^{pD}e$ by Lemma 2.1, hence (2.3) can be written as

$$e(exe+1-e)^{pD}e(exe)e(exe+1-e)^{gD}e = e(exe+1-e)^{pD}e. \tag{2.4}$$

Since $\operatorname{ind}(exe+1-e) = k$, it follows that k is the smallest positive integer such that $(exe+1-e)^k - (exe+1-e)^{k+1}(exe+1-e)^{pD} \in J(\mathcal{A})$. Therefore

$$\begin{aligned}
&(exe)^k - (exe)^{k+1}e(exe+1-e)^{pD}e \\
&= e\left((exe+1-e)^k - ((exe)^{k+1} + 1 - e)e(exe+1-e)^{pD}\right)e \\
&= e\left((exe+1-e)^k - (exe+1-e)^{k+1}(exe+1-e)^{pD}\right)e \in eJ(\mathcal{A})e.
\end{aligned}$$

Since $eJ(\mathcal{A})e \subseteq J(e\mathcal{A}e)$, we have $(exe)^k - (exe)^{k+1}e(exe+1-e)^{pD}e \in J(e\mathcal{A}e)$. Hence $exe \in (e\mathcal{A}e)^{pD}$ with $\operatorname{ind}(exe) \leq k$. Moreover, $(exe)^{pD} = e(exe+1-e)^{pD}e \in e\mathcal{A}e$.

Conversely, if exe is pseudo Drazin invertible with $\operatorname{ind}(exe) = s$ in $e\mathcal{A}e \subseteq \mathcal{A}$, then from Lemmas 2.1 and 2.3 it follows that

$$\begin{aligned}
(exe+1-e)^{pD} &= (exe)^{pD} + (1-e)^{pD} \\
&= (exe)^{pD} + 1 - e
\end{aligned}$$

since $exe(1-e) = (1-e)exe = 0$. Furthermore,

$$\begin{aligned}
&(exe+1-e)^s - (exe+1-e)^{s+1}(exe+1-e)^{pD} \\
&= ((exe)^s + 1 - e) - \Big((exe)^{s+1} + 1 - e\Big)\Big((exe)^{pD} + 1 - e\Big) \\
&= ((exe)^s + 1 - e) - \Big((exe)^{s+1}(exe)^{pD} + 1 - e\Big) \\
&= (exe)^s - (exe)^{s+1}(exe)^{pD} \in J(e\mathcal{A}e) \subseteq J(\mathcal{A}),
\end{aligned}$$

then $\mathrm{ind}(exe+1-e) \le s$.

Hence, it can be derived that $\mathrm{ind}(exe+1-e) = \mathrm{ind}(exe)$ from the proof. □

Theorem 2.6. *Let $x \in \mathcal{A}$. Then $exe+1-e \in \mathcal{A}^{gD}$ if and only if $exe \in (e\mathcal{A}e)^{gD}$. In this case,*

$$(exe)^{gD} = e(exe+1-e)^{gD}e \in e\mathcal{A}e,$$

and

$$(exe+1-e)^{gD} = (exe)^{gD} + 1 - e \in e\mathcal{A}e + 1 - e.$$

Proof. Suppose that $exe+1-e \in \mathcal{A}^{gD}$. Then an argument similar to the one in Theorem 2.5 shows that

$$(exe)e(exe+1-e)^{gD}e = e(exe+1-e)^{gD}e(exe),$$
$$e(exe+1-e)^{gD}e(exe)e(exe+1-e)^{gD}e = e(exe+1-e)^{gD}e.$$

Let $s = (exe)-(exe)^2e(exe+1-e)^{gD}e$ and $t = (exe+1-e)-(exe+1-e)^2(exe+1-e)^{gD}$. Then

$$\begin{aligned}
s &= (exe) - (exe)^2 e(exe+1-e)^{gD}e \\
&= e\Big((exe+1-e) - ((exe)^2 + 1 - e)e(exe+1-e)^{gD}\Big)e \\
&= e\Big((exe+1-e) - (exe+1-e)^2(exe+1-e)^{gD}\Big)e \\
&= ete.
\end{aligned}$$

Since t commutes with e, it follows that

$$\|s^n\|^{\frac{1}{n}} = \|et^ne\|^{\frac{1}{n}} \le \|e\|^{\frac{1}{n}} \cdot \|t^n\|^{\frac{1}{n}} \cdot \|e\|^{\frac{1}{n}}.$$

Then $t \in \mathcal{A}^{qnil}$ implies that s is quasinilpotent, and hence $exe \in (e\mathcal{A}e)^{gD}$. Moreover, $(exe)^{gD} = e(exe+1-e)^{gD}e \in e\mathcal{A}e$.

Conversely, if exe is generalized Drazin invertible in $e\mathcal{A}e \subseteq \mathcal{A}$, then by Lemmas 2.2 and 2.4 it follows that

$$\begin{aligned}
(exe+1-e)^{gD} &= (exe)^{gD} + (1-e)^{gD} \\
&= (exe)^{gD} + 1 - e
\end{aligned}$$

since $exe(1-e) = (1-e)exe = 0$. □

Remark 2.7. In [16], Patrício and Puystjens proved the relevant version for Drazin invertibility. Here we can give another proof of this result.

Indeed, let $\mathrm{i}(a)$ denote the Drazin index of a. Suppose that $exe+1-e \in \mathcal{A}^D$ with $\mathrm{i}(a) = k$. Then using the similar argument as in the proof of Theorem 2.5, we can know that the necessity suffices to prove that $(exe)^k - (exe)^{k+1}e(exe+1-e)^De = 0$. In fact,

$$\begin{aligned}
&(exe)^k - (exe)^{k+1}e(exe+1-e)^De \\
&= e\Big((exe+1-e)^k - (exe+1-e)^{k+1}(exe+1-e)^D\Big)e
\end{aligned}$$

and $(exe+1-e)^k-(exe+1-e)^{k+1}(exe+1-e)^D=0$. It follows that exe is Drazin invertible and $\mathrm{i}(exe)\le \mathrm{i}(exe+1-e)$. Conversely, we can obtain that $exe+1-e$ is Drazin invertible with $\mathrm{i}(exe+1-e)\le \mathrm{i}(exe)$ analogously. This completes the proof.

3. The relation between generalized invertibility in two operator semigroups

In this section, let X, Y be two complex Banach spaces and $\mathscr{B}(X,Y)$ denote the set of all bounded linear operators from X to Y. Write $\mathscr{B}(X)=\mathscr{B}(X,X)$. Then $\mathscr{B}(X)$, the algebra of all bounded linear operators on X, is a Banach algebra, with respect to the usual operator norm. The symbol I_X (resp., I_Y) denotes the identity operator of $\mathscr{B}(X)$ (resp., $\mathscr{B}(Y)$), and is its unit. Let $A\in\mathscr{B}(X,Y)$. Recall that $A\in\mathscr{B}(X,Y)$ is regular if there exists $A^-\in\mathscr{B}(Y,X)$ such that $AA^-A=A$. The operator $A^-\in\mathscr{B}(Y,X)$ is called an inner inverse of A. It is well known that $A\in\mathscr{B}(X,Y)$ is regular if and only if $\mathscr{R}(A)$ is closed and complemented in Y and $\mathscr{N}(A)$ is complemented in X, where $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote range and null space of $A\in\mathscr{B}(X,Y)$, respectively.

Let $E\in\mathscr{B}(X)$ such that $E^2=E$. Then we establish the relation of pseudo Drazin invertibility (resp., generalized Drazin invertibility) between the operators in the semigroup $E\mathscr{B}(X)E+I_X-E$ and the semigroup $E\mathscr{B}(X)E$ in the previous section, which can be applied to relate pseudo Drazin invertibility (resp., generalized Drazin invertibility) between the operators in the semigroup $AA^-\mathscr{B}(Y)AA^-+I_Y-AA^-$ and the semigroup $A^=A\mathscr{B}(X)A^=A+I_X-A^=A$ assuming that $A^-,A^=\in\mathscr{B}(Y,X)$ are inner inverses of $A\in\mathscr{B}(X,Y)$. To beginning, we give an auxiliary lemma.

Lemma 3.1. *Let $A\in\mathscr{B}(X,Y)$ be regular and $A^-,A^=\in\mathscr{B}(Y,X)$ be its inner inverses. Let $E=AA^-$, $F=A^=A$, $\mathcal{R}=\mathscr{B}(X)$ and $\mathcal{S}=\mathscr{B}(Y)$. Then $\varphi: E\mathcal{S}E\to F\mathcal{R}F$ defined by $EME\mapsto F(A^-MA)F$ is a ring isomorphism.*

Proof. Obviously φ is well-defined. Let $EM_1E, EM_2E\in E\mathcal{S}E$. Then it is easy to check that $\varphi(EM_1E+EM_2E)=\varphi(EM_1E)+\varphi(EM_2E)$. In addition, since $EA=A$ and $AF=A$, it follows that

$$\begin{aligned}\varphi(EM_1E\cdot EM_2E)&=F(A^-M_1EEM_2A)F\\&=F(A^-M_1A)F\cdot F(A^-M_2A)F\\&=\varphi(EM_1E)\cdot\varphi(EM_2E)\end{aligned}$$

and $\varphi(E)=\varphi(EEE)=F(A^-EA)F=FA^-A=F$. Hence φ is a ring homomorphism.

Let $\psi: F\mathcal{R}F\to E\mathcal{S}E$ defined by $FNF\mapsto E(ANA^=)E$. Analogously, it can de derived that ψ is also a ring homomorphism. Moreover, we can obtain that

$$\varphi\psi(FNF)=\varphi(E(ANA^=)E)=F(A^-ANA^=A)F=FNF,$$
$$\psi\varphi(EME)=\psi(F(A^-MA)F)=E(AA^-MAA^=)E=EME.$$

Therefore, φ is a ring isomorphism. □

Theorem 3.2. *Let $A\in\mathscr{B}(X,Y)$ be regular and $A^-,A^=\in\mathscr{B}(Y,X)$ be its inner inverses, and $B\in\mathscr{B}(Y)$. Then the following conditions are equivalent:*

(1) $\Gamma=AA^-BAA^-+I_Y-AA^-$ *is pseudo Drazin invertible with* $\mathrm{ind}(\Gamma)=k$.
(2) $\Omega=A^=AA^-BA+I_X-A^=A$ *is pseudo Drazin invertible with* $\mathrm{ind}(\Omega)=k$.

In this case

$$\Gamma^{pD}=A\Omega^{pD}A^=AA^-+I_Y-AA^-$$

and

$$\Omega^{pD}=A^=AA^-\Gamma^{pD}A+I_X-A^=A.$$

Proof. Let $\Gamma_0 = AA^- BAA^-$ and $\Omega_0 = A^= AA^- BA$. Then

$$\varphi(\Gamma_0) = A^= A(A^- BA)A^= A = \Omega_0,$$
$$\psi(\Omega_0) = \psi(A^= AA^- BAA^= A) = AA^-(AA^- BAA^=)AA^- = \Gamma_0,$$

where φ and ψ are defined as Lemma 3.1.

(1)⇒(2). Assume that Γ is pseudo Drazin invertible with $\mathrm{ind}(\Gamma) = k$. Then by Theorem 2.5, Γ_0 is pseudo Drazin invertible with $\mathrm{ind}(\Gamma_0) = k$ in $AA^- \mathscr{B}(Y)AA^-$. Moreover, $\Gamma_0^{pD} = AA^- \Gamma^{pD} AA^-$. Since φ is a ring isomorphism by Lemma 3.1, it follows that $\varphi(\Gamma_0^{pD})$ is the pseudo Drazin inverse of $\varphi(\Gamma_0)$ in $A^= A\mathscr{B}(X)A^= A$ and $\mathrm{ind}\,(\varphi(\Gamma_0)) = k$. Hence,

$$\Omega_0^{pD} = \varphi(\Gamma_0^{pD}) = \varphi(AA^- \Gamma^{pD} AA^-) = A^= A(A^- \Gamma^{pD} A)A^= A = A^= AA^- \Gamma^{pD} A.$$

Now, from Theorem 2.5, we can obtain that

$$\begin{aligned}\Omega^{pD} &= \Omega_0^{pD} + I_X - A^= A\\ &= A^= AA^- \Gamma^{pD} A + I_X - A^= A,\end{aligned}$$

and $\mathrm{ind}(\Omega) = k$.

(2)⇒(1). The proof can be completed by the method analogous to that used above. □

Remark 3.3. In [16, Proposition 5], Patrício and Puystjens presented the relevant result for Drazin invertibility, which can be proved by an alternative method.

Indeed, the proof is similar to Theorem 3.2. We define φ and ψ as Lemma 3.1, and let $\Gamma_0 = AA^- BAA^-$ and $\Omega_0 = A^= AA^- BA$. Then $\varphi(\Gamma_0) = \Omega_0$ and $\psi(\Omega_0) = \Gamma_0$. It follows that Γ_0 is Drazin invertible with Drazin index k if and only if Ω_0 is Drazin invertible with Drazin index k by the method analogous to the proof (1)⇒(2) of Theorem 3.2. Then this completes the proof that Γ is Drazin invertible with Drazin index k if and only if Ω is Drazin invertible with Drazin index k according to [16, Theorem 1]. Furthermore, their representations of Drazin inverses can be given analogously.

In fact, by an analogous method of Theorem 3.2, we can also obtain the relevant result for generalized Drazin invertibility by Lemma 3.1. Here we can give an alternative method to prove it with the help of operator norm.

Theorem 3.4. *Let $A \in \mathscr{B}(X, Y)$ be regular and $A^-, A^= \in \mathscr{B}(Y, X)$ be its inner inverses, and $B \in \mathscr{B}(Y)$. Then the following conditions are equivalent:*

(1) *$\Gamma = AA^- BAA^- + I_Y - AA^-$ is generalized Drazin invertible.*

(2) *$\Omega = A^= AA^- BA + I_X - A^= A$ is generalized Drazin invertible.*

In this case

$$\Gamma^{gD} = A\Omega^{gD} A^= AA^- + I_Y - AA^-$$

and

$$\Omega^{gD} = A^= AA^- \Gamma^{gD} A + I_X - A^= A.$$

Proof. (1)⇒(2). Assume that Γ is generalized Drazin invertible. Then by Theorem 2.6 we get that $\Gamma_0 = AA^- BAA^-$ has a generalized Drazin inverse Γ_0^{gD} in $AA^- \mathscr{B}(Y)AA^-$. It follows that

$$\Gamma_0^{gD} AA^- BAA^- \Gamma_0^{gD} = \Gamma_0^{gD}, \tag{3.1}$$
$$\Gamma_0^{gD} AA^- BAA^- = AA^- BAA^- \Gamma_0^{gD}. \tag{3.2}$$

Multiplying (3.1) and (3.2), respectively, on the left side by $A^= AA^-$ and on the right side by A, we have

$$(A^= AA^- \Gamma_0^{gD} A)(A^= AA^- BA)(A^= AA^- \Gamma_0^{gD} A) = A^= AA^- \Gamma_0^{gD} A,$$
$$(A^= AA^- \Gamma_0^{gD} A)A^= AA^- BA = A^= AA^- BA(A^= AA^- \Gamma_0^{gD} A).$$

Let

$$S = (A^{=}AA^{-}BA) - (A^{=}AA^{-}BA)^2(A^{=}AA^{-}\Gamma_0^{gD}A),$$
$$T = (AA^{-}BAA^{-}) - (AA^{-}BAA^{-})^2\Gamma_0^{gD}.$$

Now it is enough to show that S is quasinilpotent. In fact,

$$\begin{aligned} S &= (A^{=}AA^{-}BA) - A^{=}AA^{-}(BAA^{-})^2\Gamma_0^{gD}A \\ &= A^{=}\left((AA^{-}BAA^{-}) - AA^{-}(BAA^{-})^2\Gamma_0^{gD}\right)A \\ &= A^{=}TA. \end{aligned}$$

Since $AA^{=}T = T$, it follows that

$$\begin{aligned} \|S^n\|^{\frac{1}{n}} &= \|(A^{=}TA)^n\|^{\frac{1}{n}} \\ &= \|A^{=}T^nA\|^{\frac{1}{n}} \\ &\leq \|A^{=}\|^{\frac{1}{n}}\cdot\|T^n\|^{\frac{1}{n}}\cdot\|A\|^{\frac{1}{n}}. \end{aligned}$$

Then $\lim\limits_{n\to\infty}\|T^n\|^{\frac{1}{n}} = 0$ yields $\lim\limits_{n\to\infty}\|S^n\|^{\frac{1}{n}} = 0$. We conclude that S is quasinilpotent, and hence $A^{=}AA^{-}\Gamma_0^{gD}A$ is the generalized Drazin inverse of $A^{=}AA^{-}BA$ in $A^{=}A\mathscr{B}(X)A^{=}A$. By Theorem 2.6 it follows that

$$\begin{aligned} \Omega^{gD} &= A^{=}AA^{-}\Gamma_0^{gD}A + I_X - A^{=}A \\ &= A^{=}AA^{-}\Gamma^{gD}A + I_X - A^{=}A \end{aligned}$$

since $\Gamma_0^{gD} = AA^{-}\Gamma^{gD}AA^{-}$.

(2)⇒(1). If Ω is generalized Drazin invertible, then by Theorem 2.6 it can be derived that $\Omega_0 = A^{=}AA^{-}BA$ has a generalized Drazin inverse Ω_0^{gD} in $A^{=}A\mathscr{B}(X)A^{=}A$. It follows that

$$\Omega_0^{gD}A^{=}AA^{-}BA\Omega_0^{gD} = \Omega_0^{gD}, \tag{3.3}$$
$$\Omega_0^{gD}A^{=}AA^{-}BA = A^{=}AA^{-}BA\Omega_0^{gD}. \tag{3.4}$$

Multiplying (3.3) and (3.4), respectively, on the right side by $A^{=}AA^{-}$ and on the left side by A, we have

$$(A\Omega_0^{gD}A^{=}AA^{-})(AA^{-}BAA^{-})(A\Omega_0^{gD}A^{=}AA^{-}) = A\Omega_0^{gD}A^{=}AA^{-},$$
$$(A\Omega_0^{gD}A^{=}AA^{-})AA^{-}BAA^{-} = AA^{-}BAA^{-}(A\Omega_0^{gD}A^{=}AA^{-}).$$

Let

$$P = (AA^{-}BAA^{-}) - (AA^{-}BAA^{-})^2(A\Omega_0^{gD}A^{=}AA^{-}),$$
$$Q = (A^{=}AA^{-}BA) - (A^{=}AA^{-}BA)^2\Omega_0^{gD}.$$

Then it suffices to prove that P is quasinilpotent. Indeed, since $A^{=}A\Omega_0 = \Omega_0A^{=}A$, we obtain $A^{=}A\Omega_0^{gD} = \Omega_0^{gD}A^{=}A$ by Lemma 2.2. It follows that

$$\begin{aligned} P &= (AA^{-}BAA^{-}) - AA^{-}(BAA^{-})^2(A\Omega_0^{gD}A^{=}AA^{-}) \\ &= (AA^{-}BAA^{-}) - AA^{-}(BAA^{-})^2(AA^{=}A\Omega_0^{gD}A^{-}) \\ &= A\left(A^{=}AA^{-}BA - A^{=}AA^{-}(BAA^{-})^2A\Omega_0^{gD}\right)A^{-} \\ &= AQA^{-}. \end{aligned}$$

Since $QA^-A = Q$, we have

$$\begin{aligned}\|P^n\|^{\frac{1}{n}} &= \|(AQA^-)^n\|^{\frac{1}{n}}\\ &= \|AQ^nA^-\|^{\frac{1}{n}}\\ &\le \|A\|^{\frac{1}{n}}\cdot \|Q^n\|^{\frac{1}{n}} \cdot \|A^-\|^{\frac{1}{n}}.\end{aligned}$$

Then $\lim\limits_{n\to\infty} \|Q^n\|^{\frac{1}{n}} = 0$ implies $\lim\limits_{n\to\infty} \|P^n\|^{\frac{1}{n}} = 0$. It follows that P is quasinilpotent, and hence $A\Omega_0^{gD}A^=AA^-$ is the generalized Drazin inverse of AA^-BAA^- in $AA^-\mathscr{B}(Y)AA^-$. Therefore by Theorem 2.6, we obtain that

$$\begin{aligned}\Gamma^{gD} &= A\Omega_0^{gD}A^=AA^- + I_Y - AA^-\\ &= A\Omega^{gD}A^=AA^- + I_Y - AA^-\end{aligned}$$

since $\Omega_0^{gD} = A^=A\Omega^{gD}A^=A$, which completes the proof. □

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